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TAUBERIAN CONSTANTS FOR SOME DOUBLE HAUSDORFF MATRICES

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Abstract. Mirza and Thorpe [9], obtained the Tauberian constant for $C(1,1)$ summability for sequences satisfying the boundedness condition $(m^2+n^2)a_{mn}=O(1)$. In this paper we investigate the analogous questions for double Hausdorff matrices.

The problem of finding Tauberian constants for singly infinite matrices was solved completely for generalized Hausdorff matrices by Jakimovski and Leviatan [6]. For doubly infinite matrices the situation is far more complex.

Knopp [8] showed that

$$(m^2+n^2)a_{mn}=O(1) \tag{1}$$

is a Tauberian condition for $C(1,1)$; i.e., if $\{s_{mn}\}$ is a sequence which is $C(1,1)$-summable and satisfies (1), then $\{s_{mn}\}$ is convergent.

Kloosterman [7] showed that (1) is a Tauberian condition for $C(\alpha,\beta)$-summability for α and β positive integers.

Mizra and Thorpe [9] obtained the best Tauberian constant for $C(1,1)$ under condition (1).

It is not known whether (1) is a Tauberian condition for all regular double Hausdorff matrices. Nevertheless, using condition (1) we obtain, in Theorem 1, a very useful estimate for a large class of double Hausdorff matrices. Theorem 2 gives upper and lower bounds on the Tauberian constants.

Theorem 4 gives an explicit representation for the value of the Tauberian constant when H is a nonnnegative factorable double Hausdorff matrix.

Tauberian constants are also computed for some specific factorable double Hausdorff matrices, when one of the factors is the identity matrix.

A double sequence $\{s_{mn}\}$ is said to converge, in the Pringsheim sense, to a number s if, for each $\epsilon>0$, there exists a positive integer $K(\epsilon)$ such that $|s_{mn}-s|<\epsilon$ whenever $m,n\geq K(\epsilon)$.

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The notation $s_{mn} = O(1)$ means that $\{s_{mn}\}$ is a bounded sequence.

A regular double infinite Hausdorff matrix H has entries

$$h_{mnij} = \binom{m}{i}\binom{n}{j}\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\binom{m-i}{r}\binom{n-j}{s}\mu_{i+r,j+s},$$
$$0 \le i \le m, \quad 0 \le j \le n,$$

where

$$\mu_{mn} = \int_0^1\int_0^1 s^m t^n dg(s,t),$$

$g(s,t) \in BV([0,1]\times[0,1])$, and satisfies

$$g(s,0) = g(s,0+) = g(0+,t) = g(0,t) = 0, 0 \le s,t \le 1,$$

$$g(1,1) - g(1,0) - g(0,1) + g(0,0) = 1.$$

See, for example, [1].

We shall call a regular double Hausdorff matrix factorable if its mass function $g(s,t)$ can be factored into the form $g(s,t) = f(s)g(t)$, where $f, g, \in BV[0,1]$. As a consequence, each term h_{mnij} factors into the product $h'_{mi}h''_{nj}$,where (h'_{mi}) and (h''_{nj}) are each singly infinite Hausdorff matrices.

Examples of factorable Hausdorff matrices are the double Cesáro matrices $C(\alpha,\beta)$, double Euler matrices, double Hölder matrices, and double gamma matrices.

Define

$$t_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} h_{mnij}s_{ij}, \tag{2}$$

where

$$s_{mn} := \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}.$$

Theorem 1. *For every sequence $\{s_{mn}\}$ satisfying (1) and, for every regular double Hausdorff matrix which satisfies*

$$\int_0^1\int_0^1 \frac{|g(x,y)|}{xy}dxdy < \infty,$$

$$\int_0^1 \frac{|g(x,1)|}{x}dx \quad \text{and} \quad \int_0^1 \frac{|g(1,y)|}{y}dy \quad \text{exist}, \tag{3}$$

there is a constant M such that

$$\limsup_{m,n\to\infty}|s_{mn} - t_{mn}| \le M \limsup_{m,n\to\infty}|(m^2+n^2)a_{mn}|.$$

To prove this theorem we will need the following lemmas.

Lemma 1. [9] *Let B be a doubly infinite matrix satisfying the conditions*

$$\lim_{m,n\to\infty}\sum_{i=0}^{\infty}|b_{mnij}|=0 \quad \text{for each} \quad j=0,1,\dots, \tag{4}$$

$$\lim_{m,n\to\infty}\sum_{j=0}^{\infty}|b_{mnij}|=0 \quad \text{for each} \quad i=0,1,\dots, \quad \text{and} \tag{5}$$

$$\limsup_{m,n\to\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|b_{mnij}|=M<\infty. \tag{6}$$

Then each bounded sequence $s=\{s_{ij}\}$ has a transform

$$y_{mn}(s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{mnij}s_{ij}$$

such that

$$\limsup_{m,n\to\infty}|y_{mn}(s)|\le M\limsup_{m,n\to\infty}|s_{mn}|. \tag{7}$$

Moreover, there is a bounded sequence s, which is real if the b_{mnij} are real, and for which equality holds in (7).

Lemma 2. *Let A be a regular doubly infinite matrix, $\{s_{mn}\}$ a bounded null sequence. Then*

$$\lim_{m,n\to\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{mnij}s_{ij}|=0.$$

For any integers J and K,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}=\sum_{i=0}^{J}\sum_{j=0}^{\infty}+\sum_{i=J+1}^{\infty}\sum_{j=0}^{K}+\sum_{i=J+1}^{\infty}\sum_{j=K+1}^{\infty}.$$

The proof is obvious since $\{s_{mn}\}$ is a null sequence and A is regular.

To prove Theorem 1, using (1),

$$\begin{aligned}
s_{mn} - t_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} - \sum_{r=0}^{m}\sum_{s=0}^{n} h_{mnrs} \sum_{i=0}^{r}\sum_{j=0}^{s} a_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} - \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \sum_{r=i}^{m}\sum_{s=j}^{n} h_{mnrs} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \Big[1 - \sum_{r=i}^{m}\sum_{s=j}^{n} h_{mnrs}\Big] \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \Big[\sum_{r=0}^{i-1}\sum_{s=0}^{j-1} + \sum_{r=0}^{i-1}\sum_{s=j}^{n} + \sum_{r=i}^{m}\sum_{s=0}^{j-1}\Big] h_{mnrs} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \Big[\sum_{r=0}^{i-1}\sum_{s=0}^{n} + \sum_{r=0}^{m}\sum_{s=0}^{j-1} - \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\Big] h_{mnrs}.
\end{aligned}$$

Define $u_{ij} = (i^2 + j^2)a_{ij}$. Then

$$s_{mn} - t_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{u_{ij}}{i^2 + j^2} \Big[\sum_{r=0}^{i-1}\sum_{s=0}^{n} + \sum_{r=0}^{m}\sum_{s=0}^{j-1} - \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\Big] h_{mnrs},$$

where it is understood that $u_{ij}/(i^2 + j^2) = 0$ whenever $i = j = 0$.

Define B by

$$b_{mnij} = \begin{cases} \frac{1}{i^2+j^2}\Big[\sum_{r=0}^{i-1}\sum_{s=0}^{n} + \sum_{r=0}^{m}\sum_{s=0}^{j-1} - \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\Big] h_{mnrs}, \\ \qquad\qquad , 1 \le i \le m, 1 \le j \le n, \\ \frac{1}{j^2}\sum_{r=0}^{m}\sum_{s=0}^{j-1} h_{mnrs}, \quad i = 0, 1 \le j \le n, \\ \frac{1}{i^2}\sum_{r=0}^{i-1}\sum_{s=0}^{n} h_{mnrs}, \quad 1 \le i \le m, j = 0, \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \tag{8}$$

We shall first show that B satisfies (5).

Since $b_{mn00} = 0$,

$$\begin{aligned}
\sum_{j=0}^{n} |b_{mn0j}| = \sum_{j=1}^{n} |b_{mn0j}| &= \sum_{j=1}^{n} \frac{1}{j^2}\Big|\sum_{r=0}^{m}\sum_{s=0}^{j-1} h_{mnrs}\Big| \\
&\le \sum_{r=0}^{m}\sum_{s=0}^{n-1} |h_{mnrs}| \sum_{j=s+1}^{n} \frac{1}{j^2},
\end{aligned}$$

which tends to zero by Lemma 2, since $\{\sum_{j=s+1}^{\infty} 1/j^2\}$ is a bounded null sequence and H is regular.

Similarly,

$$\lim_{m,n\to\infty} \sum_{i=1}^{n} |b_{mni0}| = 0.$$

For $i > 0$,

$$\begin{aligned}\sum_{j=0}^{n} |b_{mnij}| &= \sum_{j=0}^{n} \Big| \frac{1}{i^2+j^2} \Big[\sum_{r=0}^{i-1}\sum_{s=0}^{n} + \sum_{r=0}^{m}\sum_{s=0}^{j-1} - \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} \Big] h_{mnrs} \Big| \\ &\le \sum_{j=0}^{n} \frac{1}{i^2+j^2} \Big[\sum_{r=0}^{i-1}\sum_{s=0}^{n} + \sum_{r=0}^{m}\sum_{s=0}^{j-1} + \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} \Big] |h_{mnrs}| \\ &= I_4 + I_5 + I_6, \quad \text{say.}\end{aligned}$$

$$\begin{aligned}I_4 &= \sum_{j=0}^{n} \frac{1}{i^2+j^2} \sum_{r=0}^{i-1}\sum_{s=0}^{n} |h_{mnrs}| \le \sum_{j=0}^{n} \frac{1}{1+j^2} \sum_{r=0}^{i-1}\sum_{s=0}^{n} |h_{mnrs}| \\ &\le \frac{\pi^2}{3} \sum_{r=0}^{i-1}\sum_{s=0}^{n} |h_{mnrs}|,\end{aligned}$$

which tends to zero as $m, n \to \infty$, since H is regular.

$$I_5 = \sum_{j=1}^{n} \frac{1}{i^2+j^2} \sum_{r=0}^{m}\sum_{s=0}^{j-1} |h_{mnrs}| \le \sum_{r=0}^{m}\sum_{s=0}^{n} |h_{mnrs}| \sum_{j=s+1}^{n} \frac{1}{j^2},$$

which tends to zero by Lemma 2.

$$I_6 = \sum_{j=1}^{n} \frac{1}{i^2+j^2} \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} |h_{mnrs}| \le \sum_{r=0}^{i-1}\sum_{s=0}^{n} |h_{mnrs}| \sum_{j=s+1}^{n} \frac{1}{j^2},$$

which tends to zero by Lemma 2.

Similarly, (4) is satisfied.

To verify condition (6), since $b_{mn00} = 0$,

$$\sum_{i=0}^{m}\sum_{j=0}^{n} |b_{mnij}| = \sum_{j=1}^{n} |b_{mn0j}| + \sum_{i=1}^{m} |b_{mni0}| + \sum_{i=1}^{m}\sum_{j=1}^{n} |b_{mnij}|.$$

Since conditions (4) and (5) are satisfied, the first two sums have limit zero.

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n}|b_{mnij}| &= \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{i^2+j^2}\Big|\sum_{r=0}^{i-1}\sum_{s=0}^{n}+\sum_{r=0}^{m}\sum_{s=0}^{j-1}-\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}h_{mnrs}\Big|\\ &\leq I_7+I_8+I_9, \quad \text{say.}\end{aligned} \tag{9}$$

Define

$$p_{mnrs}(x,y)=\binom{m}{r}\binom{n}{s}x^r(1-x)^{m-r}y^s(1-y)^{n-s}.$$

Then

$$\begin{aligned}\sum_{r=0}^{i-1}\sum_{s=0}^{n}h_{mnrs} &= \sum_{r=0}^{i-1}\sum_{s=0}^{n}\int_0^1\int_0^1 p_{mnrs}(x,y)dg(x,y)\\ &= \int_0^1\sum_{r=0}^{i-1}\binom{m}{r}x^r(1-x)^{m-r}\int_0^1 dg(x,y).\end{aligned}$$

Define

$$f(x)=\int_0^x\int_0^1 dg(u,y).$$

Then $f(x)=g(x,1), f(0^+)=f(0), f(1)=1$, and $f\in BV[0,1]$.
Consequently,

$$\sum_{r=0}^{i-1}\sum_{s=0}^{n}h_{mnrs}=\sum_{r=0}^{i-1}h'_{mr},$$

where h'_{mr} are the entries of the Hausdorff matrix generated by f.

The integral

$$\int_0^1\frac{|f(x)|}{x}dx$$

exists by hypothesis.

Lemma 3.

$$J:=\limsup_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{i^2+j^2}\Big|\sum_{r=0}^{i-1}h'_{mr}\Big|\leq\frac{\pi}{2}\int_0^1\frac{|g(x,1)|}{x}dx.$$

Proof.

$$\sum_{j=1}^{\infty}\frac{1}{i^2+j^2}\leq\int_0^\infty\frac{dy}{i^2+y^2}=\frac{\pi}{2i}.$$

We wish to show that

$$\lim_{m\to\infty}\sum_{i=1}^{m}\frac{1}{i}\Big|\sum_{r=0}^{i-1}h'_{mr}\Big|\le\int_0^1\frac{|g(x,1)|}{x}dx,$$

where

$$h'_{mr}=\binom{m}{r}\int_0^1 x^r(1-x)^{m-r}dg(x,1).$$

Using the argument in [5],

$$\int_x^1\frac{i}{t}\binom{m}{i}t^i(1-t)^{m-i}dt=\sum_{r=0}^{i-1}\binom{m}{r}x^r(1-x)^{m-r}.$$

Integrating by parts,

$$\begin{aligned}\sum_{i=1}^{m}\frac{1}{i}\Big|\sum_{r=0}^{i-1}h'_{mr}\Big| &= \sum_{i=1}^{m}\frac{1}{i}\Big|\int_0^1\Big(\int_x^1\frac{i}{t}\binom{m}{i}t^i(1-t)^{m-i}dt\Big)dg(x,1)\Big|\\ &= \sum_{i=1}^{m}\frac{1}{i}\Big|\int_0^1 g(x,1)\frac{i}{x}\binom{m}{i}x^i(1-x)^{m-i}dx\Big|\\ &= \sum_{i=1}^{m}\binom{m}{i}\Big|\int_0^1\frac{g(x,1)}{x}x^i(1-x)^{m-i}dx\Big|\\ &\le \int_0^1\frac{|g(x,1)|}{x}\Big(\sum_{i=1}^{m}\binom{m}{i}x^i(1-x)^{m-i}\Big)dx\\ &\le \int_0^1\frac{|g(x,1)|}{x}dx.\end{aligned}$$

Thus

$$J\le\frac{\pi}{2}\int_0^1\frac{|g(x,1)|}{x}dx.$$

$$\begin{aligned}\limsup_{m,n\to\infty}I_7 &=\limsup_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{i^2+j^2}\Big|\sum_{r=0}^{i-1}\sum_{s=0}^{n}h_{mnrs}\Big|\\ &=\limsup_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{i^2+j^2}\Big|\sum_{r=0}^{i-1}h'_{mr}\Big|\\ &\le\frac{\pi}{2}\int_0^1\frac{|g(x,1)|}{x}dx<\infty.\end{aligned}$$

With $g(1,y)$ denoting the mass function corresponding to h''_{ns}, where

$$\sum_{r=0}^{m}\sum_{s=0}^{j-1} h_{mnrs} = \sum_{s=0}^{j-1} h''_{ns},$$

it then follows, by hypothesis, that

$$\limsup_{m,n\to\infty} I_8 \le \frac{\pi}{2}\int_0^1 \frac{|g(1,y)|}{y}dy < \infty. \tag{10}$$

To evaluate I_9, using an argument of Delange [2]

$$\begin{aligned}\sum_{r=0}^{i-1}\sum_{s=0}^{j-1} h_{mnrs} &= \int_0^1\int_0^1 \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} p_{mnrs}(x,y)dg(x,y)\\ &= \int_0^1\int_0^1 \Big(\int_x^1 \frac{i}{u}\binom{m}{i}u^i(1-u)^{m-i}du\Big)\times\\ &\qquad \Big(\int_y^1 \frac{j}{v}\binom{n}{j}v^j(1-v)^{n-j}dx\Big)dg(x,y),\end{aligned}$$

which, on integrating the double Stieltjes integral by parts, gives

$$ij\int_0^1\int_0^1 \frac{1}{x}\binom{m}{i}x^i(1-x)^{m-i}\frac{1}{y}\binom{n}{j}y^j(1-y)^{n-j}g(x,y)dxdy.$$

Using 1.7.1 of Delange [2],

$$\sum_{r=0}^{i-1}\sum_{s=0}^{j-1} h_{mnrs} = ij\int_0^1\int_0^1 p_{mnij}(x,y)\frac{g(x,y)}{xy}dxdy.$$

$$\begin{aligned}I_9 &\le \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{ij}{i^2+j^2}\int_0^1\int_0^1 p_{mnij}(x,y)\frac{|g(x,y)|}{xy}dydx\\ &\le \frac{1}{2}\int_0^1\int_0^1\sum_{i=1}^{m}\sum_{j=1}^{n} p_{mnij}(x,y)\frac{|g(x,y)|}{xy}dydx\\ &= \frac{1}{2}\int_0^1\int_0^1 \frac{|g(x,y)|}{xy}dydx < \infty.\end{aligned}$$

□

As an immediate corollary we obtain upper and lower bounds on M.

Corollary 1. *Let H be a regular double Hausdorff matrix with nonnegative entries, which satisfies* (3). *Then*

$$\frac{\pi}{4}\max\Big(\int_0^1 \frac{g(x,1)}{x}dx, \int_0^1 \frac{g(1,y)}{y}dy\Big) \\ \le M \le \frac{\pi}{2}\Big(\int_0^1 \frac{g(x,1)}{x}dx + \int_0^1 \frac{g(1,y)}{y}dy\Big). \tag{11}$$

Proof. From the proof of Theorem 1 it is clear that the value of M comes from $I_7 - I_9$. From (9) it follows immediately that
$M \le \limsup_{m,n}(I_7 + I_8)$, which gives the right hand side of (11).

To prove the left hand side, note that, from (9),

$$\Big[\sum_{r=0}^{m}\sum_{s=0}^{j-1} - \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\Big] h_{mnrs} = \sum_{r=i}^{m}\sum_{s=0}^{j-1} h_{mnrs} \ge 0.$$

Therefore $M \ge \limsup_{m,n} I_7$. Similarly, $M \ge \limsup_{m,n} I_8$. The left hand side of (11) now follows from (8) and (9). □

For certain double Hausdorff matrices we can obtain a much stronger result. Since $C(\alpha,\beta)$ is factorable, if $\{x_{mn}\}$ is summable $C(\alpha,\beta)$ then it is summable $C(\alpha',\beta')$ for $\alpha' \ge \alpha, \beta' \ge \beta$. The proof of this fact is similar to the proof of the corresponding one-dimensional result in [4], Theorem 43.

Theorem 2. *Let $\alpha, \beta \ge 0$. The conditions*

$$p_{mn} = (m+1)\sum_{k=0}^{n} a_{mk} = O(1), \quad q_{mn} = (n+1)\sum_{j=0}^{m} a_{jn} = O(1) \tag{12}$$

are Tauberian conditions for $C(\alpha,\beta)$.

Proof. Since $\{s_{mn}\}$ is bounded and summable $C(\alpha,\beta)$, it is summable $C([\alpha]+1, [\beta]+1)$. But (12) implies that $\{s_{mn}\}$ is convergent. Therefore (12) is a Tauberian conditions for $C(\alpha,\beta)$. □

Corollary 2. *Let $\alpha, \beta \ge 0$. Then* (1) *is a Tauberian condition for $C(\alpha,\beta)$.*

Proof. Theorem 2 of [7] states that (1) implies (12). □

Corollary 3. *Let H be a double Hausdorff matrix such that $C(\alpha,\beta)$, for some $\alpha,\beta \ge 0$, is stronger than H for bounded sequences. Then* (1) *is a Tauberian condition for H.*

The double gamma method is generated by $\mu_{mn} = ab/(m+a)(n+b)$.

Corollary 4. *Let H be a double gamma method for $a, b > 0$. Then* (1) *is a Tauberian condition for H.*

Proof. Since H is equivalent to $C(1,1)$, the result follows from Corollary 3. $\square$

The double Hölder method of order α, β, written $H(\alpha, \beta)$, is generated by $\mu_{mn} = (m+1)^{-\alpha}(n+1)^{-\beta}$.

Corollary 5. *Let H be a double Hölder method of order $\alpha, \beta \geq 0$. Then* (1) *is a Tauberian condition for H.*

Proof. The proof follows from Corollary 2, since $H(\alpha, \beta)$ is equivalent to $C(\alpha, \beta)$. $\square$

For factorable nonnegative Hausdorff matrices it is possible to obtain an explicit representation for M.

Theorem 3. *Let H be a factorable double Hausdorff matrix with nonnegative entries. Then*

$$\begin{aligned} M = \limsup_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{i^2+j^2}\Big[i\int_0^1 \frac{\alpha(x)}{x} p_{mi}(x)dx \\ + j\int_0^1 \frac{\beta(y)}{y} p_{nj}(y)dy \\ - ij\int_0^1 \frac{\alpha(x)}{x} p_{mi}(x)dx \int_0^1 \frac{\beta(y)}{y} p_{nj}(y)dy \Big]. \end{aligned} \tag{13}$$

Proof. We may remove the absolute values from (9), since the entries of H are nonnegative. Since H is factorable, the mass function for H can be written in the form $\alpha(x)\beta(y)$. Thus

$$\sum_{r=0}^{i-1}\sum_{s=0}^{n} h_{mnrs} = \sum_{r=0}^{i-1} h'_{mr},$$

which can be written as

$$i\int_0^1 \frac{\alpha(x)}{x} p_{mi}(x)dx,$$

where

$$p_{mi}(x) = \binom{m}{i} x^i (1-x)^{m-i}.$$

Similarly,

$$\sum_{r=0}^{m}\sum_{s=0}^{j-1} h_{mnrs} = j\int_0^1 \frac{\beta(y)}{y} q_{nj}(y)dy,$$

where

$$q_{nj}(y) = \binom{n}{j} y^j (1-y)^{n-j},$$

and (13) follows. $\square$

For example, if $H = C(1,1)$, it is easy to verify that the quantity in brackets in (13) becomes

$$\frac{i}{m+1} + \frac{j}{n+1} - \frac{ij}{(m+1)(n+1)},$$

which is the form which appears in [9]. One can also use (6) to obtain the same expression, since, as was shown in the proof of Theorem 1, only the entries in the first line of b_{mnij} contribute to the value of M.

For the Cèsaro methods $C(\alpha, \beta)$ for positive integers α and β,

$$h_{mnrs} = \frac{\alpha(m!)\Gamma(m+\alpha-r)\beta(n!)\Gamma(n+\beta-s)}{(m-r)!\Gamma(m+\alpha+1)(n-s)!\Gamma(n+\beta+1)}.$$

Thus

$$\begin{aligned}\sum_{r=0}^{i-1}\sum_{s=0}^{n} h_{mnrs} &= \frac{\alpha(m!)}{\Gamma(m+\alpha+1)}\sum_{r=0}^{i-1}\frac{\Gamma(m+\alpha-r)}{\Gamma(m-r+1)} \\ &= 1 - \frac{(m-i+1)\cdots(m)}{(m-i+1+\alpha)\cdots(m+\alpha)}.\end{aligned}$$

for $i \geq \alpha + 1$.

Since α is an integer, the above sum may be written in the form

$$\begin{aligned} i\sum_{j=1}^{\alpha}\frac{1}{m+j} &- i^2 \sum \prod_{h\neq j}^{\alpha} \frac{1}{(m+h)(m+j)} \\ &+ i^3 \sum \prod_{h\neq j\neq k} \frac{1}{(m+h)(m+j)(m+k)} \\ &+ \cdots + \frac{(-1)^{\alpha+1} i^{\alpha}}{\prod_{j=1}^{\alpha}(m+j)}. \end{aligned} \tag{14}$$

Similarly,

$$\begin{aligned}\sum_{r=0}^{m}\sum_{s=0}^{j-1} h_{mnrs} &= \frac{\beta(n!)}{\Gamma(n+\beta+1)}\sum_{s=0}^{j-1}\frac{\Gamma(n+\beta-s)}{\Gamma(n-s+1)} \\ &= 1 - \frac{(n-j+1)\cdots(n)}{(n-j+1+\beta)\cdots(n+\beta)},\end{aligned}$$

which can be written in the form, for $j \geq \beta + 1$,

$$\begin{aligned} j\sum_{i=1}^{\beta} \frac{1}{n+i} - j^2 \sum \prod_{i \neq h} \frac{1}{(n+i)(n+h)} \\ + j^3 \sum \prod_{i \neq h \neq k} \frac{1}{(n+1)(n+h)(n+k)} \\ + \cdots + \frac{(-1)^{\beta+1} j^{\beta}}{\prod_{i=1}^{\beta}(n+i)}. \end{aligned} \tag{15}$$

Finally,

$$\begin{aligned} \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} h_{mnrs} \Big[i \sum_{j=1}^{\alpha} \frac{1}{m+j} \\ - i^2 \sum \prod_{i \neq k} \frac{1}{(n+i)(m+k)} \\ + \cdots + \frac{(-1)^{\alpha+1} i^{\alpha}}{\prod_{j=1}^{\alpha}(m+j)} \Big] \Big[j \sum_{i=1}^{\beta} \frac{1}{n+i} \\ - j^2 \sum \prod_{i \neq k} \frac{1}{(n+i)(n+k)} \\ + \cdots + \frac{(-1)^{\beta+1} j^{\beta}}{\prod_{i=1}^{\beta}(n+i)} \Big]. \end{aligned} \tag{16}$$

The following result appears in [3].

Lemma 4. *If f_x, f_y and f_{xy} are continuous, then*

$$\begin{aligned} \Big| \sum_{i=1}^{m}\sum_{j=1}^{n} f(i,j) - \int_1^{m+1}\int_1^{n+1} f(x,y)dxdy \Big| \\ \leq \int_1^{m+1}\int_1^{n+1} (|f_x| + |f_y| + |f_{xy}|)dxdy. \end{aligned} \tag{17}$$

We shall now show that this Lemma of Hardy also applies to $C(\alpha, \beta)$ for α, β positive integers.

Define f by

$$f(x,y) = \frac{x}{x^2+y^2} \sum_{j=1}^{\alpha} \frac{1}{m+j}.$$

Then

$$|f_x| \le \frac{\alpha}{m+1}\frac{|y^2-x^2|}{(x^2+y^2)^2} \le \frac{\alpha}{(m+1)(x^2+y^2)} = k_1(x,y), \quad \text{say},$$

$$|f_y| \le \frac{\alpha}{(m+1)}\frac{|2xy|}{(x^2+y^2)^2} \le k_1(x,y), \quad \text{and}$$

$$|f_{xy}| \le \frac{\alpha}{(m+1)}\frac{2|y(3x^2-y^2)|}{(x^2+y^2)^3} \le \frac{6\alpha|y|}{(m+1)(x^2+y^2)^2} = k_2(x,y), \quad \text{say}.$$

Using the argument in [9],

$$\int_0^{m+1}\int_0^{n+1}(2k_1+k_2)dxdy \to 0 \quad \text{as} \quad m,n\to\infty.$$

Each of the other terms is represented by a finite sum of functions of the form

$$g(x,y) = \frac{(-1)^{k+1}}{(m+i_1)(m+i_2)\cdots(m+i_k)}\frac{x^k}{x^2+y^2},$$

where each i_r satisfies $1 \le i_r \le \alpha, 1 \le r \le k, 2 \le k \le \alpha$. Then

$$g_x = \frac{(-1)^{k+1}}{(m+i_1)(m+i_2)\cdots(m+i_k)}\frac{x^{k-1}((k-2)x^2+ky^2)}{(x^2+y^2)^2}.$$

Since $1 \le x \le m+1$, $1 \le y \le n+1$,

$$\begin{aligned}|g_x| &\le \frac{kx^{k-1}}{(m+1)^k}\Big[\frac{x^2}{(x^2+y^2)^2}+\frac{y^2}{(x^2+y^2)^2}\Big]\\ &\le \frac{k}{(m+1)(x^2+y^2)} \le kk_1, \text{ say}.\end{aligned}$$

Therefore

$$\int_1^{m+1}\int_1^{n+1}|g_x|dydx \to 0 \quad \text{as} \quad m,n\to\infty.$$

$$|g_y| = \Big|\frac{-2yx^k}{(m+i_1)(m+i_2)\cdots(m+i_k)(x^2+y^2)^2}\Big| \le k_2,$$

so

$$\int_1^{m+1}\int_1^{n+1}|g_y|dydx \to 0 \quad \text{as} \quad m,n\to\infty,$$

$$\begin{aligned}|g_{xy}| &= \frac{2yx^{k-1}|(k-4)x^2+ky^2|}{(m+i_1)(m+i_2)\cdots(m+i_k)(x^2+y^2)^3}\\ &\le \frac{2y}{(m+1)}\frac{k+4}{(x^2+y^2)^2} \le (k+4)k_2(x,y), \quad \text{say}.\end{aligned}$$

and

$$\int_1^{m+1}\int_1^{n+1}|g_{xy}|dydx \to 0 \quad \text{as} \quad m,n\to\infty.$$

This shows that Lemma 4 can be applied to the first term in (13), and a similar argument applies to the second and third terms.

To find M for a factorable Hausdorff method one first performs the indicated integrations in (13). The resulting series are then estimated by integrations, using the result of Lemma 4, and involving many more terms, in order to obtain the best constant for $C(\alpha,\beta)$. One then replaces $(m+1)/(n+1)$ in the answer by λ and then determines the maximum of this function of λ.

For $C(1,1)$,

$$f(\lambda)=\frac{1}{4\lambda}\log(1+\lambda^2)+\frac{\lambda}{4}\log\Big(1+\frac{1}{\lambda^2}\Big)+\arctan\lambda+\arctan\Big(\frac{1}{\lambda}\Big)$$

which has an absolute maximum $M=(1/2)\log 2+\pi/2$ at $\lambda=1$.

For $C(2,2)$, from (13),

$$\begin{aligned}
&i\int_0^1\frac{1-(1-x)^2}{x}\binom{m}{i}x^i(1-x)^{m-i}dx\\
&=i\binom{m}{i}\Big[\int_0^1 x^{i-1}(1-x)^{m-i}dx-\int_0^1 x^{i-1}(1-x)^{m-i+2}dx\Big]\\
&=i\binom{m}{i}\Big[\frac{\Gamma(i)\Gamma(m-i+1)}{\Gamma(m+1)}-\frac{\Gamma(i)\Gamma(m-i+3)}{\Gamma(m+3)}\Big]\\
&=1-\frac{(m-i+1)(m-i+2)}{(m+1)(m+2)}\\
&=\frac{i}{m+1}+\frac{i}{m+2}-\frac{i^2}{(m+1)(m+2)}.
\end{aligned}$$

Similarly,

$$j\int_0^1\frac{\beta(y)}{y}q_{nj}(y)dy=\frac{j}{n+1}+\frac{j}{n+2}-\frac{j^2}{(n+1)(n+2)},$$

and

$$\begin{aligned}
ij\int_0^1\frac{\alpha(x)}{x}p_{mi}(x)dx\int_0^1\frac{\beta(y)}{y}q_{nj}(y)dy&\\
=\Big(\frac{i}{m+1}+\frac{i}{m+2}-\frac{i^2}{(m+1)(m+2)}\Big)&\times\\
\times\Big(\frac{j}{n+1}+\frac{j}{n+2}-\frac{j^2}{(n+1)(n+2)}\Big)&.
\end{aligned}$$

Then the quantity in brackets in (13) is equal to

$$\begin{aligned}\frac{2i}{m+1}-&\frac{i^2}{(m+1)^2}+\frac{2j}{n+1}-\frac{j^2}{(n+1)^2}-\frac{4ij}{(m+1)(n+1)}\\&+\frac{2ij^2}{(m+1)(n+1)^2}+\frac{2i^2j}{(m+1)^2(n+1)}\\&+\frac{i^2j^2}{(m+1)^2(n+1)^2}+\epsilon(m,n),\end{aligned}\tag{18}$$

where

$$\begin{aligned}\epsilon(m,n)={}&i\Big(\frac{1}{m+2}-\frac{1}{m+1}\Big)-i^2\Big(\frac{1}{(m+1)(m+2)}-\frac{1}{(m+1)^2}\Big)\\&+j\Big(\frac{1}{n+2}-\frac{1}{n+1}\Big)-j^2\Big(\frac{1}{(n+1)(n+2)}-\frac{1}{(n+1)^2}\Big)\\&+ij\Big(\frac{3}{(m+1)(n+1)}-\frac{1}{(m+2)(n+1)}\\&\qquad-\frac{1}{(m+1)(n+2)}-\frac{1}{(m+2)(n+2)}\Big)\\&+ij^2\Big(\frac{-2}{(m+1)(n+1)^2}+\frac{1}{(m+1)(n+1)(n+2)}\\&\qquad+\frac{1}{(m+2)(n+1)(n+2)}\Big)\\&+i^2j\Big(\frac{-2}{(m+1)^2(n+1)}+\frac{1}{(m+1)(m+2)(n+1)}\\&\qquad+\frac{1}{(m+1)(m+2)(n+2)}\Big)\\&-i^2j^2\Big(\frac{1}{(m+1)^2(n+1)^2}-\frac{1}{(m+1)(m+2)(n+1)(n+2)}\Big).\end{aligned}$$

From (18),

$$\begin{aligned}I_1:={}&\frac{2}{m+1}\int_0^{m+1}\int_0^{n+1}\frac{xdydx}{x^2+y^2}\\={}&\Big(\frac{n+1}{m+1}\Big)\log\Big[1+\Big(\frac{m+1}{n+1}\Big)^2\Big]+2\arctan\Big(\frac{n+1}{m+1}\Big).\end{aligned}$$

$$\begin{aligned}I_2:={}&-\frac{1}{(m+1)^2}\int_0^{m+1}\int_0^{n+1}\frac{x^2dydx}{x^2+y^2}\\={}&-\frac{1}{2}\Big(\frac{n+1}{m+1}\Big)+\frac{1}{2}\Big(\frac{n+1}{m+1}\Big)^2\arctan\Big(\frac{m+1}{n+1}\Big)\end{aligned}$$

$$-\frac{1}{2}\arctan\Big(\frac{n+1}{m+1}\Big).$$

$$\begin{aligned}I_3 :=&\frac{2}{n+1}\int_0^{m+1}\int_0^{n+1}\frac{ydydx}{x^2+y^2}\\ =&\Big(\frac{m+1}{n+1}\Big)\log\Big[1+\Big(\frac{n+1}{m+1}\Big)^2\Big]+2\arctan\Big(\frac{m+1}{n+1}\Big).\end{aligned}$$

$$\begin{aligned}I_4 :=& -\frac{1}{(n+1)^2}\int_0^{m+1}\int_0^{n+1}\frac{y^2dydx}{x^2+y^2}\\ =& -\frac{1}{2}\Big(\frac{m+1}{n+1}\Big)+\frac{1}{2}\Big(\frac{m+1}{n+1}\Big)^2\arctan\Big(\frac{n+1}{m+1}\Big)\\ &-\frac{1}{2}\arctan\Big(\frac{m+1}{n+1}\Big).\end{aligned}$$

$$\begin{aligned}I_5 :=&\frac{-4}{(m+1)(n+1)}\int_0^{m+1}\int_0^{n+1}\frac{xydydx}{x^2+y^2}\\ =& -\Big(\frac{n+1}{m+1}\Big)\log\Big[1+\Big(\frac{m+1}{n+1}\Big)^2\Big]\\ &-\Big(\frac{m+1}{n+1}\Big)\log\Big[1+\Big(\frac{n+1}{m+1}\Big)^2\Big].\end{aligned}$$

$$\begin{aligned}I_6 :=& -\frac{2}{(m+1)(n+1)^2}\int_0^{m+1}\int_0^{n+1}\frac{xy^2dydx}{x^2+y^2}\\ =&\frac{1}{3}\Big(\frac{n+1}{m+1}\Big)\log\Big[1+\Big(\frac{m+1}{n+1}\Big)^2\Big]+\frac{2}{3}\Big(\frac{m+1}{n+1}\Big)\\ &-\frac{2}{3}\Big(\frac{m+1}{n+1}\Big)^2\arctan\Big(\frac{n+1}{m+1}\Big).\end{aligned}$$

$$\begin{aligned}I_7 :=& -\frac{2}{(m+1)^2(n+1)}\int_0^{m+1}\int_0^{n+1}\frac{x^2ydydx}{x^2+y^2}\\ =&\frac{1}{3}\Big(\frac{m+1}{n+1}\Big)\log\Big[1+\Big(\frac{n+1}{m+1}\Big)^2\Big]+\frac{2}{3}\Big(\frac{n+1}{m+1}\Big)\\ &-\frac{2}{3}\Big(\frac{n+1}{m+1}\Big)^2\arctan\Big(\frac{m+1}{n+1}\Big).\end{aligned}$$

Finally,

$$\begin{aligned}
I_8 :=& \frac{1}{(m+1)^2(n+1)^2}\int_0^{m+1}\int_0^{n+1}\frac{x^2y^2\,dydx}{x^2+y^2}\\
=& \frac{1}{4}\Big(\frac{n+1}{m+1}\Big)+\frac{1}{4}\Big(\frac{m+1}{n+1}\Big)-\frac{1}{4}\Big(\frac{m+1}{n+1}\Big)^2\arctan\Big(\frac{n+1}{m+1}\Big)\\
&-\frac{1}{4}\Big(\frac{n+1}{m+1}\Big)^2\arctan\Big(\frac{m+1}{n+1}\Big).
\end{aligned}$$

Then, adding $I_1 - I_8$, and replacing $(m+1)/(n+1)$with λ, we obtain

$$\begin{aligned}
f(\lambda) = &\frac{1}{3}\Big(\frac{1}{\lambda}\log(1+\lambda^2)+\lambda\log\Big(1+\frac{1}{\lambda^2}\Big)\Big)\\
&+\frac{3}{2}\Big(\arctan\lambda+\arctan\Big(\frac{1}{\lambda}\Big)\Big)\\
&+\frac{5}{12}\Big(\lambda+\frac{1}{\lambda}-\frac{1}{\lambda^2}\arctan\lambda-\lambda^2\arctan\Big(\frac{1}{\lambda}\Big)\Big),
\end{aligned}$$

which has a maximum value M at $\lambda = 1$. The value of M is

$$M = \frac{2}{3}\log 2+\frac{5}{6}+\frac{13\pi}{24}.$$

It is also necessary to show that

$$\limsup_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\epsilon(m,n)}{i^2+j^2} = 0.$$

The expression for $\epsilon(m,n)$ contains eight terms. We shall show that the limit is zero for only the sixth term. Proofs for the other terms are similar.

The sixth term in $\epsilon(m,n)$ is

$$\begin{aligned}
+\,ij^2\Big(&\frac{-2}{(m+1)(n+1)^2}+\frac{1}{(m+1)(n+1)(n+2)}\\
&+\frac{1}{(m+2)(n+1)(n+2)}\Big).
\end{aligned}$$

We shall apply Lemma 4 to the function $f(x,y) = xy^2/(x^2+y^2)$.

$$f_x = \frac{y^2(y^2-x^2)}{(x^2+y^2)^2},\quad f_y = \frac{2x^3y}{(x^2+y^2)^2},\quad f_{xy} = \frac{2x^2y(3y^2-x^2)}{(x^2+y^2)^3}.$$

Treating each term in turn,

$$
\begin{aligned}
&\frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}|f_x|dxdy\\
&\qquad\le \frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}\frac{y^2}{x^2+y^2}dxdy\\
&\qquad\le \frac{mn}{(m+1)(n+1)^2}\to 0 \quad \text{as}\, m,n\to\infty,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}|f_y|dxdy\\
&\qquad\le \frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}\frac{x^2}{x^2+y^2}dxdy\\
&\qquad\le \frac{mn}{(m+1)(n+1)^2}\to 0 \quad \text{as }\ m,n\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}|f_{xy}|dxdy\\
&\qquad\le \frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}\frac{3x}{x^2+y^2}dxdy,
\end{aligned}
$$

since

$$
|f_{xy}| = \Big|\frac{2x^2y(3y^2-x^2)}{(x^2+y^2)^3}\Big| \le \frac{6x^2y}{(x^2+y^2)^2} \le \frac{3x}{x^2+y^2}.
$$

Therefore

$$
\begin{aligned}
&\frac{1}{(m+1)(n+1)^2}\int_1^{m+1}\int_1^{n+1}\frac{3x}{x^2+y^2}dxdy\\
&\le \frac{3}{(m+1)(n+1)^2}\int_0^{m+1}\arctan\Big(\frac{n+1}{x}\Big)dx\\
&\le \frac{3(m+1)\pi/2}{(m+1)(n+2)^2}\to 0 \quad \text{as } m,n\to\infty.
\end{aligned}
$$

The sixth term of $\epsilon(m,n)$ takes the form

$$
\begin{aligned}
&\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\frac{ij^2}{i^2+j^2}\Big(\frac{-2}{(m+1)(n+1)^2}\\
&\qquad + \frac{1}{(m+1)(n+1)(n+2)} + \frac{1}{(m+2)(n+1)(n+2)}\Big).
\end{aligned}
$$

Using I_6, the above expression is equal to

$$\Big(-1+\frac{(n+1)}{2(n+2)}+\frac{1}{2}\Big(\frac{m+1}{m+2}\Big)\Big(\frac{n+1}{n+2}\Big)\Big)\times$$
$$\Big\{\frac{1}{3}\Big(\frac{n+1}{m+1}\Big)\log\Big[1+\Big(\frac{m+1}{n+1}\Big)^2\Big]+\frac{2}{3}\Big(\frac{m+1}{n+1}\Big)$$
$$-\frac{2}{3}\Big(\frac{m+1}{n+1}\Big)^2\arctan\Big(\frac{n+1}{m+1}\Big)\Big\}.$$

Setting $\lambda=(m+1)/(n+1)$ in the quantity in braces gives

$$\Big\{\frac{1}{3\lambda}\log(1+\lambda^2)+\frac{2}{3}\lambda-\frac{2}{3}\lambda^2\arctan\Big(\frac{1}{\lambda}\Big)\Big\},$$

which is a continuous function which is bounded on $(0,\infty)$, since it tends to 0 as $\lambda\to 0+$ and as $\lambda\to\infty$.

The first expression in parentheses tends to 0 as $m,n\to\infty$. Therefore the sixth term of $\epsilon(m,n)\to 0$ as $m,n\to\infty$.

It is reasonable to conjecture that $f(\lambda)$, for $C(\alpha,\alpha)$, for α a positive integer, has its maximum at $\lambda=1$.

For α and β different positive integers there appears to be no pattern for the maximum value of λ.

The double Euler method is factorable, with mass functions

$$f(s)=\begin{cases}0, 0\le s<a,\\ 1, a\le s\le 1,\end{cases} \tag{19}$$

$$g(t)=\begin{cases}0, 0\le t<b,\\ 1, b\le t\le 1.\end{cases} \tag{20}$$

However it is not possible to obtain Tauberian constants for double Euler methods, since it is not possible to write the partial row sums in closed form.

We shall now examine the situation for a factorable double Hausdorff method when one of the factors is the identity matrix.

Theorem 4. *Let H be a regular factorable double Hausdorff matrix with series satisfying (1), and with one of the factors the identity matrix. Then the value of M for H is*

$$M\le\frac{\pi}{2}\int_0^1\frac{|f(s)|}{s}ds. \tag{21}$$

Proof. From (8),

$$b_{mnij}=\begin{cases}\frac{1}{i^2+j^2}\sum_{r=0}^{i-1}h'_{mr}, \quad 1\le i\le m, 1\le j\le n,\\ \frac{1}{i^2}\sum_{r=0}^{i-1}h'_{mr}, \quad j=0, 1\le i\le m,\\ 0, \quad \text{otherwise}.\end{cases}$$

The value of M comes from Lemma 3. $\square$

Corollary 6. *Suppose that H factors into the product of $C(\alpha)$ and the identity matrix, where $\alpha > 0$. Then*

$$M = \frac{\pi}{2}\int_0^1 \frac{[1-(1-s)^\alpha]}{s} ds. \tag{22}$$

Proof. Suppose that the second factor is the identity matrix. Then, in (13), $\alpha(x) = 1-(1-x)^\alpha$ and

$$\beta(y) = \begin{cases} 0, 0 \le y < 1, \\ 1, \quad y = 1. \end{cases} \tag{23}$$

Since $p_{nj}(1) = 0$, the series portion of the right hand side of (13) reduces to

$$\limsup_{m,n\to\infty} \sum_{i=1}^m \sum_{j=1}^n \frac{i}{i^2+j^2} \int_0^1 \frac{\alpha(x)}{x} p_{mi}(x)dx = M \le \frac{\pi}{2}\int_0^1 \frac{\alpha(x)}{x} dx \tag{24}$$

by Theorem 4.

It remains to show that the opposite inequality is true. The left hand side of (24) is greater than or equal to

$$\lim_{m\to\infty} \int_0^1 F_m(x)dx,$$

where

$$F_m(x) = \frac{\alpha(x)}{x}\sum_{i=1}^m p_{mi}(x)s_i = \frac{\alpha(x)}{x}\sum_{i=1}^m \binom{m}{i} x^i(1-x)^{m-i}s_i,$$

and

$$s_i = \sum_{j=1}^\infty \frac{i}{i^2+j^2} \to \frac{\pi}{2} \quad \text{as} \quad i\to\infty.$$

But, for each fixed $x \in (0,1]$,

$$\lim_{m\to\infty} F_m(x) = \frac{\pi}{2}\frac{\alpha(x)}{x},$$

since the matrix transformation defining $F_m(x)$ is equivalent to the Euler method $(E,(1-x)/x)$,which is a regular method. Also, for each $m \ge 1$,

$$|F_m(x)| \le \frac{\alpha(x)}{x}\sup_{i\ge 1} s_i \in L(0,1).$$

By the dominated convergence theorem,

$$\lim_{m\to\infty} \int_0^1 F_m(x)dx = \frac{\pi}{2}\int_0^1 \frac{\alpha(x)}{x}dx.$$

Using a result of Pringsheim, for any bounded positive sequence a_{mn},

$$\limsup_{m,n\to\infty} a_{mn} \geq \limsup_{m\to\infty} \limsup_{n\to\infty} a_{mn},$$

which gives the desired result.

If the identity matrix is the first factor, simply interchange the definitions of the functions α and β.

$\square$

Corollary 7. *If H factors into the product of the gamma method of order $\alpha > 0$ and the identity matrix, then*

$$M = \frac{\pi}{2\alpha}.$$

Proof. For the gamma method, $f(s) = s^\alpha$. Therefore

$$M = \frac{\pi}{2}\int_0^1 s^{\alpha-1}ds = \frac{\pi}{2\alpha}.$$

$\square$

Corollary 8. *If H factors into the product of an Euler method of order q and the identity matrix, then*

$$M = -\frac{\pi}{2}\log\alpha,$$

where $q := (1-\alpha)/\alpha$.

Proof. Since the mass function for an Euler method is positive, one has equality in (21). $\square$

If one performs the same calculations for $(C, 1, 2)$ summability, then

$$f(\lambda) = \frac{1}{2}\lambda\log\left(1+\frac{1}{\lambda^2}\right) - \frac{1}{6\lambda}\log(1+\lambda^2) - \frac{5\lambda}{6} + \frac{1}{2}\arctan\lambda + \frac{\pi}{2} + \frac{5\lambda^2}{6}\arctan\left(\frac{1}{\lambda}\right).$$

This function increases from $\pi/2$ to $3\pi/4$ on $(0,\infty)$, which suggests that the Tauberian constant for this case is $3\pi/4$.

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