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AN APPLICATION OF MITTAG-LEFFLER LEMMA TO THE L.F ALGEBRA OF $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-TEMPERED FUNCTIONS ON $\mathbb{R}^+$

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ABSTRACT. In this note, we show that a $\mathcal{C}^{(\infty)}$ N-tempered function f on $\mathbb{R}^+$ can be extended as a $\mathcal{C}^{(\infty)}$ function $\widetilde{f}$ in $]-\rho,+\infty[+\mathbb{R}i$ $(\forall\rho>0)$ such that

$$D^\alpha \frac{\partial}{\partial \bar{z}}\widetilde{f}(z)|_{\{z=x\}}=0, \quad \forall\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2, \quad \forall x\in\mathbb{R}^+.$$

To get this result, we use Mittag-Leffler Lemma.

1. Introduction

Let f be a $\mathcal{C}^{(\infty)}$ function on $\mathbb{R}^+$ and N be a positive integer. f is N-tempered if

$$\sup_{x\in\mathbb{R}^+}\delta_0^N(x)|f^{(n)}(x)|<+\infty, \quad \forall n\in\mathbb{N},$$

where $\delta_0(x)=\dfrac{1}{\sqrt{1+x^2}}$.

It is clair that the vector space denoted $\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ of N-tempered functions on $\mathbb{R}^+$ equipped with the family of norms $(\|\ \|_{N,n})_{n\in\mathbb{N}}$ defined by

$$f\in\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})\to\|f\|_{N,n}=\sup_{0\le k\le n}\sup_{x\in\mathbb{R}^+}\delta_0^N(x)|f^{(k)}(x)|$$

is a Frèchet space.

Let N, N' be two positive integers such that $N\le N'$, the natural injection mapping

$$\Pi_{N,N'}:\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})\hookrightarrow\mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

is obviously continuous.

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Consider the bilinear mapping

$$(f,g) \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) \times \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) \to f\cdot g \in \mathcal{C}_{N+N'}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}).$$

Applying Leibniz formula

$$D^n\big(f(x)g(x)\big) = \sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} f^{(n-p)}(x)\cdot g^{(p)}(x).$$

So

$$\delta_0^{N+N'}(x)|D^n(f(x)g(x))| \leq C_n \sup_{x\in\mathbb{R}^+}\sup_{0\leq k\leq n} \delta_0^N(x)|f^{(k)}(x) \\ \big|\sup_{x\in\mathbb{R}^+}\sup_{0\leq p\leq n}\delta_0^{N'}(x)|g^{(p)}(x)\big|.$$

Then

$$\|f\cdot g\|_{N+N'} \leq C_n\|f\|_N\cdot\|g\|_{N'},$$

where

$$C_n = \sum_{p=0}^{n}\begin{pmatrix} n \\ p \end{pmatrix} \text{ and } \begin{pmatrix} n \\ p \end{pmatrix} \text{ is the combination coefficient.}$$

The inequality just above shows the continuity of the bilinear mapping

$$(f,g) \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) \times \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) \to f\cdot g \in \mathcal{C}_{N+N'}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}).$$

Consider the vector space $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) = \bigcup_{N\in\mathbb{N}}\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$ equipped with the inductive limit topology of Frèchet spaces $\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is an L.F space. It is also a unital commutative complex topological algebra since the bilinear mapping

$$(f,g) \in \bigcup_{N\in\mathbb{N}}\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) \\ \times \bigcup_{N\in\mathbb{N}}\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) \to f\cdot g \in \bigcup_{N\in\mathbb{N}}\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

is continuous.

First, for $k \in \mathbb{N}^*$ we establish that a $\mathcal{C}^{(\infty)}$ function f on an nonempty bounded open interval $]a,b[\subset\mathbb{R}$ can be extended as a function $\widetilde{f}_k$ of class $\mathcal{C}^{(k)}$ in $]a,b[+\mathbb{R}i$ such that

$$D^\alpha\frac{\partial}{\partial\bar{z}}\widetilde{f}_k(z)|_{\{z=x\}} = 0, \forall\alpha = (\alpha_1,\alpha_2)\in\mathbb{N}^2, /|\alpha| = \alpha_1+\alpha_2 \leq k-1, \forall x\in]a,b[,$$

where

$$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}) \quad\text{and}\quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}, \quad \alpha = (\alpha_1,\alpha_2)\in\mathbb{N}^2.$$

At second, we use Mittag-Leffler lemma to get a $\mathcal{C}^{(\infty)}$ extension $\widetilde{f}_\infty$ of f in $]a,b[+\mathbb{R}i$ such that

$$D^\alpha \frac{\partial}{\partial \overline{z}} \widetilde{f}_\infty(z)|_{\{z=x\}} = 0, \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, \forall x \in]a,b[. \tag{1.1}$$

At last, we apply the result to the topological algebra $\bigcup_{N\in\mathbb{N}} \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$.

B.Drost[2] used other technics to study extensions of ultra-differentiable functions satisfying(1.1) on totally real compact submanifolds of open sets in $\mathbb{C}^n$.

In a future note, we will show that the algebra $\bigcup_{N\in\mathbb{N}} \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ equipped with its natural vector bornology is useful for functional calculus.

2. Notations and preliminary results

Let $]a,b[\subset \mathbb{R}$ be a nonempty bounded open interval and $k \in \mathbb{N}^*$. $\mathcal{C}^{(\infty)}$ $(]a,b[,\mathbb{C})$ denotes the vector space of $\mathcal{C}^{(\infty)}$ functions on $]a,b[$ equipped with its natural topology. It is a Frèchet space.

$\mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C})$ denotes the vector space of functions f of class $\mathcal{C}^{(k)}$ $(k \in \mathbb{N}^*)$ in $]a,b[+\mathbb{R}i$ such that

(1) $f|_{]a,b[} \in \mathcal{C}^{(\infty)}(]a,b[,\mathbb{C})$.
(2) $D^\alpha \frac{\partial}{\partial \overline{z}} f(z)|_{\{z=x\}} = 0, \forall \alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2, /|\alpha| = \alpha_1 + \alpha_2 \leq k - 1, \forall x \in]a,b[$.

On $\mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C})$, we consider the family of semi-norms defined by

$$f \in \mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C}) \to \|f\|_{n,K_1,K_2} = \sup_{0\leq p\leq n} \sup_{x\in K_1} |f^{(p)}(x)| + \sup_{|\alpha|\leq k-1} \sup_{z\in K_2} |D^\alpha \frac{\partial}{\partial \overline{z}} f(z)|$$

where $n \in \mathbb{N}$, K_1 is a compact set in $]a,b[$, K_2 is a compact set in $]a,b[+\mathbb{R}i$.

$\mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C})$ equipped with the family of semi-norms $\| \ \|_{n,K_1,K_2}$ is a Frèchet space. The restriction mapping

$$\mathcal{R}_k : Qh_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C}) \to \mathcal{C}^{(\infty)}(]a,b[,\mathbb{C})$$

is obviously continuous.

We will show that the restriction $\mathcal{R}_k$ is surjective.

Lemma 2.1. *Let φ be a $\mathcal{C}^{(\infty)}$ function with compact support in $]-1,+1[$ equal one on $[-\frac{1}{2},+\frac{1}{2}]$ and $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of strictly increasing positive real numbers such that $\lim_{n\to+\infty} \lambda_n = +\infty$. Then, there exists a sequence of $\mathcal{C}^{(\infty)}$ functions $(\varphi_n)_{n\in\mathbb{N}}$ satisfying*

(1) $\varphi_n \in \mathcal{C}^{(\infty)}\left(]-\frac{1}{\lambda_n},+\frac{1}{\lambda_n}[\right)$ $(\forall n \in \mathbb{N})$ *with compact support.*

(2) $\varphi_n(x) = 1$ *if* $|x| \leq \frac{1}{2\lambda_n}$, $(\forall n \in \mathbb{N})$.
(3) $supp(\varphi_{n+1}) \subset supp(\varphi_n), \forall n \in \mathbb{N}$.

Proof. For each positive integer n, consider the function defined by

$$\varphi_n(x) = \varphi(\lambda_n \cdot x).$$

The sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ satisfies Lemma 2.1. □

Lemma 2.2. *Let* $f \in \mathcal{C}^{(\infty)}(]a,b[,\mathbb{C})$ *with compact support* $K \subset]a,b[$ *and* $k \in \mathbb{N}^*$. *Then,* f *can be extended as a function* $\widetilde{f_k}$ *in* $\mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C})$.

Proof. For each positive integer $n \in \mathbb{N}$, put

$$M_n = \sup_{x\in K} |f^{(n)}(x)|.$$

Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying

$$\delta_n > sup(1, M_n), \quad \forall n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ and fix $k \in \mathbb{N}^*$. Consider the sequence

$$\lambda_{n,k} = \sum_{i=0}^{n+k} \delta_i.$$

Then $\lim_{n\to+\infty} \lambda_{n,k} = +\infty$.

Let $(\varphi_{n,k})_{n\in\mathbb{N}}$ be a sequence of $\mathcal{C}^{(\infty)}$ functions satisfying Lemma 2.1 above constructed with the sequence of positive real numbers $(\lambda_{n,k})_{n\in\mathbb{N}}$ above.

Consider the series

$$\sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(x)\varphi_{n,k}(y)(y\cdot i)^n.$$

The series is locally finite. It converges at each point $z = x + yi \in]a,b[+\mathbb{R}i$. So it defines a function $\widetilde{f_k}$ on $]a,b[+\mathbb{R}i$.
Obviously, if $y = 0$, we have $\widetilde{f_k}(x) = f(x), \quad \forall x \in]a,b[$.
Since the series is locally finite, $\widetilde{f_k}$ is a $\mathcal{C}^{(\infty)}$ function at each point $z = x+yi$ such that $y \neq 0$.
At each point $z = x + yi$ such that $y = 0$, $\widetilde{f_k}$ is of class $\mathcal{C}^{(k)}$. In fact:
Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2 \ / \ |\alpha| = \alpha_1 + \alpha_2 \leq k-1$. The derivative of order α of the general term of the series is

$$D^\alpha\Big(f^{(n)}(x)\varphi_{n,k}(y)(yi)^n\Big) = f^{(n+\alpha_1)}(x)D^{\alpha_2}\big(\varphi_{n,k}(y)(yi)^n\big).$$

By Leibniz formula we have

$$D^{\alpha_2}\big(\varphi_{n,k}(y)y^n\big) = \sum_{p=0}^{\alpha_2} \begin{pmatrix} \alpha_2 \\ p \end{pmatrix} \varphi_{n,k}^{(\alpha_2-p)}(y)\cdot n(n-1)\cdots(n-1-p)y^{n-p},$$

where $\begin{pmatrix} \alpha_2 \\ p \end{pmatrix}$ is the combination coefficient.

So,

$$\big|D^{\alpha_2}\big(\varphi_{n,k}(y)y^n\big)\big|$$

$$\leq n!\Big[\sup_{y\in[-\frac{1}{\lambda_{n,k}},\frac{1}{\lambda_{n,k}}]}\sup_{0\leq p\leq k}\big|\varphi^{(p)}(\lambda_{n,k}\cdot y)\big|\Big]\sum_{p=0}^{k-1}\begin{pmatrix} k-1 \\ p \end{pmatrix}\Big(\frac{1}{\lambda_{n,k}}\Big)^{n+1-k}.$$

So

$$\big|D^{\alpha_2}\big(\varphi_{n,k}(y)y^n\big)\big| \leq n!C_k\Big(\frac{1}{\lambda_{n,k}}\Big)^{n+1-k},$$

where $C_k = \sup_{y\in[-\frac{1}{\lambda_{n,k}},\frac{1}{\lambda_{n,k}}]}\sup_{0\leq p\leq k}\big|\varphi^{(p)}(\lambda_{n,k}\cdot y)\big|\sum_{p=0}^{k-1}\begin{pmatrix} k-1 \\ p \end{pmatrix}$.

And so

$$\frac{1}{n!}\big|D^{\alpha}\big(f^{(n)}(x)\varphi_{n,k}(y)(yi)^n\big)\big| \leq M_{n+\alpha_1}C_k\Big(\frac{1}{\lambda_{n,k}}\Big)^{n+1-k} \leq C_k\Big(\frac{1}{\delta_0}\Big)^{n-k}.$$

The series $\sum_{n=0}^{+\infty}\frac{1}{n!}\big|D^{\alpha}\big(f^{(n)}(x)\varphi_{n,k}(y)(yi)^n\big)\big|$ is dominated by the convergent geometrical series $C_k\sum_{n=0}^{+\infty}(\frac{1}{\delta_0})^{n-k} = C_k\frac{(\delta_0)^{k+1}}{\delta_0-1}$.

So, the series

$$\sum_{n=0}^{+\infty}\frac{1}{n!}D^{\alpha}\Big(f^{(n)}(x)\varphi_{n,k}(y)(y\cdot i)^n\Big)$$

converges uniformly in $K+\mathbb{R}i$, $\forall\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2/|\alpha|=\alpha_1+\alpha_2\leq k$.

Thus $\widetilde{f}_k$ is of class $\mathcal{C}^{(k)}$ in $]a,b[+\mathbb{R}i$.

From the calculation

$$\frac{\partial}{\partial\overline{z}}\widetilde{f}_k(x+yi) = \frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big)\widetilde{f}_k(x+yi)$$

$$=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{1}{n!}f^{(n+1)}(x)\big(\varphi_{n,k}(y)-\varphi_{n+1,k}(y)\big)(iy)^n+\frac{i}{2}\sum_{n=0}^{+\infty}\frac{1}{n!}f^{(n)}(x)\varphi'_{n,k}(y)(iy)^n.$$

we get

(1) $\widetilde{f}_k|_{]a,b[}=f$.

(2) $D^{\alpha}\frac{\partial}{\partial\overline{z}}\widetilde{f}_k(z)|_{\{z=x\}}=0, \forall\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2, /|\alpha|=\alpha_1+\alpha_2\leq k-1, \forall x\in]a,b[$.

This ends the proof of Lemma 2.2. □

Theorem 2.3. *The restriction mapping*

$$\mathcal{R}_k : \mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C}) \to \mathcal{C}^{(\infty)}(]a,b[,\mathbb{C})$$

is surjective.

Proof. Let $(U_j)_{j\in J}$ be a locally finite covering of the interval $]a,b[$. Each U_j is relatively compact in $]a,b[$ and $(\phi_j)_{j\in J}$ be a partition of unity subordinated to the covering $(U_j)_{j\in J}$.
Let $f \in \mathcal{C}^{(\infty)}(]a,b[,\mathbb{C})$ and $j \in J$. Put $\phi_j \cdot f = f_j$.
f_j is a $\mathcal{C}^{(\infty)}$ function in $]a,b[$ with compact support. By lemma 2.2, f_j can be extended as a $\widetilde{f}_{k,j} \in \mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C})$.
The function $\widetilde{f}_k = \sum_{j\in J} \widetilde{f}_{k,j} \in \mathcal{Q}h_\infty^{(k)}(]a,b[+\mathbb{R}i,\mathbb{C})$. □

We want an extension $\widetilde{f}_\infty$ of f of class $\mathcal{C}^{(\infty)}$ in $]a,b[+\mathbb{R}i$ satisfying

$$D^\alpha \frac{\partial}{\partial \overline{z}} \widetilde{f}_\infty(z)|_{\{z=x\}} = 0, \forall \alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2, \forall x \in]a,b[.$$

To get this result, we need the following Mittag-Leffler Lemma.

Lemma 2.4. *For $n \in \mathbb{N}$, let E''_n, E_n, E'_n be Frèchet spaces. $i_n : E''_n \to E_n$; $s_n : E_n \to E'_n$ be continuous linear mappings such that for each positive integer $n \in \mathbb{N}$ the following sequence*

$$0 \to E''_n \to E_n \to E'_n \to 0$$

is exact.
Let $u''_n : E''_{n+1} \to E''_n$; $u_n : E_{n+1} \to E_n$; $u'_n : E'_{n+1} \to E'_n$ be continuous linear mappings such that

$$u_n \circ i_{n+1} = i_n \circ u''_n, \quad u'_n \circ s_{n+1} = s_n \circ u_n.$$

If each linear mapping u''_n has a dense range, then the following sequence

$$0 \to \varprojlim_n E''_n \to \varprojlim_n E_n \to \varprojlim_n E'_n \to 0$$

is exact.

We use Mittag-Leffler Lemma as follows:
Let K be a nonempty compact set in $]a,b[$, $\mathcal{C}_0^{(\infty)}(K,\mathbb{C})$ the Frèchet space of $\mathcal{C}^{(\infty)}$ functions with support in K and $\mathcal{Q}h_\infty^{(k)}(K+\mathbb{R}i,\mathbb{C})$ ($k \in \mathbb{N}^*$) the Frèchet space of functions f of class $\mathcal{C}^{(k)}$ with support in $K+\mathbb{R}i$ such that

(1) $f|_K \in \mathcal{C}_0^{(\infty)}(K,\mathbb{C})$.

(2) $D^{\alpha}\dfrac{\partial}{\partial \overline{z}}f(z)|_{\{z=x\}} = 0, \forall x \in K, \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2/|\alpha| = \alpha_1 + \alpha_2 \leq k-1.$

For each positive integer $k \in \mathbb{N}^*$, the restriction linear mapping

$$\mathcal{R}_k : \mathcal{Q}h_{\infty}^{(k)}(K + \mathbb{R}i, \mathbb{C}) \to \mathcal{C}_0^{(\infty)}(K, \mathbb{C})$$

is obviously continuous.
The kernel $Ker(\mathcal{R}_k)$ is a closed subspace of $\mathcal{Q}h_{\infty}^{(k)}(K+\mathbb{R}i, \mathbb{C})$. It is a Frèchet subspace of $\mathcal{Q}h_{\infty}^{(k)}(K + \mathbb{R}i, \mathbb{C})$.
Consider the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle u''_{k-1}} & & \uparrow{\scriptstyle u_{k-1}} & & \uparrow{\scriptstyle j} & & \\
0 & \longrightarrow & Ker(\mathcal{R}_k) & \xrightarrow{i_k} & \mathcal{Q}h_{\infty}^{(k)}(K+\mathbb{R}i,\mathbb{C}) & \xrightarrow{\mathcal{R}_k} & \mathcal{C}_0^{(\infty)}(K,\mathbb{C}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle u''_{k}} & & \uparrow{\scriptstyle u_{k}} & & \uparrow{\scriptstyle j} & & \\
0 & \longrightarrow & Ker(\mathcal{R}_{k+1}) & \xrightarrow{i_{k+1}} & \mathcal{Q}h_{\infty}^{(k+1)}(K+\mathbb{R}i,\mathbb{C}) & \xrightarrow{\mathcal{R}_{k+1}} & \mathcal{C}_0^{(\infty)}(K,\mathbb{C}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle u''_{k+1}} & & \uparrow{\scriptstyle u_{k+1}} & & \uparrow{\scriptstyle j} & & \\
0 & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}$$

where
$u_k : \mathcal{Q}h_{\infty}^{(k+1)}(K + \mathbb{R}i, \mathbb{C}) \to \mathcal{Q}h_{\infty}^{(k)}(K + \mathbb{R}i, \mathbb{C})$ the continuous canonical injection.
$j : \mathcal{C}_0^{(\infty)}(K, \mathbb{C}) \to \mathcal{C}_0^{(\infty)}(K, \mathbb{C})$ the continuous identity mapping.
$u''_k : Ker(\mathcal{R}_{k+1}) \to Ker(\mathcal{R}_k)$ the continuous canonical injection.
$i_k : Ker(\mathcal{R}_k) \to \mathcal{Q}h_{\infty}^{(k)}(K + \mathbb{R}i, \mathbb{C})$ the continuous canonical injection.
These continuous morphisms satisfy for every positive integer $k \in \mathbb{N}^*$.

$$i_k \circ u''_k = u_k \circ i_{k+1} \quad \text{and} \quad \mathcal{R}_k \circ u_k = j \circ \mathcal{R}_{k+1}.$$

Each linear morphism u''_k ($k \in \mathbb{N}$) has obviously a dense range.
Using Mittag-Leffler lemma, we get the following exact sequence

$$0 \to \varprojlim_k Ker(\mathcal{R}_k) \to \varprojlim_k \mathcal{Q}h_{\infty}^{(k)}(K + \mathbb{R}i, \mathbb{C}) \to \mathcal{C}_0^{(\infty)}(K, \mathbb{C}) \to 0.$$

This means that each function $f \in \mathcal{C}_0^{(\infty)}(K, \mathbb{C})$ can be extended as a $\mathcal{C}^{(\infty)}$ function $\widetilde{f}_\infty$ in $K + \mathbb{R}i$ such that

$$D^\alpha \frac{\partial}{\partial \overline{z}} \widetilde{f}_\infty(z)|_{\{z=x\}} = 0, \forall x \in K, \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2.$$

Hence, $\widetilde{f}_\infty \in \mathcal{Q}h_\infty^{(\infty)}(K + \mathbb{R}i, \mathbb{C})$.

Theorem 2.5. *Let $f \in \mathcal{C}^{(\infty)}(]a, b[, \mathbb{C})$. It can be extended as a $\mathcal{C}^{(\infty)}$ function $\widetilde{f}_\infty$ in $]a, b[+\mathbb{R}i$ such that:*

$$D^\alpha \frac{\partial}{\partial \overline{z}} \widetilde{f}_\infty(z)|_{\{z=x\}} = 0, \forall x \in]a, b[, \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2.$$

Proof. Let $(U_j)_{j \in J}$ be a locally finite covering of the open interval $]a, b[$ and $(\varphi_j)_{j \in J}$ be a partition of unity subordinated to the covering $(U_j)_{j \in J}$.
Let $f \in \mathcal{C}^{(\infty)}(]a, b[, \mathbb{C})$, and $j \in J$. Put $f_j = \varphi_j \cdot f$.
f_j is $\mathcal{C}^{(\infty)}$ function with compact support. By Mittag-Leffler Lemma there exists a function $\widetilde{f_{\infty,j}}$ of class $\mathcal{C}^{(\infty)}$ in $supp(\varphi_j \cdot f) + \mathbb{R}i$ such that

(1) $\widetilde{f}_{\infty,j}|_{supp(\varphi_j \cdot f)} = f$.
(2) $D^\alpha \frac{\partial}{\partial \overline{z}} \widetilde{f}_{\infty,j}(z)|_{\{z=x\}} = 0, \forall x \in supp(\varphi_j), \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

The function $\widetilde{f}_\infty = \sum_{j \in J} \widetilde{f_{\infty,j}} \in \mathcal{Q}h_\infty^{(\infty)}(]a, b[+\mathbb{R}i, \mathbb{C})$ and satisfies

(1) $\widetilde{f}_\infty|_{]a,b[} = f$.
(2) $D^\alpha \frac{\partial}{\partial \overline{z}} \widetilde{f}_\infty(z)|_{\{z=x\}} = 0, \forall x \in]a, b[, \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

This ends the proof of theorem 2.5. □

3. Application to $\mathbb{N}$-tempered function algebra $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$

As seen in introduction

$$\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) = \bigcup_{N \in \mathbb{N}} \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) = \lim_{\overrightarrow{N}} \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

It is the algebra of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$ equipped with the inductive limit topology of Frèchet spaces $\mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$. An element f in this algebra means that there exists a positive integer N such that $f \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ and satisfies

$$\forall n \in \mathbb{N}, \quad \sup_{x \in \mathbb{R}^+} \delta_0^N(x) |f^{(n)}(x)| < +\infty.$$

The function f and all its derivative of any order have the same polynomially growth at infinity.

Theorem 3.1. *Let f in $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$. There exists a function $\widetilde{f}_\infty$ of class $\mathcal{C}^{(\infty)}$ in $]-\rho, +\infty[+\mathbb{R}i$ $(\forall \rho > 0)$ such that:*

(1) $\widetilde{f}_\infty|_{\mathbb{R}^+} = f$.
(2) $D^\alpha \dfrac{\partial}{\partial \overline{z}} \widetilde{f}_\infty(z)|_{\{z=x\}} = 0, \forall x \in \mathbb{R}^+, \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

Proof. Let $f \in \bigcup_{N\in\mathbb{N}} \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \Rightarrow \exists N \in \mathbb{N}$ such that $f \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$. By E. Borel theorem f can be extended as a $\mathcal{C}^{(\infty)}$ function f_1 on $\mathbb{R}$.
Let φ be a $\mathcal{C}^{(\infty)}$ function in $\mathbb{R}$ satisfying

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x \leq -\rho \quad (\rho > 0). \end{cases}$$

Multiply f_1 by the function φ we get a function F of $\mathcal{C}^{(\infty)}$ with support in $]-\rho, +\infty[$.
Let $\epsilon \in]0, \frac{1}{2}[$ and consider the open interval $]1-\epsilon, 2+\epsilon[$.
For $k \in \mathbb{Z}$, consider the open interval $2^k]1-\epsilon, 2+\epsilon[$.
The sequence of open intervals $(2^k]1-\epsilon, 2+\epsilon[)_{k\in\mathbb{Z}}$ satisfy

(1) $2^n]1-\epsilon, 2+\epsilon[\bigcap 2^m]1-\epsilon, 2+\epsilon[= \emptyset \quad \text{if} \quad |m-n| \geq 3$.
(2) $\displaystyle\bigcup_{n\in\mathbb{Z}} 2^n]1-\epsilon, 2+\epsilon[=]0, +\infty[$.

Let ϕ be a positive function in $\mathcal{C}^{(\infty)}(\mathbb{R}^+, \mathbb{C})$ with compact support in the interval $]1-\epsilon, 2+\epsilon[$ equal 1 on $[1, 2]$.
For $k \in \mathbb{Z}$ and $\rho \in]0, +\infty[$, put

$$\phi_k(x) = \phi[2^k(x+\rho)].$$

The function $\varphi(x) = \sum_{k\in\mathbb{Z}} \phi_k(x)$ is positive and of class $\mathcal{C}^{(\infty)}$ on its domain of definition.
For $k \in \mathbb{Z}$, put

$$\nu_k(x) = \frac{\phi_k(x)}{\sum_{k'\in\mathbb{Z}} \phi_{k'}(x)}.$$

Then

$$\sum_{k\in\mathbb{Z}} \nu_k(x) = 1, \quad \forall x \in]-\rho, +\infty[.$$

$(\nu_n)_{n\in\mathbb{Z}}$ is a $\mathcal{C}^{(\infty)}$ partition of unity subordinated to the open covering

$$(2^n]1-\epsilon, 2+\epsilon[-\rho)_{n\in\mathbb{Z}}$$

of the open interval $]-\rho, +\infty[$.
Let $n \in \mathbb{Z}$. Put $f_{1,n} = \nu_n \cdot F$.
We get a $\mathcal{C}^{(\infty)}$ function with compact support in $2^n]1-\epsilon, 2+\epsilon[-\rho$.

So, by Theorem 2.5 above it can be extended as a $\mathcal{C}^{(\infty)}$ function $\widetilde{f_{1,n}}$ in $\mathcal{Q}h_{\infty}^{(\infty)}\big(2^n]1-\epsilon, 2+\epsilon[-\rho+\mathbb{R}i\big)$.
The function $\widetilde{f}_{\infty} = \sum_{n\in\mathbb{Z}} \widetilde{f_{1,n}}$ satisfies

(1) $\widetilde{f}_{\infty}|_{\mathbb{R}^+} = f$.
(2) $D^{\alpha}\dfrac{\partial}{\partial \overline{z}}\widetilde{f}_{\infty}(z)|_{\{z=x\}} = 0, \forall x \in \mathbb{R}^+, \quad \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

Hence the function $\widetilde{f}_{\infty} \in \mathcal{Q}h_{\infty}^{(\infty)}\big(]-\rho, +\infty[+\mathbb{R}i\big)$.
This ends the proof of Theorem 3.1. $\square$

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