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f-BIHARMONIC AND BI-*f*-HARMONIC SUBMANIFOLDS OF PRODUCT SPACES

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Abstract. We consider f-biharmonic and bi-f-harmonic submanifolds of the product of two real space forms. We find the necessary and sufficient conditions for a submanifold to be f-biharmonic and bi-f-harmonic in a product of two real space forms.

1. Introduction

Let (M, g) and (N, h) be two Riemannian manifolds. $\varphi : M \to N$ is called a *harmonic map* if it is a critical point of the *energy functional*

$$E(\varphi) = \frac{1}{2}\int_{\Omega} \|d\varphi\|^2 \, d\nu_g,$$

where Ω is a compact domain of M. The Euler-Lagrange equation of $E(\varphi)$ is

$$\tau(\varphi) = tr(\nabla d\varphi) = 0,$$

where $\tau(\varphi)$ is the *tension field* of φ [3]. The map φ is said to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2}\int_{\Omega} \|\tau(\varphi)\|^2 \, d\nu_g,$$

where Ω is a compact domain of M. In [4], Jiang obtained the Euler-Lagrange equation of $E_2(\varphi)$. This gives us

$$\tau_2(\varphi) = tr(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.1}$$

where $\tau_2(\varphi)$ is the *bitension field* of φ and R^N is the curvature tensor of N. The map φ is said to be *f-biharmonic* if it is a critical point of the

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f-bienergy functional

$$E_{2,f}(\varphi)=\frac{1}{2}\int_{\Omega}f\left\|\tau(\varphi)\right\|^{2}d\nu_{g},$$

where Ω is a compact domain of M [5]. The Euler-Lagrange equation for the f-bienergy functional is defined by

$$\tau_{2,f}(\varphi)=f\tau_{2}(\varphi)+(\Delta f)\,\tau(\varphi)+2\nabla^{\varphi}_{gradf}\tau(\varphi)=0, \tag{1.2}$$

where $\tau_{2,f}(\varphi)$ is the f-bitension field of φ [5]. From the definition, it is trivial that any harmonic map is f-biharmonic. If the f-biharmonic map is neither harmonic nor biharmonic then we call it by *proper f-biharmonic* and if f is a constant, an f-biharmonic map turns into a biharmonic map [5].

An *f-harmonic map* with a positive function $f:M\overset{C^{\infty}}{\to}\mathbb{R}$ is a critical point of the f-energy

$$E_{f}(\varphi)=\frac{1}{2}\int_{\Omega}f\left\|d\varphi\right\|^{2}d\nu_{g},$$

where Ω is a compact domain of M. The Euler-Lagrange equation for the *f-energy functional* gives us the *f-tension field* $\tau_{f}(\varphi)$ (see [1], [8]) by

$$\tau_{f}(\varphi)=f\tau(\varphi)+d\varphi(gradf)=0. \tag{1.3}$$

The map φ is said to be *bi-f-harmonic* if it is a critical point of the bi-f-energy functional

$$E_{f}^{2}(\varphi)=\frac{1}{2}\int_{\Omega}\left\|\tau_{f}(\varphi)\right\|^{2}d\nu_{g},$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the bi-f-harmonic map equation

$$\tau_{f}^{2}(\varphi)=fJ^{\varphi}\left(\tau_{f}(\varphi)\right)-\nabla^{\varphi}_{gradf}\tau_{f}(\varphi)=0, \tag{1.4}$$

where $\tau_{f}^{2}(\varphi)$ is the bi-f-tension field of φ and J^{φ} is the Jacobi operator of the map defined by $J^{\varphi}(X)=-\left[Tr_{g}\nabla^{\varphi}\nabla^{\varphi}X-\nabla^{\varphi}_{\nabla^{M}}X-R^{N}(d\varphi,X)\,d\varphi\right]$ [8]. It is trivial that any f-harmonic map is bi-f-harmonic [8].

Eells and Sampson studied harmonic mappings of Riemannian manifolds [3]. Jiang defined biharmonic maps by using the first and second variational formulas of bienergy functional [4]. In [5], Lu defined the notion of f-biharmonic maps. He obtained the f-biharmonic map equation and studied f-biharmonicity of some special maps. In [7], Ou studied on some properties of f-biharmonic maps and f-biharmonic submanifolds. Course defined f-harmonic maps in [1]. Later, Ouakkas, Nasri and Djaa obtained some properties for f-harmonic maps between two Riemannian manifolds and defined bi-f-harmonic maps [8]. In [11], Zegga, Cherif and Djaa considered bi-f-harmonic maps and submanifolds. In [9], Roth studied biharmonic submanifolds of the product of two space forms. Motivated by the above studies, in this paper, we consider f-biharmonic and bi-f-harmonic submanifolds of the product of two space forms and obtain the necessary and sufficient conditions for a submanifold to be f-biharmonic and bi-f-harmonic in a product of two real space forms.

2. Preliminaries

Let $M^{n_1}(c_1)$ and $M^{n_2}(c_2)$ be two real space forms of constant curvatures c_1, c_2 with dimensions n_1 and n_2, respectively. Let us consider the product space $(M^{n_1}(c_1)\times M^{n_2}(c_2),\widetilde{g})$. Assume that (M^m,g) be a Riemannian manifold isometrically immersed into the product space $(M^{n_1}(c_1)\times M^{n_2}(c_2),\widetilde{g})$. Denote by $\widetilde{\nabla}$ and F the Levi-Civita connection of M^m and the product structure of the product space $(M^{n_1}(c_1)\times M^{n_2}(c_2),\widetilde{g})$, respectively. The product structure $F: TM^{n_1}(c_1)\times TM^{n_2}(c_2)\longrightarrow TM^{n_1}(c_1)\times TM^{n_2}(c_2)$ is a $(1,1)$-tensor field defined by

$$F(X_1+X_2)=X_1-X_2$$

for any vector field $X=X_1+X_2$, X_1,X_2 denote the parts of X tangent to the first and second factors, respectively. It is easy to see that F satisfies

$$F^2=I \text{ (and } F\neq I), \tag{2.1}$$

$$\widetilde{g}(FX,Y)=\widetilde{g}(X,FY), \tag{2.2}$$

$$\widetilde{\nabla}F=0, \tag{2.3}$$

(see [10]). By an easy calculation, we obtain the curvature tensor of $(M^{n_1}(c_1)$ $\times M^{n_2}(c_2),\widetilde{g})$ as

$$\begin{aligned}\widetilde{R}(X,Y)Z&=a[g(Y,Z)X-g(X,Z)Y+g(FY,Z)FX-g(FX,Z)FY]\\&+b[g(Y,Z)FX-g(X,Z)FY+g(Y,FZ)X-g(X,FZ)Y]\end{aligned} \tag{2.4}$$

with $a=\frac{c_1+c_2}{4}$ and $b=\frac{c_1-c_2}{4}$ [2].

Now let $X\in TM^m$ and $\xi\in T^{\perp}M^m$. The decompositions of FX and $F\xi$ into tangent and normal components can be written as

$$FX=kX+hX \text{ and } F\xi=s\xi+t\xi, \tag{2.5}$$

where $k:TM^m\longrightarrow TM^m$, $h:TM^m\longrightarrow T^{\perp}M^m$, $s:T^{\perp}M^m\longrightarrow TM^m$, and $t:T^{\perp}M^m\longrightarrow T^{\perp}M^m$ are $(1,1)$-tensor fields. From equations (2.1) and (2.2), it is easy to see that k and t are symmetric and satisfy the following properties:

$$k^2X=X-shX, \tag{2.6}$$

$$t^2\xi = \xi - hs\xi, \tag{2.7}$$

$$ks\xi + st\xi = 0, \tag{2.8}$$

$$hkX + thX = 0, \tag{2.9}$$

$$\widetilde{g}(hX, \xi) = \widetilde{g}(X, s\xi), \tag{2.10}$$

(for more details see [10]).

3. f-BIHARMONIC SUBMANIFOLDS OF PRODUCT SPACES

Let $\varphi : M^m \longrightarrow (M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$ be an isometric immersion from an m-dimensional Riemannian manifold (M^m, g) into the product of two space forms $M^{n_1}(c_1)$ and $M^{n_2}(c_2)$ of constant curvatures c_1, c_2 with dimensions n_1 and n_2. We shall denote by B, A, H, Δ and $\Delta^{\perp}$ the second fundamental form, the shape operator, the mean curvature vector field, the Laplacian and the Laplacian on the normal bundle of M^m in $N = (M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$, respectively.

Firstly we have the following theorem:

Theorem 3.1. *Let M^m be a Riemannian manifold isometrically immersed into the product space $N = (M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$. Then M^m is f-biharmonic if and only if the following two equations hold:*

$$\Delta^{\perp}H + trB(\cdot, A_H(\cdot)) - \frac{\Delta f}{f}H - 2\nabla^{\perp}_{grad \ln f}H$$
$$= a(mH - hsH + tr(k)tH) + b(mtH + tr(k)H) \tag{3.1}$$

and

$$\frac{m}{2} grad \, \|H\|^2 + 2tr(A_{\nabla^{\perp}_{\cdot} H} \cdot) + 2A_H grad \ln f$$
$$= a(-ksH + tr(k)sH) + b(m-1)sH. \tag{3.2}$$

Proof. Let us denote by ∇^{φ}, ∇ the Levi-Civita connections on N and M^m, respectively. Let $\{e_i\}$, $1 \leq i \leq m$ be a local geodesic orthonormal frame at $p \in M^m$. Then

$$\tau(\varphi) = tr(\nabla d\varphi) = mH. \tag{3.3}$$

From (1.1) and (3.3), we have

$$\begin{aligned}
\tau_2(\varphi) &= tr(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) \\
&= \sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} - \nabla^{\varphi}_{\nabla_{e_i}e_i})mH - \sum_{i=1}^{m} R^N(d\varphi(e_i), mH)d\varphi(e_i) \\
&= -m\left\{\Delta H + \sum_{i=1}^{m} R^N(d\varphi(e_i), H)d\varphi(e_i)\right\}.
\end{aligned} \tag{3.4}$$

By (2.4), we find

$$\sum_{i=1}^{m}(R^N(d\varphi(e_i),H)d\varphi(e_i)) \\ = a\left[-mH + F(FH)^T - tr(k)FH\right] + b\left[-mFH + (FH)^T - tr(k)H\right].$$

Using (2.5) in the above equality, we get

$$\sum_{i=1}^{m}(R^N(d\varphi(e_i),H)d\varphi(e_i)) = a\left[-mH + ksH + hsH - tr(k)sH - tr(k)tH\right] \\ + b\left[-msH - mtH + sH - tr(k)H\right]. \quad (3.5)$$

By the use of the Gauss and Weingarten formulas, we have

$$\begin{aligned}\Delta H &= -\sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}H) = -\sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}(-A_He_i + \nabla^{\perp}_{e_i}H) \\ &= -\sum_{i=1}^{m}\left\{-\nabla_{e_i}A_He_i - B(e_i,A_He_i) - A_{\nabla^{\perp}_{e_i}H}e_i + \nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}H\right\} \\ &= \sum_{i=1}^{m}\nabla_{e_i}A_He_i + \sum_{i=1}^{m}B(e_i,A_He_i) + \sum_{i=1}^{m}A_{\nabla^{\perp}_{e_i}H}e_i - \sum_{i=1}^{m}\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}H \\ &= tr(\nabla.A_H\cdot) + trB(\cdot,A_H\cdot) + tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) + \Delta^{\perp}H. \end{aligned} \quad (3.6)$$

Now we shall compute $tr(\nabla.A_H\cdot)$. In view of Gauss and Weingarten formulas, we obtain

$$\begin{aligned}\sum_{i=1}^{m}\nabla_{e_i}A_He_i &= \sum_{i,j}g(\nabla_{e_i}A_He_i,e_j)e_j = \sum_{i,j}e_ig(A_He_i,e_j)e_j \\ &= \sum_{i,j}e_ig(B(e_i,e_j),H)e_j = \sum_{i,j}e_ig(\nabla^{\varphi}_{e_j}e_i,H)e_j \\ &= \sum_{i,j}\left\{g(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}e_i,H)e_j + g(\nabla^{\varphi}_{e_j}e_i,\nabla^{\varphi}_{e_i}H)e_j\right\} \\ &= \sum_{i,j}\left\{g(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}e_i,H)e_j + g(B(e_i,e_j),\nabla^{\perp}_{e_i}H)e_j\right\} \\ &= \sum_{i,j}g(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}e_i,H)e_j + \sum_{i}A_{\nabla^{\perp}_{e_i}H}(e_i). \end{aligned} \quad (3.7)$$

Then, using the definition of the curvature tensor of N we have

$$\begin{aligned}\sum_{i,j} g(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}e_i, H)e_j &= \sum_{i,j} g(R^N(e_i,e_j)e_i + \nabla^{\varphi}_{e_j}\nabla^{\varphi}_{e_i}e_i + \nabla^{\varphi}_{[e_i,e_j]}e_i, H) \\ &= mg(\nabla_{e_j}H, H) \\ &= \frac{m}{2} grad \left\|H\right\|^2 . \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) and then using (3.7) into (3.6), we get

$$\Delta H = \frac{m}{2} grad \left\|H\right\|^2 + trB(\cdot, A_H\cdot) + 2tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) + \Delta^{\perp}H. \tag{3.9}$$

In view of equations (3.9) and (3.5) into (3.4), we obtain

$$\begin{aligned}\tau_2(\varphi) = -m\Big\{&\frac{m}{2} grad \left\|H\right\|^2 + trB(\cdot, A_H\cdot) + 2tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) + \Delta^{\perp}H \\ &+a\left[-mH + ksH + hsH - tr(k)sH - tr(k)tH\right] \\ &+b\left[-msH - mtH + sH - tr(k)H\right]\Big\}. \end{aligned} \tag{3.10}$$

Using Weingarten formula and equation (3.3), we have

$$\nabla^{\varphi}_{gradf}\tau(\varphi) = \nabla^{\varphi}_{gradf}mH = m\left(-A_H gradf + \nabla^{\perp}_{gradf}H\right). \tag{3.11}$$

Finally substituting equations (3.3), (3.10) and (3.11) into equation (1.2), we obtain

$$\begin{aligned}-fm\Big\{&\frac{m}{2} grad \left\|H\right\|^2 + trB(\cdot, A_H\cdot) + 2tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) + \Delta^{\perp}H \\ &+a\left[-mH + ksH + hsH - tr(k)sH - tr(k)tH\right] \\ &+b\left[-msH - mtH + sH - tr(k)H\right] \\ &-\frac{\Delta f}{f}H + 2A_H grad \ln f - 2\nabla^{\perp}_{grad \ln f}H\Big\} = 0. \end{aligned}$$

Hence comparing the tangential and the normal parts, we obtain the desired result. □

Corollary 3.2. *Let M^m be a Riemannian manifold isometrically immersed into the product space $N = (M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$.*

1) If FH is tangent to M^m, then M^m is f-biharmonic if and only if

$$\Delta^{\perp}H + trB(., A_H(.)) - \frac{\Delta f}{f}H - 2\nabla^{\perp}_{grad \ln f}H - \left[a(m-1) + btr(k)\right]H = 0, \tag{3.12}$$

$$\frac{m}{2} grad \left\|H\right\|^2 + 2tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) + 2A_H grad \ln f - \left[atr(k) + b(m-1)\right]FH = 0. \tag{3.13}$$

2) *If FH is normal to M^m, then M^m is f-biharmonic if and only if*

$$\Delta^{\perp}H + trB(.,A_H(.)) - \frac{\Delta f}{f}H - 2\nabla^{\perp}_{grad\ln f}H - [am + btr(k)]\,H - [atr(k) + bm]\,FH = 0, \quad (3.14)$$

$$\frac{m}{2}grad\,\|H\|^2 + 2tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) + 2A_H grad\ln f = 0. \quad (3.15)$$

Proof. 1) If FH is tangent to M^m, then by the use of (2.5) we have $FH = sH$ and $tH = 0$. So from (2.7), we have $hsH = H$ and by Theorem 3.1 we find (3.12) and (3.13).

2) If FH is normal to M^m, then $sH = 0$ and $tH = FH$. Hence from Theorem 3.1 we get (3.14) and (3.15). □

Corollary 3.3. *Let M^m be a submanifold of $S^p(r) \times S^{n-p}(r)$ of dimension $m \geq 2$ with non-zero constant mean curvature such that FH is tangent to M^m.*

1) *If M^m is proper f-biharmonic, then*

$$0 < \|H\|^2 \leq \inf\left\{\frac{\frac{1}{2r^2}(m-1) + \frac{\Delta f}{f}}{m}\right\}. \quad (3.16)$$

2) *Assume that f is an eigenfunction of the Laplacian Δ corresponding to real eigenvalue λ. Hence the equality in (3.16) occurs and M^m is proper f-biharmonic if and only if M^m is pseudo-umbilical,*

$$\nabla^{\perp}H = 0,$$

$$2A_H grad\ln f - \frac{1}{2r^2}tr(k)FH = 0$$

and

$$trB(\cdot, A_H\cdot) = \left[\frac{1}{2r^2}(m-1) + \lambda\right]H.$$

Proof. We assume that FH is tangent to M^m, then in view of (2.5) we have $FH = sH$ and $tH = 0$. Hence by (2.7), we have $hsH = H$. By the use of (3.12) we have

$$\Delta^{\perp}H + trB(.,A_H(.)) - \frac{\Delta f}{f}H - 2\nabla^{\perp}_{grad\ln f}H - \frac{1}{2r^2}(m-1)H = 0. \quad (3.17)$$

Then taking the scalar product of (3.17) with H, we find

$$g(\Delta^{\perp}H, H) + g(trB(.,A_H(.)), H) - \frac{\Delta f}{f}g(H,H) - 2g\left(\nabla^{\perp}_{grad\ln f}H, H\right) - \frac{1}{2r^2}(m-1)g(H,H) = 0.$$

Since $\|H\|$ is a constant, we have

$$g(\Delta^{\perp}H,H)=\frac{\Delta f}{f}\|H\|^2-\|A_H\|^2+\frac{1}{2r^2}(m-1)\|H\|^2.$$

Using the Bochner formula, we get

$$\left\|\nabla^{\perp}H\right\|^2+\|A_H\|^2=\left[\frac{\Delta f}{f}+\frac{1}{2r^2}(m-1)\right]\|H\|^2. \tag{3.18}$$

By the use of Cauchy-Schwarz inequality, we have $\|A_H\|^2\geq m\|H\|^4$ (see [9]). Hence we find

$$\left[\frac{\Delta f}{f}+\frac{1}{2r^2}(m-1)\right]\|H\|^2\geq m\|H\|^4+\left\|\nabla^{\perp}H\right\|^2\geq m\|H\|^4. \tag{3.19}$$

Since $\|H\|$ is a non-zero constant, we can write

$$0<\|H\|^2\leq\inf\left\{\frac{\left[\frac{\Delta f}{f}+\frac{1}{2r^2}(m-1)\right]}{m}\right\}. \tag{3.20}$$

Now, if f is an eigenfunction of the Laplacian Δ corresponding to the real eigenvalue λ, then $\frac{\Delta f}{f}=\lambda$. We can write

$$\|H\|^2=\frac{\left[\lambda+\frac{1}{2r^2}(m-1)\right]}{m}. \tag{3.21}$$

Assume that M^m is proper f-biharmonic. From (3.19), first we have $\nabla^{\perp}H=0$. Moreover substituting the equation (3.21) into (3.19) we find

$$\|A_H\|^2=\frac{\left[\lambda+\frac{1}{2r^2}(m-1)\right]^2}{m}.$$

That is, M^m is pseudo-umbilical. Then from (3.13) we have

$$2A_H grad\ln f-\frac{1}{2r^2}tr(k)FH=0.$$

In this case (3.12) turns into

$$trB(\cdot,A_H\cdot)=\left[\lambda+\frac{1}{2r^2}(m-1)\right]H.$$

This completes the proof. □

Now we consider f-biharmonic hypersurface M^m of $S^p(r)\times S^{n-p}(r)$ such that FH is tangent to M^m. Firstly we have:

Proposition 3.4. *Let M^{n-1} be a hypersurface of $S^p(r) \times S^{n-p}(r)$ with non-zero constant mean curvature such that FH is tangent to M^{n-1}. Then M^{n-1} is f-biharmonic if and only if*

$$A_H grad \ln f = \frac{1}{4r^2} tr(k) FH$$

and

$$\|B\|^2 = \frac{n-2}{2r^2} + \frac{\Delta f}{f}.$$

Proof. Assume that M^{n-1} is a hypersurface of $S^p(r) \times S^{n-p}(r)$ with non-zero constant mean curvature such that FH is tangent to M^{n-1}. Then from (3.12) we get

$$tr B(\cdot, A_H \cdot) = \left[\frac{\Delta f}{f} + \frac{1}{2r^2}(n-2)\right] H.$$

Then, by taking the scalar product with H, we have

$$\sum_{i=1}^{n-1} g\left(B(e_i, A_H e_i), H\right) = \left[\frac{\Delta f}{f} + \frac{1}{2r^2}(n-2)\right] \|H\|^2,$$

$$\sum_{i=1}^{n-1} g\left(A_H e_i, A_H e_i\right) = \left[\frac{\Delta f}{f} + \frac{1}{2r^2}(n-2)\right] \|H\|^2$$

and

$$\|A_H\|^2 = \left[\frac{\Delta f}{f} + \frac{1}{2r^2}(n-2)\right] \|H\|^2.$$

From ([10], page 71), we know that $\|A_H\|^2 = \|B\|^2$. Hence we find

$$\|B\|^2 = \frac{n-2}{2r^2} + \frac{\Delta f}{f}. \tag{3.22}$$

So the equation (3.13) is reduced to

$$A_H grad \ln f = \frac{1}{4r^2} tr(k) FH.$$

This completes the proof of the proposition. □

Proposition 3.5. *Let M^{n-1} be a proper f-biharmonic hypersurface of $S^p(r)$ $\times S^{n-1}(r)$ with non-zero constant mean curvature such that FH is tangent to M^{n-1}. Then the scalar curvature of M^{n-1} is given by*

$$Scal_{M^{n-1}} = \frac{1}{2r^2}\left\{(n-1)(n-3) - (n-2) + tr(k)^2\right\} + (n-1)^2 \|H\|^2 - \frac{\Delta f}{f}.$$

Proof. By the use of the Gauss equation, we can write

$$Scal_{M^{n-1}} = \sum_{i=1}^{m} g\left(R^N(e_i,e_j)e_j,e_i\right) + \sum_{i=1}^{m} g\left(B(e_i,e_i),B(e_j,e_j)\right) - \sum_{i=1}^{m} g\left(B(e_j,e_i),B(e_j,e_i)\right).$$

Then we compute

$$Scal_{M^{n-1}} = \sum_{i=1}^{m} g\left(R^N(e_i,e_j)e_j,e_i\right) + (n-1)^2 \|H\|^2 - \|B\|^2. \tag{3.23}$$

Using (2.4) we write

$$\sum_{i=1}^{m} g\left(R^N(e_i,e_j)e_j,e_i\right) = \frac{1}{2r^2}\Bigg\{\sum_{i=1}^{m} g(e_j,e_j)g(e_i,e_i) - \sum_{i=1}^{m} g(e_i,e_j)g(e_i,e_j) + \sum_{i=1}^{m} g(Fe_j,e_j)g(Fe_i,e_i) - \sum_{i=1}^{m} g(Fe_i,e_j)g(Fe_i,e_j)\Bigg\}.$$

Hence we find

$$\sum_{i=1}^{m} g\left(R^N(e_i,e_j)e_j,e_i\right) = \frac{1}{2r^2}\left\{(n-1)(n-3) + tr(k)^2\right\}. \tag{3.24}$$

Finally, in view of equations (3.24) and (3.22) into (3.23), we get

$$Scal_{M^{n-1}} = \frac{1}{2r^2}\left\{(n-1)(n-3) - (n-2) + tr(k)^2\right\} + (n-1)^2 \|H\|^2 - \frac{\Delta f}{f}.$$

This proves the proposition. □

4. Bi-f-harmonic submanifolds of product spaces

In this section, we consider bi-f-harmonic submanifolds of product of two real space forms. We firstly state the following theorem:

Theorem 4.1. *Let M^m be a Riemannian manifold isometrically immersed into the product space $N = (M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$. Then M^m is bi-f-harmonic if and only if the following two equations hold:*

$$\left(mf^2\right)\left(\Delta^{\perp} H\right) + \left(mf^2\right) trB(\cdot, A_H(\cdot)) - fm\left(\Delta f\right)H - (3mf)\nabla^{\perp}_{gradf} H$$

$$-ftrB\left(\cdot, \nabla_{\cdot} gradf\right) - ftr\nabla^{\perp}_{\cdot} B\left(\cdot, gradf\right) - m\|gradf\|^2 H - B\left(gradf, gradf\right)$$

$$= \left(mf^2\right)\{a\left[mH - hsH + tr(k)tH + tr(k)hgradf + hkgradf\right] + b\left[mtH + tr(k)H + (m-1)hgradf\right]\}$$

and

$$\frac{(mf)^2}{2} grad\,\|H\|^2 + 2\left(mf^2\right) tr(A_{\nabla^{\perp}_{\cdot} H}\cdot) + 3\,(mf)\,A_H gradf$$
$$+fRicci^M\,(gradf) + fgrad\,(\Delta f) + ftrA_{B(\cdot,gradf)}\,(\cdot) - \frac{1}{2}grad\left(\|gradf\|^2\right)$$
$$= \left(mf^2\right)\left\{a\left[-ksH + tr(k)sH + (m-1)gradf + tr(k)kgradf - k^2 gradf\right]\right.$$
$$\left. +b\left[(m-1)sH + m\,(kgradf) + tr(k)gradf\right]\right\}.$$

Proof. Let us denote by ∇^{φ}, ∇ the Levi-Civita connections on N and M^m, respectively. Let $\{e_i\}$, $1 \leq i \leq m$ be a local geodesic orthonormal frame at $p \in M^m$.

From the equations (1.3) and (3.3), we find

$$\tau_f(\varphi) = fmH + d\varphi\,(gradf) = fmH + gradf. \tag{4.1}$$

Then, we can write

$$\sum_{i=1}^{m}(R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i)) = mf\sum_{i=1}^{m}(R^N(H, d\varphi(e_i))d\varphi(e_i))$$
$$+\sum_{i=1}^{m}(R^N(gradf, d\varphi(e_i))d\varphi(e_i)). \tag{4.2}$$

Using the equation (2.4), we obtain

$$\sum_{i=1}^{m}(R^N(gradf, d\varphi(e_i))d\varphi(e_i))$$
$$= a\left[(m-1)\,(gradf) + tr(k)\,(Fgradf) - F\,(Fgradf)^T\right]$$
$$+ b\left[(m-1)\,(Fgradf) + tr(k)\,(gradf) - (Fgradf)^T\right].$$

Using (2.5) in the above equality, we get

$$\sum_{i=1}^{m}(R^N(gradf, d\varphi(e_i))d\varphi(e_i)) = a\left[(m-1)\,(gradf) + tr(k)\,(kgradf)\right.$$
$$\left. +tr(k)\,(hgradf) - k^2 gradf + hkgradf\right]$$
$$+ b\left[m\,(kgradf) + (m-1)\,(hgradf) + tr(k)gradf\right]. \tag{4.3}$$

In view of equations (3.5) and (4.3) into equation (4.2), we obtain

$$\sum_{i=1}^{m}(R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i)) = a\left[mH - ksH - hsH + tr(k)sH + tr(k)tH\right.$$
$$+(m-1)\,(gradf) + tr(k)\,(kgradf) + tr(k)\,(hgradf)$$

$$-k^2 gradf + hkgradf\big] + b\left[msH + mtH - sH + tr(k)Hm\left(kgradf\right) \right. \\ \left. +(m-1)\left(hgradf\right) + tr(k)gradf\right]. \tag{4.4}$$

By the use of the Gauss and Weingarten formulas, we have

$$\sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau_f(\varphi) - \nabla^{\varphi}_{\nabla_{e_i}e_i}\tau_f(\varphi)) = \sum_{i=1}^{m}\left[\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\left(fmH + gradf\right)\right]$$

$$= m\sum_{i=1}^{m}\nabla^{\varphi}_{e_i}\left(e_i\left(f\right)H + f\nabla^{\varphi}_{e_i}H\right) + \sum_{i=1}^{m}\nabla^{\varphi}_{e_i}\left(\nabla_{e_i}gradf + B\left(e_i, gradf\right)\right)$$

$$= m\sum_{i=1}^{m}\left\{e_i\left(e_i\left(f\right)\right)H + 2e_i\left(f\right)\nabla^{\varphi}_{e_i}H + f\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}H\right\} + \sum_{i=1}^{m}\nabla_{e_i}\nabla_{e_i}gradf$$

$$+\sum_{i=1}^{m}\left\{B\left(e_i, \nabla_{e_i}gradf\right) - A_{B(e_i,gradf)}\left(e_i\right) + \nabla^{\perp}_{e_i}B\left(e_i, gradf\right)\right\}. \tag{4.5}$$

Using the equation (3.9), we can write

$$\sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}H) = -\frac{m}{2}grad\left\|H\right\|^2 - trB(\cdot, A_H\cdot) - 2tr(A_{\nabla^{\perp}_{\cdot}H}\cdot) - \Delta^{\perp}H. \tag{4.6}$$

In view of equation (4.6) into (4.5), we obtain

$$\sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau_f(\varphi) - \nabla^{\varphi}_{\nabla_{e_i}e_i}\tau_f(\varphi)) = m\left(\Delta f\right)H + 2m\sum_{i=1}^{m}\nabla^{\varphi}_{gradf}H$$

$$-\frac{m^2 f}{2}grad\left\|H\right\|^2 - \left(mf\right)trB(\cdot, A_H\cdot) - 2\left(mf\right)tr(A_{\nabla^{\perp}_{\cdot}H}\cdot)$$

$$-\left(mf\right)\Delta^{\perp}H + \sum_{i=1}^{m}\nabla_{e_i}\nabla_{e_i}gradf + \sum_{i=1}^{m}B\left(e_i, \nabla_{e_i}gradf\right)$$

$$-\sum_{i=1}^{m}A_{B(e_i,gradf)}\left(e_i\right) + \sum_{i=1}^{m}\nabla^{\perp}_{e_i}B\left(e_i, gradf\right). \tag{4.7}$$

Using Gauss and Weingarten formulas, we have

$$\nabla^{\varphi}_{gradf}\tau_f(\varphi) = \nabla^{\varphi}_{gradf}\left(fmH + gradf\right)$$

$$= m\left\|gradf\right\|^2 H - \left(mf\right)A_H gradf + \left(mf\right)\nabla^{\perp}_{gradf}H$$

$$+\frac{1}{2}grad\left(\left\|gradf\right\|^2\right) + B\left(gradf, gradf\right). \tag{4.8}$$

Finally substituting (4.4), (4.7) and (4.8) into equation (1.4) and comparing the tangential and the normal parts, we obtain the desired result. □

Corollary 4.2. *Let M^m be a Riemannian manifold isometrically immersed into the product space $N = (M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$.*

1) *If FH and $Fgradf$ are tangent to M^m, then M^m is bi-f-harmonic if and only if*

$$\left(mf^2\right)\left(\Delta^{\perp}H\right)+\left(mf^2\right)trB(\cdot,A_H(\cdot))-fm\left(\Delta f\right)H-\left(3mf\right)\nabla^{\perp}_{gradf}H$$
$$-ftrB\left(\cdot,\nabla_{\cdot}gradf\right)-ftr\nabla^{\perp}_{\cdot}B\left(\cdot,gradf\right)-m\left\|gradf\right\|^2H-B\left(gradf,gradf\right)$$
$$=\left(mf^2\right)\left\{a\left[\left(m-1\right)H\right]+b\left[tr(k)H\right]\right\}$$

and

$$\frac{(mf)^2}{2}grad\left\|H\right\|^2+2\left(mf^2\right)tr(A_{\nabla^{\perp}_{\cdot}H}\cdot)+3\left(mf\right)A_Hgradf$$
$$+fRicci^M\left(gradf\right)+fgrad\left(\Delta f\right)+ftrA_{B(\cdot,gradf)}\left(\cdot\right)-\frac{1}{2}grad\left(\left\|gradf\right\|^2\right)$$
$$=\left(mf^2\right)\left\{a\left[tr(k)\left(Fgradf\right)+(m-2)gradf+tr(k)FH\right]\right.$$
$$\left.+b\left[(m-1)FH+m\left(Fgradf\right)+tr(k)gradf\right]\right\}.$$

2) *If FH is tangent to M^m and $Fgradf$ is normal to M^m, then M^m is bi-f-harmonic if and only if*

$$\left(mf^2\right)\left(\Delta^{\perp}H\right)+\left(mf^2\right)trB(\cdot,A_H(\cdot))-fm\left(\Delta f\right)H-\left(3mf\right)\nabla^{\perp}_{gradf}H$$
$$-ftrB\left(\cdot,\nabla_{\cdot}gradf\right)-ftr\nabla^{\perp}_{\cdot}B\left(\cdot,gradf\right)-m\left\|gradf\right\|^2H-B\left(gradf,gradf\right)$$
$$=\left(mf^2\right)\left\{a\left[\left(m-1\right)H+tr(k)\left(Fgradf\right)\right]\right.$$
$$\left.+b\left[tr(k)H+\left(m-1\right)\left(Fgradf\right)\right]\right\}$$

and

$$\frac{(mf)^2}{2}grad\left\|H\right\|^2+2\left(mf^2\right)tr(A_{\nabla^{\perp}_{\cdot}H}\cdot)+3\left(mf\right)A_Hgradf$$
$$+fRicci^M\left(gradf\right)+fgrad\left(\Delta f\right)+ftrA_{B(\cdot,gradf)}\left(\cdot\right)-\frac{1}{2}grad\left(\left\|gradf\right\|^2\right)$$
$$=\left(mf^2\right)\left\{a\left[tr(k)FH+(m-1)gradf\right]+b\left[(m-1)FH+tr(k)gradf\right]\right\}.$$

3) *If FH is normal to M^m and $Fgradf$ is tangent to M^m, then M^m is bi-f-harmonic if and only if*

$$\left(mf^2\right)\left(\Delta^{\perp}H\right)+\left(mf^2\right)trB(\cdot,A_H(\cdot))-fm\left(\Delta f\right)H-\left(3mf\right)\nabla^{\perp}_{gradf}H$$
$$-ftrB\left(\cdot,\nabla_{\cdot}gradf\right)-ftr\nabla^{\perp}_{\cdot}B\left(\cdot,gradf\right)-m\left\|gradf\right\|^2H-B\left(gradf,gradf\right)$$
$$=\left(mf^2\right)\left\{a\left[mH+tr(k)FH\right]+b\left[mFH+tr(k)H\right]\right\}$$

and

$$\frac{(mf)^2}{2}grad\left\|H\right\|^2+2\left(mf^2\right)tr(A_{\nabla^{\perp}_{\cdot}H}\cdot)+3\left(mf\right)A_Hgradf$$
$$+fRicci^M\left(gradf\right)+fgrad\left(\Delta f\right)+ftrA_{B(\cdot,gradf)}\left(\cdot\right)-\frac{1}{2}grad\left(\left\|gradf\right\|^2\right)$$
$$=\left(mf^2\right)\left\{a\left[(m-2)gradf+tr(k)\left(Fgradf\right)\right]\right.$$

$$+b\left[m\left(Fgradf\right)+tr(k)gradf\right]\right\}.$$

4) *If FH and $Fgradf$ are normal to M^m, then M^m is bi-f-harmonic if and only if*

$$\left(mf^2\right)\left(\Delta^{\perp}H\right)+\left(mf^2\right)trB(\cdot,A_H(\cdot))-fm\left(\Delta f\right)H-\left(3mf\right)\nabla^{\perp}_{gradf}H$$

$$-ftrB\left(\cdot,\nabla_{\cdot}gradf\right)-ftr\nabla^{\perp}_{\cdot}B\left(\cdot,gradf\right)-m\left\|gradf\right\|^2H-B\left(gradf,gradf\right)$$
$$=\left(mf^2\right)\left\{a\left[mH+tr(k)FH+tr(k)\left(Fgradf\right)\right]\right.$$
$$\left.+b\left[mFH+tr(k)H+(m-1)Fgradf\right]\right\}$$

and

$$\frac{(mf)^2}{2}grad\left\|H\right\|^2+2\left(mf^2\right)tr(A_{\nabla^{\perp}_{\cdot}H}\cdot)+3\left(mf\right)A_Hgradf$$

$$+fRicci^M\left(gradf\right)+fgrad\left(\Delta f\right)+ftrA_{B(\cdot,gradf)}\left(\cdot\right)-\frac{1}{2}grad\left(\left\|gradf\right\|^2\right)$$
$$=\left(mf^2\right)\left\{a\left[(m-1)gradf\right]+b\left[tr(k)gradf\right]\right\}.$$

Proof. 1) If FH and $Fgradf$ are tangent to M^m, then by the use of (2.5) we have $FH=sH$, $tH=0$, $Fgradf=kgradf$ and $hgradf=0$. So from equations (2.6), (2.7) and (2.9), we have $hsH=H$, $k^2gradf=gradf$ and $hkgradf=0$. By Theorem 4.1 we find the result.

2) If FH is tangent and $Fgradf$ is normal to M^m, then $tH=0$, $sH=FH$, $H=hsH$, $ksH=0$, $kgradf=0$ and $Fgradf=hgradf$. Hence from Theorem 4.1, we get the result.

3) If FH is normal and $Fgradf$ is tangent to M^m, then $sH=0$, $FH=tH$, $Fgradf=kgradf$, $hgradf=0$, $k^2gradf=gradf$ and $hkgradf=0$. Using Theorem 4.1, we obtain the result.

4) If FH and $Fgradf$ are normal to M^m, then $sH=0$, $FH=tH$, $kgradf=0$ and $Fgradf=hgradf$. By Theorem 4.1 we find the result. □

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References

[1] N. Course, *f-harmonic maps*, Thesis, University of Warwick, 2004.
[2] F. Dillen and D. Kowalczyk, *Constant angle surfaces in product spaces*, J. Geom. Phys., 62 (6) (2012), 1414–1432.
[3] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), 109–160.
[4] G. Y. Jiang, 2-*harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A, 7 (4) (1986), 389–402.
[5] W-J. Lu, *On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds*, Sci. China Math., 58 (2015), 1483–1498.

[6] Y-L. Ou, *On f-biharmonic maps and f-biharmonic submanifolds*, Pacific J. Math., 271 (2) (2014), 461–477.
[7] Y-L. Ou, *f-harmonic morphisms between Riemannian manifolds*, Chin. Ann. Math. Ser. B, 35 (2) (2014), 225–236.
[8] S. Ouakkas, R. Nasri and M. Djaa, *On the f-harmonic and f-biharmonic maps*, JP J. Geom. Topol., 10 (1) (2010), 11–27.
[9] J. Roth, *A note on biharmonic submanifolds of product spaces*, J. Geom., 104 (2) (2013), 375–381.
[10] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.
[11] K. Zegga, A. M. Cherif and M. Djaa, *On the f-biharmonic maps and submanifolds*, Kyungpook Math. J., 55 (1) (2015), 157–168.

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