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CERTAIN TYPES OF FUNCTIONS VIA $\beta\theta$-OPEN SETS

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ABSTRACT. New classes of functions, called somewhat $\beta\theta$-continuous, somewhat $\beta\theta$-open and hardly $\beta\theta$-open functions, has been defined and studied by making use of $\beta\theta$-open sets. Characterizations and properties of these functions are presented.

1. Introduction and Preliminaries

Is common viewpoint of many topologists that generalized open sets are important ingredients in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of β-open sets or semipreopen sets introduced by Abd El Monsef et al. [1] and Andrijević [2] respectively.

In 2003, Noiri [11] used this notion and the β-closure [1] of a set to introduce the concepts of $\beta\theta$-open and $\beta\theta$-closed sets which provide a formulation of the $\beta\theta$-closure of a set in a topological space. Caldas, jafari and Ekici [3, 4, 5, 6, 8] continued the work of Noiri and defined other concepts utilizing $\beta\theta$-closed sets.

In this paper somewhat $\beta\theta$-continuous, somewhat $\beta\theta$-open and hardly $\beta\theta$-open functions are introduced and get results which similar the results for somewhat continuous, somewhat open and hardly open functions.

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset A of a space (X,τ) is called β-open [1] if $A \subset cl(int(cl(A)))$, where $cl(A)$ and $int(A)$ denote the closure and the interior of

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A, respectively. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by $\beta cl(A)$. The β-interior $\beta int(A)$ of a subset $A \subset X$ is the union of all β-open sets contained in A. The $\beta\theta$-closure of A [11], denoted by $\beta cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $\beta cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X,\tau)$ with $x \in O$. The set $\{x \in X : \beta cl_\theta(O) \subset A \ for \ some \ O \in \beta(X,x)\}$ is called the $\beta\theta$-interior of A and is denoted by $\beta int_\theta(A)$. A subset A is said to be $\beta\theta$-closed [11] if $A = \beta cl_\theta(A)$. The complement of a $\beta\theta$-closed set is said to be $\beta\theta$-open. The family of all $\beta\theta$-open (resp. $\beta\theta$-closed) subsets of X is denoted by $\beta\theta O(X,\tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X,\tau)$). We set $\beta\theta O(X,x) = \{U : x \in U \in \beta\theta O(X,\tau)\}$ and $\beta\theta C(X,x) = \{U : x \in U \in \beta\theta C(X,\tau)\}$.

Lemma 1.1. [11] *For any subset A of X:*

(1) $\beta cl_\theta(\beta cl_\theta(A)) = \beta cl_\theta(A)$.
(2) $\beta cl_\theta(A)$ *is $\beta\theta$-closed.*
(3) *Intersection of arbitrary collection of $\beta\theta$-closed set in X is $\beta\theta$-closed.*
(4) $\beta cl_\theta(A)$ *is the intersection of all $\beta\theta$-closed sets each containing A.*

For some more information concerning $\beta\theta$-open sets and their applications we refer the interested reader to [3], [4], [5], [7], [8] and [11].

2. Somewhat $\beta\theta$-continuous functions

Definition 2.1. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be somewhat $\beta\theta$-continuous provided that if for $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$ then there is $U \in \beta\theta O(X,\tau)$ of X such that $U \neq \emptyset$ and $U \subset f^{-1}(V)$.

Definition 2.2. A function $f : (X,\tau) \to (Y,\sigma)$ is called somewhat continuous [9], if for $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$ there exists an open set U of X such that $U \neq \emptyset$ and $U \subset f^{-1}(V)$.

Example 2.3. ([9], Example 1) Let $X = \{a,b,c\}$ with topologies $\tau = \{\emptyset,\{b\},\{a,c\},X\}$ and $\sigma = \{\emptyset,\{a,b\},X\}$. Then the identity function $f : (X,\tau) \to (Y,\sigma)$ is somewhat continuous. But f is not continuous.

Example 2.4. Let $X = \{a,b,c\}$ with topologies $\tau = \{\emptyset,\{c\},\{a,b\},X\}$ and $\sigma = \{\emptyset,\{a\},X\}$. Define a function $f : (X,\tau) \to (X,\sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is somewhat $\beta\theta$-continuous. But f is not somewhat continuous.

Definition 2.5. A subset E of a topological space (X,τ) is said to be $\beta\theta$-dense in X if there is no proper $\beta\theta$-closed set C in X such that $E \subset C \subset X$.

Theorem 2.6. *For a surjective function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(i) *f is somewhat $\beta\theta$-continuous;*
(ii) *If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper $\beta\theta$-closed subset F of X such that $f^{-1}(C) \subset F$;*
(iii) *If E is an $\beta\theta$-dense subset of X, then $f(E)$ is a dense subset of Y.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y\backslash C$ is an open set in Y such that $f^{-1}(Y\backslash C) = X\backslash f^{-1}(C) \neq \emptyset$. By (i), there exists a $\beta\theta$-open set U in X such that $U \neq \emptyset$ and $U \subset f^{-1}(Y\backslash C) = X\backslash f^{-1}(C)$. This means that $f^{-1}(C) \subset X\backslash U$ and $X\backslash U = F$ is a proper $\beta\theta$-closed set in X.
(ii) $\Rightarrow$ *(i)*: Let $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$. Then $Y\backslash V$ is closed and $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V) \neq X$. By (ii), there exists a proper $\beta\theta$-closed set F of X such that $f^{-1}(Y\backslash V) \subset F$. This implies that $X\backslash F \subset f^{-1}(V)$ and $X\backslash F \in \beta\theta O(X)$ with $X\backslash F \neq \emptyset$.
(ii) $\Rightarrow$ *(iii)*: Let E be an $\beta\theta$-dense set in X. Suppose that $f(E)$ is not dense in Y. Then there exists a proper closed set C in Y such that $f(E) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. By (ii), there exists a proper $\beta\theta$-closed subset F such that $E \subset f^{-1}(C) \subset F \subset X$. This is a contradiction to the fact that E is $\beta\theta$-dense in X.
(iii) $\Rightarrow$ *(ii)*: Suppose (ii) is not true. This means that there exists a closed set C in Y such that $f^{-1}(C) \neq X$ but there is not proper $\beta\theta$-closed set F in X such that $f^{-1}(C) \subset F$. This means that $f^{-1}(C)$ is $\beta\theta$-dense in X. But by (iii), $f(f^{-1}(C)) = C$ must be dense in Y, which is a contradiction to the choice of C. □

Definition 2.7. If X is a set and τ and τ^* are topologies on X, then τ is said to be $\beta\theta$-equivalent (resp. equivalent [9]) to τ^* provided if $U \in \tau$ and $U \neq \emptyset$ then there is a $\beta\theta$-open (resp. open) set V in (X, τ^*) such that $V \neq \emptyset$ and $V \subset U$ and if $U \in \tau^*$ and $U \neq \emptyset$ then there is a $\beta\theta$-open (resp. open) set V in (X, τ) such that $V \neq \emptyset$ and $V \subset U$.

Now consider the identity function $f : (X, \tau) \to (X, \tau^*)$ and assume that τ and τ^* are $\beta\theta$-equivalent. Then $f : (X, \tau) \to (X, \tau^*)$ and $f^{-1} : (X, \tau^*) \to (X, \tau)$ are somewhat $\beta\theta$-continuous. Conversely, if the identity function $f : (X, \tau) \to (X, \tau^*)$ is somewhat $\beta\theta$-continuous in both directions, then τ and τ^* are $\beta\theta$-equivalent.

Theorem 2.8. *If $f : (X, \tau) \to (Y, \sigma)$ is a surjection somewhat $\beta\theta$-continuous and τ^* is a topology for X, which is $\beta\theta$-equivalent to τ, then $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat $\beta\theta$-continuous.*

Proof. Let V be an open subset of Y such that $f^{-1}(V) \neq \emptyset$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat $\beta\theta$-continuous, there exists a nonempty $\beta\theta$-open set U in (X, τ) such that $U \subset f^{-1}(V)$. But by hypothesis τ^* is $\beta\theta$-equivalent to τ. Therefore, there exists a $\beta\theta$-open set U^* in (X, τ^*) such that

$U^* \subset U$. But $U \subset f^{-1}(V)$. Then $U^* \subset f^{-1}(V)$; hence $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat $\beta\theta$-continuous. □

Theorem 2.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat $\beta\theta$-continuous surjection and σ^* be a topology for Y, which is equivalent to σ. Then $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat $\beta\theta$-continuous.*

Proof. Let V^* be an open subset of (Y, σ^*) such that $f^{-1}(V^*) \neq \emptyset$. Since σ^* is equivalent to σ, there exists a nonempty open set V in (Y, σ) such that $V \subset V^*$. Now $\emptyset \neq f^{-1}(V) \subset f^{-1}(V^*)$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat $\beta\theta$-continuous, there exists a nonempty $\beta\theta$-open set U in (X, τ) such that $U \subset f^{-1}(V)$. Then $U \subset f^{-1}(V^*)$; hence $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat $\beta\theta$-continuous. □

Theorem 2.10. *If $f : (X, \tau) \to (X, \sigma)$ is somewhat $\beta\theta$-continuous and $g : (X, \sigma) \to (X, \eta)$ is continuous, then $gof : (X, \tau) \to (Z, \eta)$ is somewhat $\beta\theta$-continuous.*

3. Two new forms of openness

In this section we define and characterize two new forms of openness the somewhat $\beta\theta$-open and hardly $\beta\theta$-open functions.

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat $\beta\theta$-open provided that if $U \in \tau$ and $U \neq \emptyset$, then there exists a $\beta\theta$-open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.

Example 3.2. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Clearly $\beta\theta O(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then the identity function $f = I : (X, \tau) \to (X, \tau)$ is somewhat $\beta\theta$-open.

Example 3.3. Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{c\}, \{a, b\}, X\}$. Consider the identity function $f = I : (X, \tau) \to (X, \sigma)$. Then f is somewhat $\beta\theta$-open. But f is not somewhat open.

Theorem 3.4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is somewhat $\beta\theta$-open;*
(ii) *If D is a $\beta\theta$-dense subset of Y, then $f^{-1}(D)$ is a dense subset of X.*

Proof. $(i) \Rightarrow (ii)$: Suppose D is a $\beta\theta$-dense set in Y. We want to show that $f^{-1}(D)$ is a dense subset of X. Suppose that $f^{-1}(D)$ is not dense in X. Then there exists a closed set B in X such that $f^{-1}(D) \subset B \subset X$. By (i) and since that $X \backslash B$ is open, there exists a nonempty $\beta\theta$-open subset E in Y such that $E \subset f(X \backslash B)$. Therefore $E \subset f(X \backslash B) \subset f(f^{-1}(Y \backslash D)) \subset Y \backslash D$.

That is, $D \subset Y\backslash E \subset Y$. Now, $Y\backslash E$ is a $\beta\theta$-closed set and $D \subset Y\backslash E \subset Y$. This implies that D is not a $\beta\theta$-dense set in Y, which is a contradiction. Therefore, $f^{-1}(D)$ is a dense subset of X.
$(ii) \Rightarrow (i)$: Suppose D is a nonempty open subset of X. We want to show that $\beta int_\theta(f(D)) \neq \emptyset$. Suppose $\beta int_\theta(f(D)) = \emptyset$. Then $\beta cl_\theta(Y\backslash f(D)) = Y$. Therefore, by (ii) $f^{-1}(Y\backslash f(D))$ is dense in X. But $f^{-1}(Y\backslash f(D)) \subset X\backslash D$. Now $X\backslash D$ is closed. Therefore $f^{-1}(Y\backslash f(D)) \subset X\backslash D$ gives $X = cl(f^{-1}((Y\backslash f(D)) \subset X\backslash D$. This implies that $D = \emptyset$ which is contrary to $D \neq \emptyset$. Therefore $\beta int_\theta(f(D)) \neq \emptyset$. This proves that f is somewhat $\beta\theta$-open. □

Theorem 3.5. *For a bijective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is somewhat $\beta\theta$-open;*
(ii) *If C is a closed subset of X such that $f(C) \neq Y$, then there is a $\beta\theta$-closed subset F of Y such that $F \neq Y$ and $f(C) \subset F$.*

Proof. $(i) \Rightarrow (ii)$: Let C be any closed subset of X such that $f(C) \neq Y$. Then $X\backslash C$ is an open set in X and $X\backslash C \neq \emptyset$. Since f is somewhat $\beta\theta$-open there exists a $\beta\theta$-open set V in Y such that $V \neq \emptyset$ and $V \subset f(X\backslash C)$. Put $F = Y\backslash V$. Clearly F is $\beta\theta$-closed in Y and we claim $F \neq Y$. If $F = Y$, then $V = \emptyset$ which is a contradiction. Since $V \subset f(X\backslash C)$, $f(C) = (Y\backslash f(X\backslash C)) \subset Y\backslash V = F$.
$(ii) \Rightarrow (i)$: Let U be any nonempty open subset of X. Then $C = X\backslash U$ is closed set in X and $f(X\backslash U) = f(C) = Y\backslash f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is a $\beta\theta$-closed set F of Y such that $F \neq Y$ and $f(C) \subset F$. Clearly $V = Y\backslash F \in \beta\theta O(Y, \sigma)$ and $V \neq \emptyset$. Also $V = Y\backslash F \subset Y\backslash f(C) = Y\backslash f(X\backslash U) = f(U)$. □

Theorem 3.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be somewhat $\beta\theta$-open and A be any open subset of X. Then the restriction function $f_A : (X, \tau_A) \to (Y, \sigma)$ is somewhat $\beta\theta$-open.*

Proof. Let $U \in \tau_A$ such that $U \neq \emptyset$. Since U is open in A and A is open in X, U is open in X and since by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is a somewhat $\beta\theta$-open function there exists a $\beta\theta$-open set V of Y such that $V \subset f(U)$. Thus, for any open set U of A with $U \neq \emptyset$. there exists a $\beta\theta$-open set V of Y such that $V \subset f(U)$. which implies f_A is a somewhat $\beta\theta$-open function. □

Theorem 3.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $X = A \cup B$, where $A, B \in \tau$. If the restriction function $f_A : (X, \tau_A) \to (Y, \sigma)$ and $f_B : (X, \tau_B) \to (X, \sigma)$ are somewhat $\beta\theta$-open, then f is somewhat $\beta\theta$-open.*

Proof. Let U be any open subset of X such that $U \neq \emptyset$. Since $X = A \cup B$, either $A \cap U \neq \emptyset$ or $B \cap U \neq \emptyset$ or both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. Since U

is open in X, U is open in both A and B.
Case (i): Suppose that $A \cap U \neq \emptyset$ where $A \cap U$ is open in A. Since $f_A : (X, \tau_A) \to (Y, \sigma)$ is somewhat $\beta\theta$-open, there exists a $\beta\theta$-open set V of Y such that $V \subset f(U \cap A) \subset f(U)$, which implies that f is a somewhat $\beta\theta$-open function.
Case (ii): Suppose that $B \cap U \neq \emptyset$ where $B \cap U$ is open in B. Since $f_B : (X, \tau_B) \to (Y, \sigma)$ is somewhat $\beta\theta$-open, there exists a $\beta\theta$-open set V of Y such that $V \subset f(U \cap B) \subset f(U)$, which implies that f is a somewhat $\beta\theta$-open function.
Case (iii): Suppose that both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. Then by case (i) and (ii) f is a somewhat $\beta\theta$-open function. □

Definition 3.8. A topological space (X, τ) is said to be $\beta\theta$-resolvable (resp. resolvable [10]) if there exists a $\beta\theta$-dense (resp. dense) set A in (X, τ) such that $X \backslash A$ is also $\beta\theta$-dense (resp. dense) in (X, τ). A space (X, τ) is called $\beta\theta$-irresolvable (resp. irresolvable) if it is not $\beta\theta$-resolvable (resp. resolvable).

Theorem 3.9. *For a topological space (X, τ), the following statements are equivalent:*

(i) (X, τ) *is $\beta\theta$-resolvable;*
(ii) (X, τ) *has a pair of $\beta\theta$-dense sets A and B such that $A \subset (X \backslash B)$.*

Proof. $(i) \Rightarrow (ii)$: Suppose that (X, τ) is $\beta\theta$-resolvable. There exists an $\beta\theta$-dense set A in (X, τ) such that $X \backslash A$ is $\beta\theta$-dense in (X, τ). Set $B = X \backslash A$, then we have $A = X \backslash B$.
$(ii) \Rightarrow (i)$: Suppose that the statement (ii) holds. Let (X, τ) be $\beta\theta$-irresolvable. Then $X \backslash B$ is not $\beta\theta$-dense and $\beta cl_\theta(A) \subset \beta cl_\theta(X \backslash B) \neq X$. Hence A is not $\beta\theta$-dense. This contradicts the assumption. □

Theorem 3.10. *For a topological space (X, τ), the following statements are equivalent:*

(i) (X, τ) *is $\beta\theta$-irresolvable (resp. irresolvable);*
(ii) *For any $\beta\theta$-dense (resp. dense) set A in X, $\beta int_\theta(A) \neq \emptyset$ (resp $int(A) \neq \emptyset$).*

Proof. We prove the first statement since the proof of the second is similar.
$(i) \Rightarrow (ii)$: Let A be any $\beta\theta$-dense set of X. Then we have $\beta cl_\theta(X \backslash A) \neq X$, hence $\beta int_\theta(A) \neq \emptyset$.
$(ii) \Rightarrow (i)$: Suppose that (X, τ) is a $\beta\theta$-resolvable space. There exists a $\beta\theta$-dense set A in (X, τ) such that $X \backslash A$ is also $\beta\theta$-dense in (X, τ). It follows that $\beta int_\theta(A) = \emptyset$, which is a contradiction; hence (X, τ) is $\beta\theta$-irresolvable. □

Theorem 3.11. *If $f : (X, \tau) \to (Y, \sigma)$ is a somewhat $\beta\theta$-open function and $\beta int_\theta(B) = \emptyset$ for a nonempty subset B of Y, then $int(f^{-1}(B) = \emptyset$.*

Proof. Let B be a nonempty set in Y such that $\beta int_\theta(B) = \emptyset$. Then $\beta cl_\theta(Y\backslash B) = Y$. Since f is somewhat $\beta\theta$-open and $Y\backslash B$ is $\beta\theta$-dense in Y, by Theorem 3.3, $f^{-1}(Y\backslash B)$ is dense in X. Then $cl(X\backslash f^{-1}(B)) = X$. Hence $int(f^{-1}(B)) = \emptyset$. □

Theorem 3.12. *If $f : (X, \tau) \to (Y, \sigma)$ be a somewhat β-open function. If X is irresolvable, then Y is $\beta\theta$-irresolvable.*

Proof. Let B be a nonempty set in Y such that $\beta cl_\theta(B) = Y$. We show that $\beta int_\theta(B) \neq \emptyset$. Suppose not, i.e., $\beta int_\theta(B) = \emptyset$, then $\beta cl_\theta(Y\backslash B) = Y$. Since f is somewhat $\beta\theta$-open and $Y\backslash B$ is $\beta\theta$-dense in Y, we have by Theorem 3.4, $f^{-1}(Y\backslash B)$ is dense in X. Then $int(f^{-1}(B)) = \emptyset$. Now, since B is β-dense in Y and using again Theorem 3.3 $f^{-1}(B)$ is dense in X. Therefore by Theorem 3.10 we have that $int(f^{-1}(B)) \neq \emptyset$, which is a contradiction. Hence we must have $\beta int_\theta(B) \neq \emptyset$ for all $\beta\theta$-dense sets B in Y. Hence by Theorem 3.8, Y is $\beta\theta$-irresolvable. □

Definition 3.13. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be hardly $\beta\theta$-open provided that for each $\beta\theta$-dense subset A of Y that is contained in a proper open set, $f^{-1}(A)$ is $\beta\theta$-dense in X.

Example 3.14. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the identity function $f = I : (X, \tau) \to (X, \tau)$ is hardly $\beta\theta$-open.

Example 3.15. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is hardly $\beta\theta$-open. But f is not hardly open.

We have the following characterizations of hardly $\beta\theta$-openness.

Theorem 3.16. *A function $f : (X, \tau) \to (Y, \sigma)$ is hardly $\beta\theta$-open if and only if $\beta int_\theta(f^{-1}(A)) = \emptyset$ for each set $A \subset Y$ having the property that $\beta int_\theta(A) = \emptyset$ and A containing a nonempty closed set.*

Proof. Assume f is hardly $\beta\theta$-open. Let $A \subset Y$ such that $\beta int_\theta(A) = \emptyset$ and let F be a nonempty closed set contained in A. Since $\beta int_\theta(A) = \emptyset$, $Y\backslash A$ is $\beta\theta$-dense in Y. Because $F \subset A$, $Y\backslash A \subset Y\backslash F \neq Y$. Therefore $f^{-1}(Y\backslash A)$ is $\beta\theta$-dense in X. Thus $X = \beta cl_\theta(f^{-1}(Y\backslash A)) = \beta cl_\theta(X\backslash f^{-1}(A)) = X\backslash \beta int_\theta$ $(f^{-1}(A))$ which implies that $\beta int_\theta(f^{-1}(A)) = \emptyset$.

For the converse implication assume that $\beta int_\theta(f^{-1}(A)) = \emptyset$ for every $A \subset Y$ having the property that $\beta int_\theta(A) = \emptyset$ and A contains a nonempty closed set. Let A be a $\beta\theta$-dense subset of Y, that is contained in the proper open set U. Then $\beta int_\theta(Y\backslash A) = \emptyset$ and $\emptyset \neq Y\backslash U \subset Y\backslash A$. Thus $Y\backslash A$ contains a nonempty closed set and hence $\beta int_\theta(f^{-1}(Y\backslash A)) = \emptyset$. Then $\emptyset = \beta int_\theta(f^{-1}(Y\backslash A)) = \beta int_\theta(X\backslash f^{-1}(A)) = X\backslash \beta cl_\theta(f^{-1}(A))$ and hence $f^{-1}(A)$ is $\beta\theta$-dense in X. □

Theorem 3.17. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $\beta int_\theta(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $\beta int_\theta(A) \neq \emptyset$ and there exists a nonempty closed set F for which $f^{-1}(F) \subset A$ then f is hardly $\beta\theta$-open.*

Proof. Let $B \subset U \subset Y$ where B is $\beta\theta$-dense and U is a proper open set. Let $A = f^{-1}(Y \backslash B)$ and $F = Y \backslash U$, obviously $f^{-1}(F) = f^{-1}(Y \backslash U) \subset f^{-1}(Y \backslash B) = A$. Also $\beta int_\theta(f(A)) = \beta int_\theta(f(f^{-1}(Y \backslash B))) \subset \beta int_\theta(Y \backslash B) = \emptyset$. Therefore we must have that $\emptyset = \beta int_\theta(A) = \beta int_\theta(f^{-1}(Y \backslash B)) = \beta int_\theta (X \backslash f^{-1}(B))$ which implies that $f^{-1}(B)$ is $\beta\theta$-dense. It follows that f is hardly $\beta\theta$-open. □

Theorem 3.18. *If $f : (X, \tau) \to (Y, \sigma)$ is hardly $\beta\theta$-open, then $\beta int_\theta(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $\beta int_\theta(A) \neq \emptyset$ and $f(A)$ contains a nonempty closed set.*

Proof. Let $A \subset X$ such that $\beta int_\theta(A) \neq \emptyset$ and let F be a nonempty closed set for which $F \subset f(A)$. Suppose $\beta int_\theta(f(A)) = \emptyset$. Then $Y \backslash f(A)$ is $\beta\theta$-dense in Y and $Y \backslash f(A) \subset Y \backslash F$ where $Y \backslash F$ is a proper open set. Since f is hardly $\beta\theta$-open, $f^{-1}(Y \backslash f(A))$ is $\beta\theta$-dense in X. But $f^{-1}(Y \backslash f(A)) = X \backslash f^{-1}(f(A))$ and hence $\beta int_\theta(f^{-1}(f(A))) = \emptyset$. It follows that $\beta int_\theta(A) = \emptyset$ which is a contradiction. □

Theorems 3.16 and 3.17 are reversible provided that f is surjective. Thus we have the following characterization for surjective hardly $\beta\theta$-open functions.

Theorem 3.19. *If $f : (X, \tau) \to (Y, \sigma)$ is surjective, then the following conditions are equivalent:*

(i) *f is hardly $\beta\theta$-open.*
(ii) *$\beta int_\theta(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $\beta int_\theta(A) \neq \emptyset$ and there exists a nonempty closed $F \subset Y$ such that $F \subset f(A)$.*
(iii) *$\beta int_\theta(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $\beta int_\theta(A) \neq \emptyset$ and there exists a nonempty closed set $F \subset Y$ such that $f^{-1}(F) \subset A$.*

Proof. $(i) \Rightarrow (ii)$: Theorem 3.17.
$(ii) \Rightarrow (iii)$: Since f is surjective $f^{-1}(F) \subset A$ implies that $F \subset f(A)$.
$(iii) \Rightarrow (i)$: Theorem 3.16. □

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