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COMMON FIXED POINT RESULTS OF DAS-NAIK AND GERAGHTY TYPES IN ν-GENERALIZED METRIC SPACES

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Abstract. In this paper, common fixed point results in ν-generalized metric spaces of Branciari are deduced under several types of contractive conditions.

1. Introduction

There have been several attempts to extend metric fixed point theory to more general settings. Branciari introduced in 2000 one of such generalizations, when he replaced the triangle inequality by a so-called *polygonal inequality* in the following way; see also [2, 8, 10, 11, 14, 15, 16, 17].

Definiton 1.1 (Branciari [4])**.** Let X be a nonempty set, $\nu \in \mathbb{N}$ and let $d\colon X \times X \to [0, \infty)$ be a mapping. The pair (X, d) is called a *ν-generalized metric space* if the following hold:

(1) $d(x, y) = 0$ if and only if $x = y$, for $x, y \in X$;
(2) $d(x, y) = d(y, x)$, for every $x, y \in X$;
(3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$, for every set $\{x, u_1, \ldots, u_\nu, y\}$ of $\nu + 2$ elements of X that are all different.

Obviously, (X, d) is a metric space if and only if it is a 1-generalized metric space. In [14, 16], it was shown that not every generalized metric space has a compatible topology.

Definiton 1.2. [4, 2, 15] Let (X, d) be a ν-generalized metric space and $\{x_n\}$ a sequence in X.

(1) The sequence $\{x_n\}$ is said to be a *Cauchy-sequence* if $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

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(2) The sequence $\{x_n\}$ is said to *converge* to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$.
(3) The sequence $\{x_n\}$ is said to *converge to x in the strong sense* if $\{x_n\}$ is a Cauchy sequence and $\{x_n\}$ converges to x.

The space X is said to be *complete* if every Cauchy sequence in X converges.

It was shown in [12] and [13] (e.g., [13, Example 1.1]) that, among other things, a sequence in a 2-generalized metric space may converge to more than one point and that a convergent sequence may not be a Cauchy sequence, i.e., a sequence may be convergent, but not in the strong sense.

Proposition 1.3 ([15]). *Let (X, d) be a ν-generalized metric space and let $\{x_n\}$ be a sequence in X such that x_n $(n \in \mathbb{N})$ are all different. If $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ and $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$, then $\{x_n\}$ is a Cauchy sequence.*

Proposition 1.4 ([17]). *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a ν-generalized metric space (X, d) that converge to x and y in the strong sense, respectively. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Branciari proved in [4] a generalization of the Banach contraction principle. His proof was not fully correct because a ν-generalized metric space does not necessarily have a compatible topology, but it was later corrected, see [9, 12, 13, 14, 19]. A proof of the Banach contraction principle, as well as proofs of Kannan's and Ćirić's fixed point theorems, in ν-generalized metric spaces, can be found in [15].

Theorem 1.5 ([15]). *Let (X, d) be a complete ν-generalized metric space, and let T be a self-map of X. For every $x, y \in X$, let*

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Assume there exists $r \in [0, 1)$ such that $d(Tx, Ty) \le rm(x, y)$, for all $x, y \in X$. Then T has a unique fixed point z and, moreover, for any $x \in X$, the Picard iterates $T^n x$ $(n \in \mathbb{N})$ converge to z in the strong sense.

Some additional properties of Cauchy sequences in ν-generalized metric spaces, as well as some new fixed point results were recently obtained in [1].

In the present article, we deduce some common fixed point results in ν-generalized metric spaces, thus generalizing certain results from [15]. In particular, Das-Naik and Geraghty-type results are obtained for two mappings. An example shows how these results can be used.

Throughout the paper, the set of integers is denoted by $\mathbb{Z}$, the set of nonnegative integers is denoted by $\mathbb{Z}^+$, and the set of positive integers is denoted by $\mathbb{N}$.

2. Common fixed point results in ν-generalized metric spaces

2.1. **Das-Naik-type result.** Theorem 1.5 (i.e., [15, Theorem 11]) expresses the extension of classical Ćirić's theorem on quasicontractions [5] to ν-generalized metric spaces. Naturally, the results of Banach, Kannan (i.e., [15, Theorems 9, 10]), Chatterjea, Zamfirescu and Hardy-Rogers types follow as special cases.

We will prove here the corresponding common fixed point result, i.e., an extension of Das-Naik theorem [6] to ν-generalized metric spaces. We recall that two self-mappings $S, T : X \to X$ on a non-empty set X are said to be weakly compatible if they commute at their coincidence points, i.e., if $Tx = Sx$ for some $x \in X$ implies that $STx = TSx$ (see also a discussion on various types of compatible pairs in [18]).

Theorem 2.1. *Let (X, d) be a ν-generalized metric space and let $S, T\colon X \to X$ be two self-mappings such that $T(X) \subseteq S(X)$. Suppose that there exists $r \in [0, 1)$ such that for all $x, y \in X$*

$$d(Tx, Ty) \le r \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}. \tag{2.1}$$

If at least one of subspaces $T(X)$ and $S(X)$ is complete, then S and T have a unique point of coincidence. If, moreover, S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. We follow the lines of proof of [15, Theorem 11]. Starting with arbitrary $x_0 \in X$, and using that $T(X) \subseteq S(X)$, construct a Jungck sequence by $y_n = Tx_n = Sx_{n+1}$, $n \in \mathbb{Z}^+$. For $m, n \in \mathbb{Z}^+$, $m \le n$, denote

$$\delta_{m,n} = \operatorname{diam}\{\, y_k : m \le k \le n \,\}, \quad \delta_m = \operatorname{diam}\{\, y_k : m \le k \,\}.$$

Using the condition (2.1), we get that

$$\delta_{m,n} \le r\delta_{m-1,n}, \quad m, n \in \mathbb{N},\ m \le n, \tag{2.2}$$

and

$$\delta_{0,n} = \max\{\, d(y_0, y_k) : 1 \le k \le n \,\}. \tag{2.3}$$

Consider the following two possible cases.

1. $y_k = y_l$ for some $k, l \in \mathbb{N}$, $k < l$.

Then, (2.2) implies that

$$\delta_{k,l-1} = \delta_{k+1,l} \le r\delta_{k,l} = r\delta_{k,l-1}.$$

It follows that

$$\delta_k = \delta_{k,l-1} = 0,$$

which means that $Tx_{k+1} = y_{k+1} = y_k = Sx_{k+1}$, i.e., y_k is a point of coincidence of S and T.

2. y_n's are all different.

Take any $n \in \mathbb{N}$ with $n > \nu$. According to (2.3), there is some $m \in \mathbb{N}$ with $m \leq n$ such that $d(y_0, y_m) = \delta_{0,n}$. Suppose that $m > \nu$. Then

$$\begin{aligned}\delta_{0,n} = d(y_0, y_m) &\leq \sum_{j=0}^{\nu-1} d(y_j, y_{j+1}) + d(y_\nu, y_m) \\ &\leq \nu\delta_{0,\nu} + \delta_{\nu,m} \leq \nu\delta_{0,\nu} + r^\nu \delta_{0,m} \leq \nu\delta_{0,\nu} + r^\nu \delta_{0,n}.\end{aligned}$$

Therefore,

$$\delta_{0,n} \leq \frac{\nu}{1 - r^\nu}\delta_{0,\nu}.$$

If $m \leq \nu$, the previous inequality holds trivially. It follows that the set $\{\delta_{0,n} : n \in \mathbb{N}\}$ is bounded, i.e., $\delta_0 < \infty$. Now, (2.1) implies that

$$\delta_m \leq r\delta_{m-1} \leq \cdots \leq r^m \delta_0$$

for $m \in \mathbb{N}$ and it follows that the sequence $\{y_n\}$ is Cauchy. If at least one of $S(X)$, $T(X)$ is complete, it follows that it converges to Sz for some $z \in X$.

Now, using Proposition 1.4, we obtain

$$d(Tz, Sz) = \lim_{n\to\infty} d(Tz, y_n) = \lim_{n\to\infty} d(Tz, Tx_n) \leq \lim_{n\to\infty} rd(Sz, Sx_n) \to r \cdot 0 = 0.$$

Hence, $Tz = Sz = w$ is a (unique) point of coincidence of S and T.

The final conclusion is standard. □

It is clear that common fixed point results of Banach, Kannan, Chatterjea, Zamfirescu and Hardy-Rogers types follow directly as special cases. Also, the following can be deduced in the same way.

Corollary 2.2. *Suppose that all the conditions of Theorem 2.1 are fulfilled except that the condition* (2.1) *is replaced by*

$$d(Tx, Ty) \leq r \max\left\{d(Sx, Sy), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}, \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\},$$

or

$$d(Tx, Ty) \leq r \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}.$$

Then the conclusions of Theorem 2.1 *hold true.*

All the previously mentioned results have their counterparts for expansion type mappings. We state here just the simplest version.

Corollary 2.3. *Let (X, d) be a ν-generalized metric space and let $S, T\colon X \to X$ be two self-mappings such that $T(X) \supseteq S(X)$. Suppose that there exists $s > 1$ such that for all $x, y \in X$*

$$d(Tx, Ty) \geq sd(Sx, Sy). \tag{2.4}$$

If at least one of subspaces $T(X)$ and $S(X)$ is complete, then S and T have a unique point of coincidence. If, moreover, S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. Follows directly from Theorem 2.1 since

$$d(Sx, Sy) \leq \frac{1}{s} d(Tx, Ty) \text{ and } S(X) \subseteq T(X).$$

□

2.2. **Geraghty-type result.** We will prove now a common fixed point result of Geraghty-type [7] in ν-generalized metric spaces. We note that in the case $\nu = 2$, the result was proved as [9, Theorem 1]. We provide here a result for arbitrary ν.

We start by proving the following auxiliary assertion which is a ν-generalized metric version of a lemma used in several earlier situations.

Lemma 2.4. *Let (X, d) be a ν-generalized metric space and let $\{y_n\}$ be a sequence in X with distinct elements ($y_n \neq y_m$ for $n \neq m$). Suppose that $d(y_n, y_{n+1})$, $d(y_n, y_{n+2})$, ..., $d(y_n, y_{n+\nu})$ tend to 0 as $n \to \infty$ and that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε as $k \to \infty$:*

$$d(y_{m_k}, y_{n_k}), \quad d(y_{m_k}, y_{n_k+1}), \quad d(y_{m_k-1}, y_{n_k}), \quad d(y_{m_k-1}, y_{n_k+1}). \tag{2.5}$$

Proof. Since $\{y_n\}$ is not a Cauchy sequence, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$, $d(y_{m_k}, y_{n_k}) \geq \varepsilon$ and n_k is the smallest integer satisfying this inequality, i.e., $d(y_{m_k}, y_\ell) < \varepsilon$ for $m_k < \ell < n_k$.

Let us prove that the first of the sequences in (2.5) tends to ε as $k \to \infty$. Note that, by the assumption, $d(y_{m_k}, y_{m_k+1}) \to 0$, $d(y_{m_k}, y_{m_k+2}) \to 0$, ..., $d(y_{m_k}, y_{m_k+\nu}) \to 0$ as $k \to \infty$. Hence, it is impossible that $n_k = m_k + 1$, or $n_k = m_k + 2$, ..., or $n_k = m_k + \nu$ (because in either of these cases it would be impossible to have $d(y_{m_k}, y_{n_k}) \geq \varepsilon$).

Thus, we can apply the polygonal inequality to obtain

$$\begin{aligned} \varepsilon \leq d(y_{m_k}, y_{n_k}) &\leq d(y_{m_k}, y_{n_k-\nu}) + d(y_{n_k-\nu}, y_{n_k-\nu+1}) + \cdots + d(y_{n_k-1}, y_{n_k}) \\ &< \varepsilon + d(y_{n_k-\nu}, y_{n_k-\nu+1}) + \cdots + d(y_{n_k-1}, y_{n_k}) \to \varepsilon, \end{aligned}$$

as $k \to \infty$, implying that $d(y_{m_k}, y_{n_k}) \to \varepsilon$ as $k \to \infty$.

In order to prove that the second sequence in (2.5) tends to ε as $k \to \infty$, consider the following two $(\nu+2)$-polygonal inequalities:

$$\begin{aligned} d(y_{m_k}, y_{n_k+1}) \le d(y_{m_k}, y_{n_k}) &+ d(y_{n_k}, y_{n_k-1}) + \cdots \\ &+ d(y_{n_k-\nu+2}, y_{n_k-\nu+1}) + d(y_{n_k-\nu+1}, y_{n_k+1}), \\ d(y_{m_k}, y_{n_k}) \le d(y_{m_k}, y_{n_k+1}) &+ d(y_{n_k+1}, y_{n_k-\nu+1}) \\ &+ d(y_{n_k-\nu+2}, y_{n_k-\nu+1}) + \cdots + d(y_{n_k}, y_{n_k-1}), \end{aligned}$$

which imply

$$\begin{aligned} |d(y_{m_k}, y_{n_k+1}) - d(y_{m_k}, y_{n_k})| &\le d(y_{n_k}, y_{n_k-1}) + \cdots \\ &+ d(y_{n_k-\nu+2}, y_{n_k-\nu+1}) + d(y_{n_k-\nu+1}, y_{n_k+1}) \to 0 \end{aligned}$$

as $k \to \infty$. This, together with $d(y_{m_k}, y_{n_k}) \to \varepsilon$ imply that $d(y_{m_k}, y_{n_k+1}) \to \varepsilon$ as $k \to \infty$.

The proof for the other two sequences can be done in a similar way, using the following polygons:

$$(y_{m_k-1}, y_{n_k}, y_{m_k+\nu-1}, \dots, y_{m_k}),$$

resp.

$$(y_{m_k-1}, y_{n_k+1}, y_{n_k}, y_{n_k-1}, \dots, y_{n_k-\nu+1}).$$

□

As usual in Geraghty-type assertions, we will use the class $\mathcal{S}$ of real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ as } n \to \infty \text{ implies } t_n \to 0 \text{ as } n \to \infty.$$

Theorem 2.5. *Let (X, d) be a ν-generalized metric space and let $S, T : X \to X$ be two self maps such that $T(X) \subseteq S(X)$, one of these two subsets of X being complete. If, for some function $\beta \in \mathcal{S}$,*

$$d(Tx, Ty) \le \beta(d(Sx, Sy))d(Sx, Sy) \tag{2.6}$$

holds for all $x, y \in X$, then S and T have a unique point of coincidence z. Moreover, for each $x_0 \in X$, a corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = z$.

If, moreover, S and T are weakly compatible, then they have a unique common fixed point.

Proof. We will prove first that S and T cannot have more than one point of coincidence. Suppose to the contrary that there exist $w_1, w_2 \in X$ such that

$w_1 \neq w_2$, $w_1 = Su_1 = Tu_1$ and $w_2 = Su_2 = Tu_2$ for some $u_1, u_2 \in X$. Then (2.6) would imply that

$$\begin{aligned} d(w_1, w_2) &= d(Tu_1, Tu_2) \leq \beta(d(Su_1, Su_2))d(Su_1, Su_2) \\ &= \beta(d(w_1, w_2))d(w_1, w_2) < d(w_1, w_2), \end{aligned}$$

which is impossible.

In order to prove that S and T have a coincidence point, take an arbitrary $x_0 \in X$ and, similarly as in previous proofs, choose a Jungck sequence $\{y_n\}$ in X such that

$$y_n = Tx_n = Sx_{n+1}, \quad \text{for } n = 0, 1, 2, \ldots$$

Moreover, if $y_n = y_m$ for some $n \neq m$, then we choose $x_{n+1} = x_{m+1}$ (and hence also $y_{n+1} = y_{m+1}$).

If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0+1} is a coincidence point of S and T, and y_{n_0+1} is their (unique) point of coincidence.

Suppose now that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. Then, using (2.6), we get that

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Tx_n, Tx_{n+1}) \leq \beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}) \\ &= \beta(d(y_{n-1}, y_n))d(y_{n-1}, y_n) < d(y_{n-1}, y_n). \end{aligned}$$

Hence, $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive real numbers, tending to some $\delta \geq 0$. Suppose that $\delta > 0$. Then, since

$$\frac{d(y_n, y_{n+1})}{d(y_{n-1}, y_n)} \leq \beta(d(y_{n-1}, y_n)) < 1,$$

taking the limit as $n \to \infty$, we get that $\beta(d(y_{n-1}, y_n)) \to 1$. But this implies that $d(y_{n-1}, y_n) \to 0$, a contradiction. Hence,

$$d(y_{n-1}, y_n) \to 0 \text{ as } n \to \infty. \tag{2.7}$$

In a similar way, one can prove that

$$d(y_{n-i}, y_n) \to 0 \text{ as } n \to \infty, \text{ for } i = 2, 3, \ldots, \nu. \tag{2.8}$$

Suppose now that $y_n = y_m$ for some $n > m$ (and hence, by the way y_n's are chosen, $y_{n+k} = y_{m+k}$ for $k \in \mathbb{N}$). Then, (2.6) implies that

$$\begin{aligned} d(y_m, y_{m+1}) &= d(y_n, y_{n+1}) \leq \beta(d(y_{n-1}, y_n))d(y_{n-1}, y_n) \leq \cdots \\ &\leq \beta(d(y_{n-1}, y_n)) \cdots \beta(d(y_m, y_{m+1}))d(y_m, y_{m+1}) < d(y_m, y_{m+1}), \end{aligned}$$

a contradiction. Thus, in what follows, we can assume that $y_n \neq y_m$ for $n \neq m$.

In order to prove that $\{y_n\}$ is a Cauchy sequence, suppose that it is not. Then, by Lemma 2.4, using (2.7) and (2.8), we conclude that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$n_k > m_k > k$ and the sequences (2.5) tend to ε as $k \to \infty$. Using (2.6) with $x = x_{m_k}$ and $y = x_{n_k+1}$, one obtains

$$\frac{d(y_{m_k}, y_{n_k+1})}{d(y_{m_k-1}, y_{n_k})} \leq \beta(d(y_{m_k-1}, y_{n_k})) < 1.$$

Letting $k \to \infty$, it follows that $\beta(d(y_{m_k-1}, y_{n_k})) \to 1$, implying that $d(y_{m_k-1}, y_{n_k}) \to 0$, a contradiction. Thus, $\{y_n\}$ is a Cauchy sequence.

The rest of the proof is similar as for Theorem 2.1. □

Example 2.6. Let $X = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ and $d\colon X \times X \to [0, +\infty)$ be defined by:

$$\begin{aligned}
&d(x,x) = 0 \text{ for } x \in X;\\
&d(x,y) = d(y,x) \text{ for } x, y \in X;\\
&d(\alpha,\beta) = 5k,\\
&d(\alpha,\gamma) = d(\beta,\gamma) = d(\alpha,\varepsilon) = d(\gamma,\varepsilon) = k,\\
&d(\alpha,\delta) = d(\beta,\delta) = d(\gamma,\delta) = d(\beta,\varepsilon) = d(\delta,\varepsilon) = 2k,
\end{aligned}$$

where $k \in \mathbb{R}$ is chosen so that $0 < k < \frac{\log 2}{2}$, i.e., $e^{-2k} > e^{-\log 2} = \frac{1}{2}$. Then it is easy to check that (X, d) is a 3-generalized metric space which is not a 2-generalized metric space since

$$d(\alpha,\beta) = 5k > 4k = d(\alpha,\gamma) + d(\gamma,\varepsilon) + d(\varepsilon,\beta).$$

Consider the following mappings $T, S\colon X \to X$.

$$T = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon \\ \gamma & \gamma & \gamma & \alpha & \gamma \end{pmatrix} \qquad S = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon \\ \alpha & \alpha & \gamma & \delta & \alpha \end{pmatrix}.$$

Then $T(X) = \{\alpha, \gamma\} \subset \{\alpha, \gamma, \delta\} = S(X)$. Take the function $\beta \in \mathcal{S}$ defined by $\beta(t) = e^{-t}$ for $t > 0$ and $\beta(0) \in [0, 1)$. Let us check that T, S satisfy the contractive condition (2.6) of Theorem 2.5. Let $x, y \in X$ with $x \neq y$ and consider the following possible cases:

1° $x, y \in \{\alpha, \beta, \gamma, \varepsilon\}$. Then $Tx = Ty = \gamma$ and $d(Tx, Ty) = 0$. Hence, (2.6) trivially holds.

2° $x \in \{\alpha, \beta, \varepsilon\}$, $y = \delta$. Then $Tx = \gamma$, $Ty = \alpha$ and $d(Tx, Ty) = k$; $Sx = \alpha$, $Sy = \delta$ and $d(Sx, Sy) = 2k$. Hence,

$$d(Tx, Ty) = k < e^{-2k} \cdot 2k = \beta(2k) \cdot 2k = \beta(d(Sx, Sy))d(Sx, Sy),$$

since $1 < e^{-2t} \cdot 2$.

3° $x = \gamma$, $y = \delta$. Then $Tx = \gamma$, $Ty = \alpha$ and $d(Tx, Ty) = k$; $Sx = \gamma$, $Sy = \delta$ and $d(Sx, Sy) = 2k$. Hence, the inequality (2.6) is again satisfied.

All the conditions of Theorem 2.5 are satisfied and T and S have a unique point of coincidence (which is γ). γ is also their unique common fixed point.

2.3. **Weak contractions.** By using Lemma 2.4 several common fixed point results of weak-type can be deduced as well. As an illustration, we will just give a sketch of proof for a ν-generalized version of [3, Theorem 2.1] (which was proved in [3] for $\nu = 2$ under additional assumption that the corresponding topology is Hausdorff).

Theorem 2.7. *Let (X, d) be a ν-generalized metric space. Suppose that $T, S\colon X \to X$ are two mappings such that $T(X) \subseteq S(X)$ and that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}(d(Sx, Tx) + d(Sy, Ty)) - \phi(d(Sx, Tx), d(Sy, Ty)), \tag{2.9}$$

where $\phi\colon [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous and $\phi(a, b) = 0$ if and only if $a = b = 0$. If at least one of subspaces $T(X)$ and $S(X)$ is complete, then S and T have a unique point of coincidence. If, moreover, S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. First of all, it is easy to prove that a point of coincidence of S and T is unique (if it exists).

Form a standard Jungck sequence $\{y_n\}$ as in the previous proofs. It can be proved, again in a standard way, that (under the assumption $d(y_n, y_{n+1}) > 0$ for each $n \in \mathbb{Z}^+$) $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for $n \in \mathbb{N}$. It easily follows that $y_n \neq y_m$ for $n \neq m$ and that the sequences

$$d(y_n, y_{n+1}), \quad \dots, \quad d(y_n, y_{n+\nu})$$

tend to 0 as $n \to \infty$. Now, using the contractive condition (2.9) and Lemma 2.4, we deduce that $\{y_n\}$ is a Cauchy sequence. The completeness of $S(X)$ (or $T(X)$) implies that

$$y_n = Sx_{n+1} \to Sz \text{ as } n \to \infty$$

for some $z \in X$.

Suppose that $Tz \neq Sz$. Then, for some $n \in \mathbb{N}$, $Tz, Sz \notin \{y_n, y_{n+1}, \dots\}$ and we can apply the polygonal inequality to obtain

$$\begin{aligned} d(Tz, Sz) &\leq d(Tz, y_n) + d(y_n, y_{n+1}) + \dots \\ &\qquad + d(y_{n+\nu-2}, y_{n+\nu-1}) + d(y_{n+\nu-1}, Sz) \\ &= d(Tz, Tx_n) + d(y_n, y_{n+1}) + \dots \\ &\qquad + d(y_{n+\nu-2}, y_{n+\nu-1}) + d(y_{n+\nu-1}, Sz) \\ &\leq \frac{1}{2}(d(Sz, Tz) + d(Sx_n, Tx_n)) - \phi(d(Sz, Tz), d(Sx_n, Tx_n)) \\ &\quad + d(y_n, y_{n+1}) + \dots + d(y_{n+\nu-2}, y_{n+\nu-1}) + d(y_{n+\nu-1}, Sz). \end{aligned}$$

Passing to the limit as $n \to \infty$ and using the properties of function ϕ, we get

$$\begin{aligned} d(Sz,Tz) &\le \frac{1}{2}(d(Sz,Tz)+0) - \phi(d(Sz,Tz),0) + (\nu-1)\cdot 0 + 0 \\ &\le \frac{1}{2} d(Sz,Tz). \end{aligned}$$

Hence, $Sz = Tz$. □

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