

## SOME PROPERTIES OF SPECIAL HYPERBOLAS IN THE ISOTROPIC PLANE

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**ABSTRACT.** In this paper we consider special hyperbolas circumscribed to the symmetral triangle of a given triangle. Considering some special cases of these hyperbolas we get equations of some special conics related to the triangles associated to a given triangle as e.g., the Jeřabek hyperbola of the tangential triangle of the given triangle and the circumscribed circle of the symmetral triangle which is also the polar circle of the given triangle. We investigate some other interesting properties of special hyperbolas circumscribed to the symmetral triangle of a given triangle.

### 1. INTRODUCTION

There are nine possible projective-metric planes, depending on whether the elliptical, parabolic or hyperbolic measurement is used to measure the distance on the line or to measure the angles in a pencil of lines. Thus, for example, in the Euclidean plane, distances on a line are measured in a parabolic way, and angles in a pencil of a line are measured in an elliptical way. The simplest of all these projective-metric planes is the parabolic-parabolic one, which is in the literature called Galilean, or most often isotropic, although in the strict sense of the word it is not actually isotropic since it has one prominent direction, in which measurement behaves differently than in all other directions. We use that name here for traditional reasons. However, a nice property of this “isotropic plane” is that the principle of duality applies thereto. Among the different properties of the Euclidean plane, some are valid and some are not valid in the isotropic plane, so it is interesting to investigate the fate of various properties of the Euclidean plane during this transition from the Euclidean to the isotropic plane. In this paper, we prove some statements about special hyperbolas which are also valid in the Euclidean plane.

Let  $P_2(\mathbb{R})$  be a real projective plane,  $\omega$  a real line in  $P_2$ , and  $A_2 = P_2 \setminus \omega$  the associated affine plane. The isotropic plane  $I_2(\mathbb{R})$  is a real affine plane  $A_2$ , where the measurement of lengths and angles is introduced with a real line  $\omega \subset P_2$  and a

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real point  $\Omega$  incidental with it. The ordered pair  $(\omega, \Omega)$  is called the absolute figure of the isotropic plane  $I_2(R)$ .

All straight lines through the point  $\Omega$  are called *isotropic lines*. Points on the line  $\omega$  are called *isotropic points*. Two points are called parallel if they are incidental with the same isotropic line.

For two non-parallel points  $P_1 = (x_1, y_1)$ ,  $P_2 = (x_2, y_2)$  the *isotropic distance* is defined by  $d(P_1, P_2) := x_2 - x_1$ . The isotropic distance is directed. For two parallel points  $P_1 = (x_1, y_1)$ ,  $P_2 = (x_1, y_2)$ , the isotropic span is defined by  $s(P_1, P_2) := y_2 - y_1$ .

The *angle* between two non-parallel lines  $p$  and  $q$  given by the equations  $y = k_p x + l_p$ ,  $y = k_q x + l_q$  is given by  $k_q - k_p$ . Note that the isotropic angle is directed.

An isotropic normal to the straight line  $p$  in the point  $P = (x_1, x_2)$ ,  $P \notin p$  is an isotropic line through  $P$ . Inversely, each straight line  $p \subset I_2$  is a normal for each isotropic straight line.

Facts about the isotropic plane can be found in [7] and [8].

We say that a triangle is *allowable* if none of its sides is isotropic. If we choose the coordinate system such that the circumscribed circle of an allowable triangle  $ABC$  has the equation  $y = x^2$  and therefore its vertices are points  $A = (a, a^2)$ ,  $B = (b, b^2)$ , and  $C = (c, c^2)$ , we say that the triangle  $ABC$  is in *standard position*. Its sides  $BC$ ,  $CA$ , and  $AB$  have equations  $y = -ax - bc$ ,  $y = -bx - ca$ , and  $y = -cx - ab$ , respectively. If  $a + b + c = 0$ , then we say that  $ABC$  is a *standard triangle*. In order to prove geometric facts for any allowable triangle, it suffices to prove them for a standard triangle [5].

Isotropic altitudes  $h_a$ ,  $h_b$ , and  $h_c$  associated with sides  $BC$ ,  $CA$ , and  $AB$  are isotropic lines passing through the vertices  $A, B$ , and  $C$ , i.e. normals to the sides  $BC$ ,  $CA$ , and  $AB$ . The points  $A_h = BC \cap h_a$ ,  $B_h = CA \cap h_b$ , and  $C_h = AB \cap h_c$  are the vertices of a triangle which is called the *orthic triangle* of a triangle  $ABC$ .

Denoting  $p = abc$  and  $q = bc + ca + ab$ , a number of useful equalities have been proved in [5], e.g.  $a^2 = bc - q$ .

The classification of conics in the isotropic plane is given in [2] and [7]. A *hyperbola* is a conic intersecting the absolute line  $\omega$  at two different real points. Any hyperbola with an isotropic line as one of its asymptotes is called a *special hyperbola*.

## 2. SOME SPECIAL HYPERBOLAS IN THE ISOTROPIC PLANE

The bisectors of angles  $A, B$ , and  $C$  of the triangle  $ABC$  with vertices  $A = (a, a^2)$ ,  $B = (b, b^2)$ , and  $C = (c, c^2)$  have the following equations  $y = \frac{a}{2}x + \frac{a^2}{2}$ ,  $y = \frac{b}{2}x + \frac{b^2}{2}$ , and  $y = \frac{c}{2}x + \frac{c^2}{2}$  [3, Theorem 1]. If the bisectors of the angles of the triangle  $ABC$  are understood as their “outer” bisectors, where their “inner” bisectors would be the isotropic lines through the points  $A, B$ , and  $C$ , then the role of the intersection of “inner” bisector has the absolute point of the plane. According to [3, Theorem 4]

the angle bisectors of the triangle  $ABC$  determine the triangle  $A_sB_sC_s$ , whose vertices are parallel to the points  $A, B$ , and  $C$  and they are the midpoints of the altitudes  $h_a, h_b$ , and  $h_c$  of the triangle  $ABC$ . The triangle  $A_sB_sC_s$  will be called the symmetrical triangle of the triangle  $ABC$ .

**Theorem 2.1.** *Every special hyperbola circumscribed to the symmetrical triangle  $A_sB_sC_s$  of a standard triangle  $ABC$  has the equation of the form*

$$Lx^2 + 2xy + 2Ly + p + qL = 0. \quad (2.1)$$

*Proof.* Every special hyperbola has the equation of the form

$$Lx^2 + 2Mxy + 2Ox + 2Py + Q = 0. \quad (2.2)$$

According to [3], we have  $A_s = (a, -\frac{1}{2}bc)$ . This point lies on the conic (2.2) under condition (2.3a)

$$a^2L - pM + 2aO - bcP + Q = 0, \quad (2.3a)$$

$$b^2L - pM + 2bO - caP + Q = 0, \quad (2.3b)$$

where condition (2.3b) is for the point  $B_s = (b, -\frac{1}{2}ca)$  on the conic (2.2). After subtracting these two equations, then dividing by  $-(a-b)$ , and because  $a+b = -c$  we get  $cL - 2O - cP = 0$ , i.e. the third of the three analogous equalities:

$$a(L - P) - 2O = 0, \quad b(L - P) - 2O = 0, \quad c(L - P) - 2O = 0.$$

These three equalities are equivalent to  $O = 0$  and  $P = L$ . Hence from (2.3a) we get

$$Q = -a^2L + pM + bcL = pM + qL,$$

which, because of the symmetry in  $a, b$ , and  $c$ , satisfies also (2.3b) and the third analogous condition for the point  $C_s = (c, -\frac{1}{2}ab)$  on the conic (2.2). The equation (2.2) turns into

$$Lx^2 + 2Mxy + 2Ly + pM + qL = 0. \quad (2.4)$$

Here  $M \neq 0$ , because otherwise (2.4) would be the equation of a circle, so we can take  $M = 1$ .  $\square$

**Corollary 2.1.** *The circumscribed circle  $\mathcal{K}_S$  of the triangle  $A_sB_sC_s$  has the equation  $y = -\frac{1}{2}x^2 - \frac{q}{2}$  (Figure 1).*

According to [1], the circle  $\mathcal{K}_S$  is the polar circle of the triangle  $ABC$ .

For  $L = 0$ , from (2.1), we get the equation  $2xy + p = 0$  of the special hyperbola circumscribed to the triangle  $A_sB_sC_s$ . By [4] it is the Jeřábek hyperbola  $J_t$  of the tangential triangle  $A_tB_tC_t$  of the triangle  $ABC$  (Figure 1).

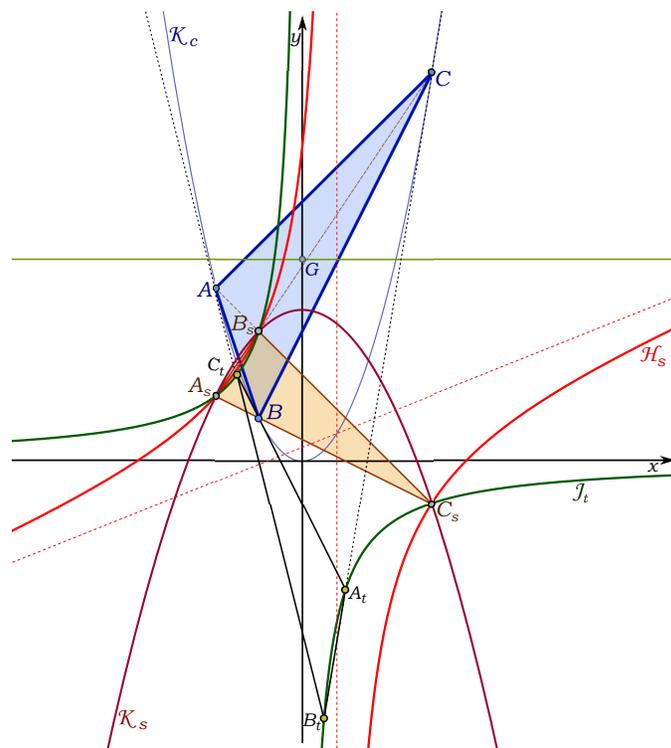


FIGURE 1. A special hyperbola  $\mathcal{H}_s$  circumscribed to  $A_s B_s C_s$ , the Jeřábek hyperbola  $\mathcal{J}_t$  of  $A_t B_t C_t$ , and the circumscribed circle  $\mathcal{K}_c$  of  $A_s B_s C_s$ .

**Theorem 2.2.** *The tangent of the special hyperbola with the equation (2.1) at its point  $T = (x, y)$  has the slope*

$$-\frac{Lx + y}{x + L}. \quad (2.5)$$

*Proof.* The point  $T = (u, v)$  has, with respect to the conic (2.1), the polar line

$$Lux + uy + xv + L(y + v) + p + qL = 0,$$

which has the slope  $-\frac{Lu + v}{u + L}$ . □

**Lemma 2.1.** *The segment joining the point  $T = (x, y)$  and its isogonal point  $T'$  with respect to a standard triangle  $ABC$  has the slope*

$$\frac{x^2 y - 2y^2 + px - qy}{x^3 + qx - p}. \quad (2.6)$$

*Proof.* By [6] we get

$$T' = \left( \frac{xy + qx - p}{y - x^2}, \frac{px - qy - y^2}{y - x^2} \right),$$

and the line  $TT'$  has the slope

$$\frac{px - qy - y^2 - y(y - x^2)}{xy + qx - p - x(y - x^2)},$$

proving (2.6). □

**Theorem 2.3.** *The tangent at an arbitrary point  $T$  of any special hyperbola circumscribed to the symmetrical triangle of an allowable triangle  $ABC$  passes through the point  $T'$  isogonal to the point  $T$  with respect to the triangle  $ABC$ .*

*Proof.* The point  $T = (x, y)$  of the special hyperbola  $\mathcal{H}_s$  satisfies (2.1). The statement of the theorem is true if the slopes given by (2.5) and (2.6) are equal. However, this equality, written in the form

$$(x^2y - 2y^2 + px - qy)(x + L) + (x^3 + qx - p)(Lx + y) = 0,$$

after rearrangement, can be written as

$$(x^2 - y)(Lx^2 + 2xy + 2Ly + p + qL) = 0,$$

and it is valid because of (2.1). □

In [3] it is shown that the triangle  $ABC$  is the orthic triangle of the triangle  $A_sB_sC_s$ . Hence, applying Theorem 2.3 to the orthic triangle  $A_hB_hC_h$  of the triangle  $ABC$  we get:

**Corollary 2.2.** *The tangent at an arbitrary point  $T$  of any special hyperbola circumscribed to an allowable triangle  $ABC$ , passes through the point  $T'$  isogonal to the point  $T$  with respect to the orthic triangle of the triangle  $ABC$ .*

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