SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.18.02.05
Vol.18 (31), No.2 (2022), 239 – 256

THE IMPACT OF THE PROPERTIES OF THE STIFFNESS MATRIX ON DEFINITE QUADRATIC EIGENVALUE PROBLEMS

ALEKSANDRA KOSTIĆ, HEINRICH VOSS, AND VALENTINA TIMOTIĆ

ABSTRACT. Waiving the positive definiteness of the leading matrix $\mathbf{A}$ in a hyperbolic quadratic eigenvalue problem $\mathbf{Q}(\lambda)\mathbf{x} = (\lambda^2\mathbf{A}+\lambda\mathbf{B}+\mathbf{C})\mathbf{x} = \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ one obtains a definite eigenvalue problem, which is known to have $2n$ eigenvalues in $\mathbb{R}\cup\{\infty\}$. One of the characterizations of the definite quadratic eigenvalue problem is the existence of parameters ξ and μ so that $\mathbf{Q}(\mu)$ is positive definite and $\mathbf{Q}(\xi)$ is negative definite, where ξ and μ are not known in advance. In this paper we consider the impact of the properties of the stiffness matrix $\mathbf{C}$ of the quadratic pencil $\mathbf{Q}(\lambda)$ on the corresponding definite quadratic eigenvalue problem and on the localization of the parameters ξ and μ.

1. INTRODUCTION

In this paper we consider the quadratic eigenvalue problem which is a special case of the nonlinear eigenvalue problem

$$\mathbf{T}(\lambda)\mathbf{x} = 0, \mathbf{x} \neq \mathbf{0} \tag{1.1}$$

where $\mathbf{T}(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, is a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, and J is a real open interval which may be unbounded.

We will introduce a notation for positive and negative definite matrices as follows. If the matrix $\mathbf{T}(\lambda)$ is positive definite we write $\mathbf{T}(\lambda) > 0$, and if $\mathbf{T}(\lambda)$ is negative definite we write $\mathbf{T}(\lambda) < 0$.

Nonlinear eigenproblems usually occur in the dynamic stability analysis of structures, fluid mechanics, vibration of fluid-solid structures, electronic behavior of quantum dots, and nonlinear integrated optics, e.g. Essentially, there are two well-known tools in literature for solving the nonlinear eigenvalue problems: linearization (for polynomial or rational eigenvalue problems) and methods based on variational characterization. Details on linearization (including structure preservation) are discussed in [3], [9], [15], [16]. More information on variational characterization of nonlinear eigenvalue problems can be found in [25] and [23]. A review of the theory of eigenvalue problems is contained in [11], [22] [24].

2010 *Mathematics Subject Classification.* 15A18, 65F15.
Key words and phrases. definite quadratic eigenvalue problem, eigenvalue, localization of parameters ξ and μ, quadratic pencil, Sylvester's law of inertia.

In this paper we consider the quadratic eigenvalue problem (QEP) defined as $\mathbf{Q}(\lambda)\mathbf{x} = 0$, $\mathbf{x} \neq \mathbf{0}$ where

$$\begin{aligned} &\mathbf{Q}(\lambda) = \mathbf{A}\lambda^2 + \mathbf{B}\lambda + \mathbf{C},\ \mathbf{A},\mathbf{B},\mathbf{C} \in \mathbb{C}^{n\times n},\ \mathbf{A} \neq \mathbf{0}, \\ &\mathbf{A} = \mathbf{A}^H,\ \mathbf{B} = \mathbf{B}^H,\ \mathbf{C} = \mathbf{C}^H. \end{aligned} \tag{1.2}$$

In particular for hyperbolic and definite quadratic problems (see [19]), which are significant examples of overdamped quadratic problems, we investigate properties of the mass matrix $\mathbf{C}$ which preserve this property.

We can summarize the results that have been published so far. Mackey et al. in [17] gave an approach to constructing linearizations of polynomial eigenvalue problems which generalize the companion forms. Higham, Tisseur and Van Dooren in [7] proved that Hermitian matrix polinomials

$$P(\lambda) = \sum_{j=0}^{l} \lambda^j \mathbf{A}_j,\ \mathbf{A}_j \in \mathbb{C}^{n\times n},\ \mathbf{A}_l \neq \mathbf{0}$$

that allow a definite linearization, are characterized with the property that there exists $\mu \in \mathbb{R} \cup \{\infty\}$ so that $P(\mu)$ is positive definite and for each $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq 0$ the scalar polynomial $f(\lambda;\mathbf{x}) = \mathbf{x}^H P(\lambda)\mathbf{x}$ has l distinct roots in $\mathbb{R} \cup \{\infty\}$. These Hermitian matrix polynomials are called definite. They proved a method for testing a quadratic eigenvalue problem for hyperbolicity and for constructing a definite linearization for quadratic hyperbolic pencils.

Another method for detecting whether a Hermitian quadratic matrix polynomial is hyperbolic is based on cyclic reduction and was first introduced by Guo and Lancaster in [6] and later on, accelerated by Guo, Higham and Tisseur in [5].

Niendorf and Voss in [19] concurrently determined parameters ξ and μ so that the matrices $\mathbf{Q}(\xi) < 0$ and $\mathbf{Q}(\mu) > 0$, which allows for a computation of a definite linearization. They used a property that all eigenvalues of a definite matrix polynomial can be characterized as minmax values of an appropriate Rayleigh functional and that the extreme eigenvalues in each of the intervals $(-\infty,\xi)$, (ξ,μ) and $(\mu,+\infty)$ are the limits of monotonically and quadratically convergent sequences.

Kostić and Voss in [12] researched eigenvalue problems and applied the Sylvester's law of inertia on the quadratic eigenvalue problem, in particular on the definite quadratic eigenvalue problems.

Kostić and Šikalo in [13] have already studied some properties of Hermitian matrices in quadratic pencils. They have obtained an improvement of the algorithm given by Niendorf and Voss in [19], by improving the process of determining the initial vector. Further improvement of the algorithm in [19] is given in [14] in the sense of determining the initial vector.

This paper is organized as follows. In Section 2, we state Sylvester's law of inertia for linear and nonlinear eigenvalue problems. We give basic definitions and properties of the hyperbolic quadratic pencil and definite quadratic pencil in Section 3. In the following Section 4, we demonstrate the impact of a matrix $\mathbf{C}$ on the definite quadratic eigenvalue problem. In Section 5 we give numerical experiments. The final Section 6 gives a review of the results obtained in the previous sections, and directions for further research.

2. Sylvester's law for nonlinear eigenvalue problems

Sylvester's law of inertia for the linear eigenvalue problem is a strong tool for locating eigenvalues of a symmetric matrix. It was first proved in 1858 by J.J. Sylvester in [21], and several different proofs can be found in textbooks [1], [4], [10], [18], [20], one of which is based on the minmax characterization of eigenvalues of Hermitian matrices.

The inertia of a Hermitian matrix $\mathbf{A}$ is the triplet of non-negative integers $In(\mathbf{A}) := (n_{p_\mathbf{A}}, n_{n_\mathbf{A}}, n_{z_\mathbf{A}})$ where $n_{p_\mathbf{A}}$, $n_{n_\mathbf{A}}$ and $n_{z_\mathbf{A}}$ are the number of positive, negative and zero eigenvalues of $\mathbf{A}$ (counting multiplicities). Sylvester's law for the linear eigenvalue problem states:

Theorem 2.1. [4] *Two Hermitian matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ are congruent (i.e., $\mathbf{A} = \mathbf{S}^H\mathbf{B}\mathbf{S}$ for some nonsingular matrix $\mathbf{S}$) if and only if they have the same inertia $In(\mathbf{A}) = In(\mathbf{B})$.*

An obvious consequence of the law of inertia is the following corollary: If $\mathbf{A}$ has an LDL^H factorization $\mathbf{A} = \mathbf{LDL}^H$, where $\mathbf{D}$ is diagonal matrix, then n_p and n_n equals the number of positive and negative entries of $\mathbf{D}$, respectively, and if only a block LDL^H factorization exists where $\mathbf{D}$ is a block diagonal matrix with 1×1 and indefinite 2×2 blocks on its diagonal, then one has to increase the number of positive and negative 1×1 blocks of $\mathbf{D}$ by the number of 2×2 blocks to get n_p and n_n, respectively. Hence, the inertia of A can be computed easily.

For the general linear eigenvalue problem

$$\mathbf{Ax} = \lambda\mathbf{Bx},\ \mathbf{x} \neq \mathbf{0}, \tag{2.1}$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$, Sylvester's law of inertia has the following form:

The following theorem is a very important for characterization of eigenvalues. Although the proof is simple and known, we prove it here for convenience.

Theorem 2.2. *If $\mathbf{B} \in \mathbb{C}^{n\times n}$ is positive definite, and $\mathbf{A} - \sigma\mathbf{B} = \mathbf{LDL}^H$ is the block diagonal LDL^H factorization of $\mathbf{A} - \sigma\mathbf{B}$ for some $\sigma \in \mathbb{R}$, from which we get the inertia $In(\mathbf{A} - \sigma\mathbf{B}) = (n_p, n_n, n_z)$ as described in the last paragraph, then the generalized eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{Bx}$ has n_n eigenvalues smaller than σ.*

Proof. Let $\chi_{A-\sigma B}(\lambda)$ and $\chi(\lambda)$ be the characteristic polynomials of the matrices $A - \sigma B$ and A, respectively. Then

$$\chi_{A-\sigma B}(\lambda) = det(A - \sigma B - \lambda B) = det(A - (\sigma + \lambda)B) = \chi_A(\sigma + \lambda), \tag{2.2}$$

holds from which we immediately obtain the proof. □

Hence, the law of inertia yields a tool to locate the eigenvalues of a Hermitian matrix or a definite matrix pencil. Combining it with bisection or the secant method one can determine all eigenvalues in a given interval or determine initial approximations for fast eigensolvers, and it can be used to check whether a method has found all eigenvalues in an interval of interest or not.

In [12] Sylvester's law was generalized to nonlinear eigenvalue problems $\mathbf{T}(\lambda)\mathbf{x} = \mathbf{0}$ allowing for a minmax characterization of its real eigenvalues, i.e. problem (1.1) satisfying the conditions (A_1) and (A_2).

(A_1) for each fixed $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, the scalar real equation $f(\lambda;\mathbf{x}) := \mathbf{x}^H\mathbf{T}(\lambda)\mathbf{x} = 0$ has at most one solution $p(\mathbf{x}) := \lambda \in J$.

(A_2) for each $\mathbf{x} \in D \subseteq \mathbb{C}^n \setminus \{0\}$, and each $\lambda \in J$ with $p(\mathbf{x}) \neq \lambda$, $(\lambda - p(\mathbf{x}))f(\lambda;\mathbf{x}) > 0$ holds, where $D \subseteq \mathbb{C}^n \setminus \{0\}$ is the set on which the functional $p(\mathbf{x})$ is defined.

For general eigenproblems the natural ordering naming the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate. If $\lambda \in J$ is an eigenvalue of $\mathbf{T}(\cdot)$, then $\mu = 0$ is an eigenvalue of the linear problem $\mathbf{T}(\lambda)\mathbf{y} = \mu\mathbf{y}$, and therefore there exists $\ell \in \mathbb{N}$ such that

$$0 = \max_{V \in H_\ell} \min_{v \in V \setminus \{0\}} \frac{v^H\mathbf{T}(\lambda)v}{\|v\|^2}$$

where H_ℓ denotes the set of all ℓ–dimensional subspaces of $\mathbb{C}^n$. In this case it is appropriate to call λ an ℓth eigenvalue of $\mathbf{T}(\cdot)$.

With this enumeration the following minmax characterization for eigenvalues was proved in [23, 25].

Theorem 2.3. *Let J be an open interval in $\mathbb{R}$, and let $\mathbf{T}(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$ such that the conditions (A_1) and (A_2) are satisfied. Then the following minmax characterization holds:*

For every $\ell \in \mathbb{N}$ there is at most one ℓth eigenvalue of $\mathbf{T}(\cdot)$ which can be characterized by

$$\lambda_\ell = \min_{V \in H_\ell,\ V \cap \mathcal{D} \neq \emptyset} \ \sup_{v \in V \cap \mathcal{D}} p(v). \tag{2.3}$$

Sylvester's law obtains the following form for the general case of a nonlinear eigenvalue problem:

Theorem 2.4. *Let $T : J \to \mathbb{C}^{n\times n}$ satisfy the conditions (A_1) and (A_2) of the minmax characterization and let $\sigma, \tau \in J$, $\sigma < \tau$. Let $(n_{p_\sigma}, n_{n_\sigma}, n_{z_\sigma})$ and $(n_{p_\tau}, n_{n_\tau}, n_{z_\tau})$ be the inertia of $\mathbf{T}(\sigma)$ and $\mathbf{T}(\tau)$, respectively. Then the inequality $n_{p_\sigma} \leq n_{p_\tau}$ holds and the eigenvalue problem (1.1) has exactly $n_{p_\tau} - n_{p_\sigma}$ eigenvalues $\lambda_{n_{p_\sigma}+1} \leq, \dots, \leq \lambda_{n_{p_\tau}}$ in (σ, τ).*

More information about application of Sylvester's law on the nonlinear eigenvalue problem can be found in [12].

3. Hyperbolic and definite pencil properties

In this paragraph we will collect well-known properties of hyperbolic and definite pencils, as well as applications of Sylvester's law of inertia for this pencil.

3.1. A. Hyperbolic pencils

A quadratic matrix pencil [2]

$$\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C},\ \mathbf{A} = \mathbf{A}^H > 0,\ \mathbf{B} = \mathbf{B}^H,\ \mathbf{C} = \mathbf{C}^H \tag{3.1}$$

is hyperbolic if for every $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq 0$ the quadratic polynomial

$$f(\lambda;\mathbf{x}) := \lambda^2\mathbf{x}^H\mathbf{A}\mathbf{x} + \lambda\mathbf{x}^H\mathbf{B}\mathbf{x} + \mathbf{x}^H\mathbf{C}\mathbf{x} = 0 \tag{3.2}$$

has two distinct real roots

$$p_+(\mathbf{x}) := -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} + \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}}, \tag{3.3}$$

$$p_-(\mathbf{x}) := -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} - \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}}. \tag{3.4}$$

The functionals (3.3) and (3.4) are called Rayleigh functionals of the quadratic matrix polynomial (3.2). They are generalizations of the Rayleigh quotient for linear eigenproblems.

Remark 3.1. If $\mathbf{C} = \mathbf{C}^H < 0$, then the quadratic matrix pencil (3.1) is hyperbolic, since

$$\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}} > 0,$$

holds and therefore equation (3.2) has two real solutions for each $x \neq 0$.

If the quadratic pencil (3.1) is not hyperbolic, then with the greatest eigenvalue λ_{n_C} of the matrix $\mathbf{C}$ we easily obtain a hyperbolic pencil

$$\mathbf{Q}(\lambda) = \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C} - \lambda_{n_C}\mathbf{I}.$$

The ranges $J_+ := p_+(\mathbb{C}^n \setminus \{\mathbf{0}\})$ and $J_- := p_-(\mathbb{C}^n \setminus \{\mathbf{0}\})$ are disjoint real intervals with $\max J_- < \min J_+$. $\mathbf{Q}(\lambda)$ is positive definite for $\lambda < \min J_-$ and $\lambda > \max J_+$, and it is negative definite for $\lambda \in (\max J_-, \min J_+)$.

(Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of eigenvalues [2], i.e. there exist $2n$ eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots \leq \lambda_{2n} \tag{3.5}$$

and

$$\lambda_j = \min_{\dim V = j} \max_{x \in V, x \neq 0} p_-(\mathbf{x}),\ \lambda_{n+j} = \min_{\dim V = j} \max_{x \in V, x \neq 0} p_+(\mathbf{x}),\ j = 1, 2, \ldots, n. \tag{3.6}$$

We now discuss the consequences of Sylvester's law of inertia for hyperbolic quadratic pencils.

Let $In(\mathbf{Q}(\sigma)) = (n_{p_\sigma}, n_{n_\sigma}, n_{z_\sigma})$ denote the inertia of $\mathbf{Q}(\sigma)$. If $n_{n_\sigma} = n$, then $\mathbf{Q}(\sigma)$ is negative definite and there are n eigenvalues which are smaller than σ, and n eigenvalues which are greater than σ. Otherwise, if $n_{p_\sigma} = n$, then $\mathbf{Q}(\sigma)$ is positive definite, and $f(\sigma;\mathbf{x}) > 0$ for each $\mathbf{x} \neq 0$.

If $n_{p_\sigma} \neq n$ and $n_{n_\sigma} \neq n$, then $\sigma \in J_- \cup J_+$ and Theorem 2.4 holds. We have to determine where σ is located, which means that we need to determine $\mathbf{x} \neq 0$ so that $f(\sigma;\mathbf{x}) = \mathbf{x}^H\mathbf{Q}(\sigma)\mathbf{x} > 0$. In the case of $\frac{\partial f}{\partial \lambda}(\sigma;\mathbf{x}) = 2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$ it follows that $p_-(\mathbf{x}) > \sigma$, therefore $\sigma < \lambda_n = \max_{x\neq 0} p_-(x)$. If we have inequalities $f(\sigma;\mathbf{x}) > 0$ and $\frac{\partial f}{\partial \lambda}(\sigma;\mathbf{x}) > 0$, it follows that $\sigma > \lambda_{n+1} = \min_{x\neq 0} p_+(x)$.

Kostić and Voss in [12] proved the following theorem on the location of eigenvalues of problem (3.1):

Theorem 3.1. *Let $\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}$ be hyperbolic, and let $(n_{p_\sigma}, n_{n_\sigma}, n_{z_\sigma})$ be the inertia of $\mathbf{Q}(\sigma)$ for $\sigma \in \mathbb{R}$.*

(1) *If $n_{n_\sigma} = n$, then there are n eigenvalues smaller than σ and n eigenvalues greater than σ.*
(2) *Let $n_{p_\sigma} = n$. If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$ for an arbitrary $\mathbf{x} \neq 0$, then there are $2n$ eigenvalues exceeding σ. If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$ for an arbitrary $\mathbf{x} \neq 0$, then all $2n$ eigenvalues are less than σ.*
(3) *For $n_{p_\sigma} = 0$ and $n_{z_\sigma} > 0$, let $x \neq 0$ be an element of the null space of $\mathbf{Q}(\sigma)$. If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$, then $\mathbf{Q}(\lambda)\mathbf{x} = 0$ has $n - n_{z_\sigma}$ eigenvalues in $(-\infty,\sigma)$ and n eigenvalues in (σ,∞), and $\sigma = \lambda_n$ with multiplicity n_{z_σ}. If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$, then $\mathbf{Q}(\cdot)$ has n eigenvalues in $(-\infty,\sigma)$ and $n - n_{z_\sigma}$ eigenvalues in (σ,∞), and $\sigma = \lambda_{n+1}$ with multiplicity n_{z_σ}.*
(4) *For $n_{p_\sigma} > 0$ and $n_{z_\sigma} = 0$ let $\mathbf{x} \neq 0$ be so that $f(\sigma;\mathbf{x}) > 0$. If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$, then $\mathbf{Q}(\cdot)$ has $n - n_{p_\sigma}$ eigenvalues in $(-\infty,\sigma)$ and $n + n_{p_\sigma}$ eigenvalues in (σ,∞). If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$, then $\mathbf{Q}(\cdot)$ has $n + n_{p_\sigma}$ eigenvalues in $(-\infty,\sigma)$ and $n - n_{p_\sigma}$ eigenvalues in (σ,∞).*
(5) *For $n_{p_\sigma} > 0$ and $n_{z_\sigma} > 0$ let $\mathbf{x} \neq 0$ be so that $f(\sigma;\mathbf{x}) > 0$. If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$, then $\mathbf{Q}(\cdot)$ has $n - n_{p_\sigma} - n_{z_\sigma}$ eigenvalues in $(-\infty,\sigma)$ and $n + n_{p_\sigma}$ eigenvalues in (σ,∞). If $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$, then $\mathbf{Q}(\cdot)$ has $n + n_{p_\sigma}$ eigenvalues in $(-\infty,\sigma)$ and $n - n_{p_\sigma} - n_{z_\sigma}$ eigenvalues in (σ,∞). In either case σ is an eigenvalue with multiplicity n_{z_σ}.*

3.2. **B. Definite Quadratic Pencils**

Higham, Mackey and Tisseur in [8] generalized the concept of hyperbolic matrix pencils to definite matrix pencils by waiving the positive definiteness of the leading matrix $\mathbf{A}$. They proved the following theorem:

Theorem 3.2. *The Hermitian matrix pencil $\mathbf{Q}(\lambda)$ is definite if and only if any two (and hence all) of the following properties hold:*

(1) *$d(\mathbf{x}) := (\mathbf{x}^H\mathbf{B}\mathbf{x})^2 - 4(\mathbf{x}^H\mathbf{A}\mathbf{x})(\mathbf{x}^H\mathbf{C}\mathbf{x}) > 0$ for each $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq 0$,*

(2) $\mathbf{Q}(\mu) > 0$ *for some* $\mu \in \mathbb{R} \cup \{\infty\}$,
(3) $\mathbf{Q}(\xi) < 0$ *for some* $\xi \in \mathbb{R} \cup \{\infty\}$.

Without loss of generality, we can assume that $\xi < \mu$ *holds.*

The following theorem contains a result about the location of the eigenvalues relative to a given parameter ξ:

Theorem 3.3. *[12] Let* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{n\times n}$ *be positive semidefinite and assume that* $d(\mathbf{x}) > 0$ *for* $\mathbf{x} \neq 0$. *Let* r *be the rank of* $\mathbf{A}$, *and* $In(\mathbf{Q}(\sigma) = (n_{p_\sigma}, n_{n_\sigma}, n_{z_\sigma})$ *be the inertia of* $\mathbf{Q}(\sigma)$ *for* $\sigma \in \mathbb{R}$. *Then the following holds:*

(1) If $n_{n_\sigma} = n$, *then there are* r *eigenvalues smaller than* σ *and* n *eigenvalues greater than* σ.

(2) For $n_{p_\sigma} = 0$ *and* $n_{z_\sigma} > 0$, *let* $\mathbf{x} \neq 0$ *be an element of the null space* $\mathbf{Q}(\sigma)$. *If* $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$, *then* $\mathbf{Q}(\cdot)$ *has* $r - n_{z_\sigma}$ *eigenvalues in* $(-\infty, \sigma)$ *and* n *eigenvalues in*$(\sigma, 0]$. *If* $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$, *then* $\mathbf{Q}(\cdot)$ *has* r *eigenvalues in* $(-\infty, \sigma)$ *and* $n - n_{p_\sigma}$ *eigenvalues in* $(\sigma, 0]$. *In either case* σ *is an eigenvalue of* $\mathbf{Q}(\cdot)$ *with multiplicity* n_z.

(3) For $n_{p_\sigma} > 0$ *let* $\mathbf{x} \neq 0$ *be so that* $f(\sigma;\mathbf{x}) > 0$. *If* $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} < 0$, *then* $\mathbf{Q}(\lambda)\mathbf{x} = 0$ *has* $r - n_{p_\sigma}$ *eigenvalues in* $(-\infty, \sigma)$ *and* $n - n_{p_\sigma} - n_{z_\sigma}$ *eigenvalues in* $(\sigma, 0]$. *If* $2\sigma\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$, *then* $\mathbf{Q}(\lambda)\mathbf{x} = 0$ *has* $r + n_{p_\sigma} - n_{z_\sigma}$ *eigenvalues in* $(-\infty, \sigma)$ *and* $n - n_{p_\sigma}$ *eigenvalues in* $(\sigma, 0]$.

Figure 1 shows the graphics of $f_1(\lambda, \mathbf{x})$ for different vectors $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq 0$, and their position related to previously defined parameters ξ, μ. As the matrix $\mathbf{A}$ does not have to be positive definite, the parabola opens up when $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$ holds, opens down when $\mathbf{x}^H\mathbf{A}\mathbf{x} < 0$ holds, and the parabola becomes a straight line if $\mathbf{x}^H\mathbf{A}\mathbf{x} = 0$ holds. We can see from Figure 1 that for each vector $\mathbf{x} \in \mathbb{C}^n$, $f(\xi, \mathbf{x}) = \xi^2\mathbf{x}^H\mathbf{A}\mathbf{x} + \xi\mathbf{x}^H\mathbf{B}\mathbf{x} + \mathbf{x}^H\mathbf{C}\mathbf{x} < 0$ and $f(\mu, \mathbf{x}) = \mu^2\mathbf{x}^H\mathbf{A}\mathbf{x} + \mu\mathbf{x}^H\mathbf{B}\mathbf{x} + \mathbf{x}^H\mathbf{C}\mathbf{x} > 0$ holds.

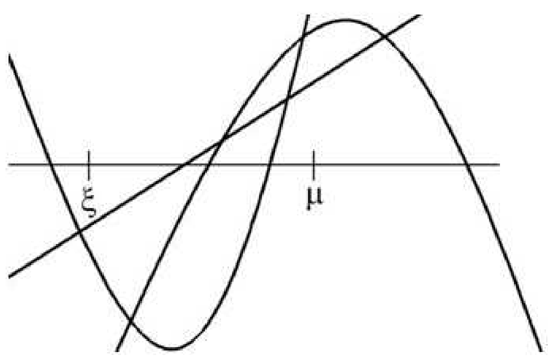

FIGURE 1. Definite quadratic matrix polinomial $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$, $\xi < \mu$.

An explicit form of a Rayleigh functional of the definite eigenvalue problem (3.1) is given by

$$p(\mathbf{x}) = \begin{cases} -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} + \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}} & \text{if } \mathbf{x}^H\mathbf{A}\mathbf{x} > 0 \\ -\frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{B}\mathbf{x}} & \text{if } \mathbf{x}^H\mathbf{A}\mathbf{x} = 0 \\ -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} - \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}} & \text{if } \mathbf{x}^H\mathbf{A}\mathbf{x} < 0. \end{cases}$$

Then the range (ξ,μ) of p contains n eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ which are all eigenvalues of positive type, i.e. $\frac{\partial f}{\partial \lambda}(\lambda,\mathbf{x}) > 0$, and which are minmax values of the functional p. n eigenvalues of negative type are contained in the intervals $[-\infty,\xi)$ and $(\mu,\infty]$. Also, in general the location of the interval (ξ,μ) is not known a priori. Papers [5] (based on cyclic reduction), [14] (based on the better determining of the initial vector, for which we used properties of the matrix **A** and the matrix **B** from the quadratic pencil), and [19] (based on safeguarded iteration) contain methods for detecting whether a quadratic pencil is definite or not.

4. Results

In this section we consider the definite quadratic matrix pencil

$$\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C},\ \mathbf{A} = \mathbf{A}^H,\ \mathbf{B} = \mathbf{B}^H,\ \mathbf{C} = \mathbf{C}^H, \tag{4.1}$$

and appropriate eigenproblem

$$\mathbf{Q}(\lambda)\mathbf{x} = 0,\ \mathbf{x} \neq 0. \tag{4.2}$$

In particular we will study properties of the matrix **C**.

We assume that there exist real parameters μ and ξ such that $\mathbf{Q}(\mu) > 0 > \mathbf{Q}(\xi)$, and without restriction we suppose that $\mu > \xi$ for otherwise we replace $Q(\lambda)$ with the quadratic pencil $Q(-\lambda)$.

Lemma 4.1. $\lambda = 0$ *is an eigenvalue of problem* (4.2) *if and only if the matrix* **C** *is singular, and* $\mathbf{x} \neq \mathbf{0}$ *is an eigenvector corresponding to* $\lambda = 0$ *if and only if* $\mathbf{C}\mathbf{x} = \mathbf{0}$.

The proof is obvious (see also [14]).

Theorem 4.1. *Let* $(\lambda,\mathbf{x})$ *be an eigenpair of the definite quadratic eigenvalue problem* $\mathbf{Q}(\lambda)\mathbf{x} = 0$.

(i) *If the matrix* **C** *is singular and* $\mathbf{C}\mathbf{x} = 0$*, then* $\mathbf{x}^H\mathbf{B}\mathbf{x} \neq 0$ *holds.*
(ii) *If the matrix* **A** *is singular and* $\mathbf{A}\mathbf{x} = 0$*, then* $\mathbf{x}^H\mathbf{B}\mathbf{x} \neq 0$ *holds.*

Proof. (i): Suppose that $\mathbf{x}^H\mathbf{B}\mathbf{x} = 0$ holds. From $\mathbf{C}\mathbf{x} = 0$ it follows $\mathbf{x}^H\mathbf{C}\mathbf{x} = 0$, and therefore $d(\mathbf{x}) = (\mathbf{x}^H\mathbf{B}\mathbf{x})^2 - 4(\mathbf{x}^H\mathbf{A}\mathbf{x})(\mathbf{x}^H\mathbf{C}\mathbf{x}) = 0$, contradicting the definiteness of the pencil $Q(\cdot)$.

The proof of (ii) is analogous. □

Lemma 4.2. *If the matrix* $\mathbf{C}$ *is singular, then* $\lambda = 0$ *is an eigenvalue of the definite quadratic pencil* (4.1) *of positive type, if and only if* $\mathbf{x}^H\mathbf{B}\mathbf{x} > 0$ *holds for each* $\mathbf{x} \neq \mathbf{0}$ *with* $\mathbf{C}\mathbf{x} = \mathbf{0}$.

Proof. The eigenvalue $\lambda = 0$ is of positive type if and only if $\frac{\partial f}{\partial \lambda}(\lambda;\mathbf{x}) = 2\lambda\mathbf{x}^H\mathbf{A}\mathbf{x} + \mathbf{x}^H\mathbf{B}\mathbf{x} > 0$ holds, and in particular we get $\mathbf{x}^H\mathbf{B}\mathbf{x} > 0$. □

Remark 4.1. Without loss of generality, we now assume that if the matrix $\mathbf{C}$ is singular, then 0 is an eigenvalue of positive type, for otherwise we can replace $\mathbf{Q}(\lambda)$ with

$$-\mathbf{Q}(\lambda) := -\lambda^2\mathbf{A} - \lambda\mathbf{B} - \mathbf{C},\ -\mathbf{A} = -\mathbf{A}^H,\ -\mathbf{B} = -\mathbf{B}^H,\ -\mathbf{C} = -\mathbf{C}^H$$

which has the same eigenvalues and eigenvectors as the initial definite problem, except now the type of eigenvalue changes.

Let us observe the case when $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$. Then $-\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} < 0$ holds, according to the definition of functional p, $p(\mathbf{x}) = p_+(\mathbf{x}) = 0$. If $\mathbf{x}^H\mathbf{A}\mathbf{x} < 0$ holds, then $-\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} > 0$, therefore $p(\mathbf{x}) = p_-(\mathbf{x}) = 0$.

Theorem 4.2. *Let* $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$*, and* $\xi < \mu$ *hold. If* $\mathbf{C}$ *is singular, then* $\mu > 0 > \xi$.

Proof. According to Remark 4.1 the eigenvalue 0 of (4.2) is of positive type. Let $\mathbf{x} \neq 0$ with $\mathbf{C}\mathbf{x} = 0$.

If $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$ holds, then $\mathbf{x}^H\mathbf{Q}\mathbf{x} < 0$ for $\lambda \in \left(-\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}, 0\right)$, and $\mu > 0 > \xi$, and if $\mathbf{x}^H\mathbf{A}\mathbf{x} < 0$ holds, then $\mathbf{x}^H\mathbf{Q}(\lambda)\mathbf{x} > 0$ for $\lambda \in \left(0, -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)$ and $\mu > 0 > \xi$.

□

Kostić and Šikalo proved the following result in [13]:

Theorem 4.3. *Let* $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$*, and* $\xi < \mu$*. Let Rank* $(\mathbf{A}) = n - p$, $0 < p < n$, *, and* $\{\mathbf{y_1}, \mathbf{y_2}, \cdots, \mathbf{y_p}\}$ *be an orthonormal basis of the null space of* $\mathbf{A}$.

Then $\mu > a_1 > c_1 > \xi$ *where*

$$a_1 := \max\left\{-\frac{\mathbf{y^H C y}}{\mathbf{y^H B y}}\ :\ \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}\right\}$$

and

$$c_1 := \min\left\{-\frac{\mathbf{y^H C y}}{\mathbf{y^H B y}}\ :\ \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y_2}, \cdots, \mathbf{y}_p\}\right\}.$$

As a corollary of the Theorem 4.2 and the Theorem 4.3 we obtain the next theorem, which gives us a better localization of the parameters ξ and μ.

Theorem 4.4. *Let the matrix* $\mathbf{C}$ *be singular, and Rank* $\mathbf{C} = n - r$*, and let* $\mathbf{x}_i$ $(i = 1, 2, \ldots, r)$ *be eigenvectors of the matrix* $\mathbf{C}$ *corresponding to eigenvalue* 0.

Let $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$*,* $\xi < \mu$*, and let* a_1*,* c_1 *be defined as in the Theorem 4.3. Then* $\mu \in (a,b)$*, and* $\xi \in (c,b)$ *hold, where*

$$a = \max(0, a_1), b = \min\left\{ -\frac{\mathbf{x_i^H B x_i}}{\mathbf{2x_i^H A x_i}} \ : \ \mathbf{x_i^H A x_i} < 0 \right\},$$
$$c = \max\left\{ -\frac{\mathbf{x_i^H B x_i}}{\mathbf{2x_i^H A x_i}} \ : \ \mathbf{x_i^H A x_i} > 0 \right\}, \ d = \min(0, c_1).$$

Proof. From Theorems 4.2 and 4.3 it follows that

$$\mu > a = \max(0, a_1), \tag{4.3}$$

and

$$\xi < d = \min(0, c_1). \tag{4.4}$$

From the proof of the Theorem 4.2 for $\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i < 0$ we obtain $\mathbf{x}_i^H \mathbf{Q}(\lambda) \mathbf{x}_i > 0$, for $\lambda \in (0, -\frac{\mathbf{x}_i^H \mathbf{B} \mathbf{x}_i}{2\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i})$. Specifically, it means that $\mu < \frac{\mathbf{x}_i^H \mathbf{B} \mathbf{x}_i}{2\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i}$, for each eigenvector $\mathbf{x}_i$ of the matrix $\mathbf{C}$, which belongs to the eigenvalue 0 and for which $\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i < 0$ holds. Therefore,

$$\mu < b = \min\left\{ -\frac{\mathbf{x}_i^H \mathbf{B} \mathbf{x}_i}{2\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i} \middle| \ \mathbf{x}_i^H \mathbf{A} \mathbf{x}_i < 0 \right\}. \tag{4.5}$$

From (4.3) and (4.5) it follows that $\mu \in (a,b)$. From the proof of the Theorem 4.2 for $\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i > 0$ we obtain $\mathbf{x}_i^H \mathbf{Q}(\lambda) \mathbf{x}_i < 0$ for $\lambda \in \left(-\frac{\mathbf{x}_i^H \mathbf{B} \mathbf{x}_i}{2\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i}, 0 \right)$. Specifically, $\xi > -\frac{\mathbf{x}_i^H \mathbf{B} \mathbf{x}_i}{2\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i}$ for each vector $\mathbf{x}_i$ which is an eigenvector of the matrix $\mathbf{C}$, which belongs to the eigenvalue 0 and for which $\mathbf{x}_i^H \mathbf{A} \mathbf{x}_i > 0$ holds. Therefore

$$\xi > c = \max\left\{ -\frac{\mathbf{x_i^H B x_i}}{\mathbf{2x_i^H A x_i}} \ : \ \mathbf{x_i^H A x_i} > 0 \right\}. \tag{4.6}$$

From (4.4) and (4.6) it follows that $\xi \in (c,d)$. □

It is already mentioned that n eigenvalues, of the definite quadratic eigenvalues problem with quadratic pencil (4.1), are in the interval (ξ, μ), and that all those eigenvalues are of positive type. In the case that $\mathbf{C}$ is singular, we can determine the number of eigenvalues in intervals $(0, \mu)$ and $(\xi, 0)$, and we can determine the multiplicity of eigenvalue $\lambda = 0$.

Theorem 4.5. *Let the matrix* $\mathbf{C}$ *be singular and let* $(n_{p_c}, n_{n_c}, n_{z_c})$ *be the inertia of the matrix* $\mathbf{C}$*. Then there are exactly* n_{n_p} *eigenvalues* $\lambda_{n_{nc}+n_{zc}+1} \leq \ldots \leq \lambda_n$ *of the quadratic eigenvalues problem* (4.2) *in the interval* $(0, \mu)$*, and* n_{n_c} *eigenvalues* $\lambda_1 \leq \ldots \leq \lambda_{n_{nc}}$ *of the quadratic eigenvalues problem* (4.2) *in the interval* $(\xi, 0)$*, and the eigenvalue* $\lambda = 0$ *of the quadratic eigenvalues problem* (4.2) *has multiplicity* n_{z_c}*.*

Proof. The proof is based on the application of Theorem 2.4, Remark 4.1 and Theorem 4.2. The inertia of $\mathbf{Q}(\mu)$ is $(n, 0, 0)$, because μ is chosen so that $\mathbf{Q}(\mu) > 0$

holds. The matrix $\mathbf{C}$ is singular, therefore $n_{z_c} > 0$. $\mathrm{In}(\mathbf{Q}(0)) = \mathrm{In}(\mathbf{C}) = (n_{p_c}, n_{n_c}, n_{z_c})$ holds. According to Remark 4.1, 0 is assumed to be an eigenvalue of positive type and according to Theorem 4.2, $\mu > 0$.

The definite quadratic eigenvalue problem is an overdamped problem and the conditions of minmax characterization hold. According to Theorem 2.4 in the interval $(0,\mu)$ we have exactly $n - n_{p_c}$ eigenvalues. Therefore, and from the fact that this multiplicity of the eigenvalue $\lambda = 0$ is n_{z_c}, and from the fact that in the interval (ξ,μ) there are n eigenvalues, it follows that the number of eigenvalues in the interval $(\xi,0)$ is equal to $n - n_{p_c} - n_{z_c}$. It is clear now, according to inertia of the matrix , that the number of eigenvalues in interval $(\xi,0)$ is n_{n_c}. □

If the matrices $\mathbf{A}$ and $\mathbf{C}$ in the definite quadratic pencil are both singular, then we obtain the following bound for the rank of the matrix $\mathbf{B}$.

Theorem 4.6. *Let $\mathbf{Q}(\cdot)$ be a definite quadratic pencil. Let the matrices $\mathbf{A}$ and $\mathbf{C}$ be singular, and let $Rank(\mathbf{A}) = n - p$, and $Rank(\mathbf{C}) = n - q$, $0 \le p < n$, $0 \le q \le n$. Then $Rank(\mathbf{B}) \ge max(p,q)$ holds.*

Proof. Let $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_n}\}$ and $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}\}$ be orthonormal bases of the vector space $\mathbb{C}^n$ consisting of eigenvectors of the corresponding matrix $\mathbf{A}$ and $\mathbf{C}$ respectively, and let $\mathbf{Ay_i} = \mathbf{0}$, $\mathbf{y_i} \neq \mathbf{0}$, $i = 1,2,\ldots,p$, and $\mathbf{Cw_i} = \mathbf{0}$, $\mathbf{w_i} \neq \mathbf{0}$, $i = 1,2,\ldots,q$. According to Theorem 4.1 $\mathbf{y_i^H B y_i} \neq 0$, $i = 1,2,\cdots,p$, respectively $\mathbf{By_i} \neq 0$, $i = 1,2,\ldots,p$. Hence, $Rank(\mathbf{B}) \ge p$. Analogously, according to Theorem 4.1 $\mathbf{w_i^H B w_i} \neq 0$, $i = 1,2,\ldots,$ $\mathbf{Cy_i} \neq \mathbf{0}$, $i = 1,2,\ldots q$, holds. Therefore, $Rank(\mathbf{B}) \ge q$. It follows that $Rank(\mathbf{B}) \ge max(p,q)$. □

Theorem 4.7. *Let $In(\mathbf{A}) = (n_{p_A}, n_{n_A}, n_{z_A})$, $In(\mathbf{B}) = (n_{p_B}, n_{n_B}, n_{z_B})$ and $In(\mathbf{C}) = (n_{p_C}, n_{n_C}, n_{z_C})$ be the inertia of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively, and assume that $n_{p_A} \cdot n_{n_A} \cdot n_{p_C} \cdot n_{n_C} \neq 0$.*

(1) *If $\mathbf{B} > 0$ and $\lambda_{1B}^2 > 4\max(\lambda_{1A}\lambda_{1C}, \lambda_{nA}\lambda_{nC})$, where $\lambda_{1A}, \lambda_{1B}$, and λ_{1C} are the smallest eigenvalues and $\lambda_{nA}, \lambda_{nB}$, and λ_{nC} are the largest eigenvalues of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively, then the quadratic pencil $\mathbf{Q}(\lambda)$ is definite.*

(2) *If $\mathbf{B} < 0$ and $\lambda_{nB}^2 > 4\max(\lambda_{1A}\lambda_{1C}, \lambda_{nA}\lambda_{nC})$, then the quadratic pencil $\mathbf{Q}(\lambda)$ is definite.*

Proof. One of the conditions of Theorem 3.2 for the pencil $\mathbf{Q}(\lambda)$ to be positive definite states that

$$\frac{d(\mathbf{x})}{(\mathbf{x}^H\mathbf{x})^2} = \left(\frac{\mathbf{x}^H\mathbf{Bx}}{\mathbf{x}^H\mathbf{x}}\right)^2 - 4\frac{\mathbf{x}^H\mathbf{Ax}}{\mathbf{x}^H\mathbf{x}} \cdot \frac{\mathbf{x}^H\mathbf{Cx}}{\mathbf{x}^H\mathbf{x}} > 0 \tag{4.7}$$

for each $\mathbf{x} \neq 0$.

(1) Let $\mathbf{B} > 0$ and $\frac{\mathbf{x}^H\mathbf{Bx}}{\mathbf{x}^H\mathbf{x}} \geqslant \lambda_{1,B}$, i.e. $\left(\frac{\mathbf{x}^H\mathbf{Bx}}{\mathbf{x}^H\mathbf{x}}\right)^2 \geqslant \lambda_{1B}^2$ for each $\mathbf{x} \neq 0$.

If $\frac{\mathbf{x}^H\mathbf{Ax}}{\mathbf{x}^H\mathbf{x}} \cdot \frac{\mathbf{x}^H\mathbf{Cx}}{\mathbf{x}^H\mathbf{x}} \leqslant 0$, then the condition (4.7) obviously holds, and two cases remain to be considered:

(a) If $\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} > 0$ and $\frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} > 0$, then

$$0 < \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \leqslant \lambda_{nA} \text{ and } 0 < \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \leqslant \lambda_{nC},$$

and therefore

$$\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \cdot \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \leqslant \lambda_{nA} \cdot \lambda_{nC},$$

and we get

$$\begin{aligned}\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right)^2 &\geqslant \lambda_{1B}^2 > 4\max(\lambda_{1A}\lambda_{1C}, \lambda_{nA}\lambda_{nC}) \\ &\geqslant 4\lambda_{nA} \cdot \lambda_{nC} \geqslant 4\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \cdot \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}. \end{aligned} \tag{4.8}$$

From (4.8) we get (4.7). From $\mathbf{B} > 0$ and (4.8) we obtain the remaining two conditions of Theorem 3.2.

(b) If $\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} < 0$ and $\frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} < 0$, then

$$0 > \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \geqslant \lambda_{1A} \text{ and } 0 > \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \geqslant \lambda_{1C},$$

and therefore

$$\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \cdot \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \leqslant \lambda_{1A} \cdot \lambda_{1C}.$$

Hence

$$\begin{aligned}\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right)^2 &\geqslant \lambda_{1B}^2 > 4\max(\lambda_{1A}\lambda_{1C}, \lambda_{nA}\lambda_{nC}) \\ &\geqslant 4\lambda_{1A} \cdot \lambda_{1C} \geqslant 4\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \cdot \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \end{aligned} \tag{4.9}$$

holds, and from (4.9) we get (4.7). From $\mathbf{B} > 0$ and (4.8) we obtain the remaining two conditions of Theorem 3.2.

(2) Let $\mathbf{B} < 0$ then

$$\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \leqslant \lambda_{nB} < 0. \tag{4.10}$$

From (4.10) we get

$$\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right)^2 \geqslant \lambda_{nB}^2, \tag{4.11}$$

the rest of the proof is analogous to (1). □

In the special case of a positive definite matrix $\mathbf{C}$, we can reduce the definite eigenvalue problem $\mathbf{Q}(\lambda) = 0$ to a hyperbolic eigenproblem by the transformation below.

Let $\mathbf{C} > 0$. Dividing the quadratic eigenproblem

$$\begin{aligned} &\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}, \quad \mathbf{C} = \mathbf{C}^{\mathbf{H}} > 0 \\ &\mathbf{Q}(\lambda)\,\mathbf{x} = 0 \end{aligned} \tag{4.12}$$

by λ^2 we obtain the following equation:

$$\left(\frac{1}{\lambda^2}\mathbf{C} + \frac{1}{\lambda}\mathbf{B} + \mathbf{A}\right)\mathbf{x} = 0.$$

If we denote $\varphi := \frac{1}{\lambda}$, and define $\overline{\mathbf{Q}}(\varphi) = \varphi^2\mathbf{C} + \varphi\mathbf{B} + \mathbf{A}$, we can write the last equation in the following form:

$$\overline{\mathbf{Q}}(\varphi) = \left(\varphi^2\mathbf{C} + \varphi\mathbf{B} + \mathbf{A}\right)\mathbf{x} = 0. \tag{4.13}$$

Hence, we have obtained a hyperbolic quadratic pencil, and we can implement the variational characterization of its eigenvalues. We will also denote the Rayleigh functionals of problem (4.13) as

$$\begin{aligned} \overline{p_+}(\mathbf{x}) &= -\frac{\mathbf{x}^{\mathbf{H}}\mathrm{Bx}}{2\mathbf{x}^{\mathbf{H}}\mathrm{Cx}} + \sqrt{\left(\frac{\mathbf{x}^{\mathbf{H}}\mathrm{Bx}}{2\mathbf{x}^{\mathbf{H}}\mathrm{Cx}}\right)^2 - \frac{\mathbf{x}^{\mathbf{H}}\mathrm{Ax}}{\mathbf{x}^{\mathbf{H}}\mathrm{Cx}}} \\ \overline{p_-}(\mathbf{x}) &= -\frac{\mathbf{x}^{\mathbf{H}}\mathrm{Bx}}{2\mathbf{x}^{\mathbf{H}}\mathrm{Cx}} - \sqrt{\left(\frac{\mathbf{x}^{\mathbf{H}}\mathrm{Bx}}{2\mathbf{x}^{\mathbf{H}}\mathrm{Cx}}\right)^2 - \frac{\mathbf{x}^{\mathbf{H}}\mathrm{Ax}}{\mathbf{x}^{\mathbf{H}}\mathrm{Cx}}}. \end{aligned} \tag{4.14}$$

The following theorem states the relation between the functionals $p_+(\mathbf{x})$ and $\overline{p_+}(\mathbf{x})$ and subsequently, the relation between the functionals $p_-(\mathbf{x})$ and $\overline{p_-}(\mathbf{x})$ is obvious.

Theorem 4.8. *Let $\overline{p_+}(\mathbf{x})$ and $\overline{p_-}(\mathbf{x})$ be defined as in* (4.7)*, and let $p_+(\mathbf{x})$ and $p_-(\mathbf{x})$ be defined as in* (3.4)*. Then the relations between those functionals are given by the following equations $p_+(\mathbf{x}) = \frac{\mathbf{1}}{\overline{p_-}(\mathbf{x})}$ and $p_-(\mathbf{x}) = \frac{\mathbf{1}}{\overline{p_+}(\mathbf{x})}$.*

The following examples demonstrate the benefit of the proposed theorems checking the definiteness of a quadratic pencil.

5. NUMERICAL EXPERIMENTS

In this section we show through numerical experiments that the application of the theorems, which are stated in section 4, contributes to a rapid determination whether a quadratic pencil is definite or not. For comparison we use the algorithm, which is given in [19], as well as the effective algorithm for the determination of initial vectors, which is given in [14]. We will merge these two algorithms into one. For a better understanding of the paper, we state this algorithm.

Algorithm 1
This algorithm finds out whether a quadratic pencil is definite, and if so, it determines an initial vector

Require: initial vector $\mathbf{x}_0$
if $\det \mathbf{B} = 0$ **then**
  $\mathbf{x}_0 = \mathbf{z}_1$
else
  determine smallest eigenvalue $\lambda_{\mathbf{1A}}$ of $\mathbf{A}$ and corr. eigenvector $\mathbf{q}_1$
  determine largest eigenvalue $\lambda_{\mathbf{nB}}$ of $\mathbf{B}$ and corr. eigenvector $\mathbf{f}_n$
  $\mathbf{x}_0 = 0.9\mathbf{q}_1 + 0.1\mathbf{f}_n$
end if
if $d(\mathbf{x}_0) := (\mathbf{x}_0^H \mathbf{B}\mathbf{x}_0)^2 - 4(\mathbf{x}_0^H \mathbf{A}\mathbf{x}_0)(\mathbf{x}_0^H \mathbf{C}\mathbf{x}_0) < 0$ **then**
  STOP: $\mathbf{Q}(\lambda)$ is not definite
end if
determine $\sigma_0 = p(\mathbf{x}_0)$
for $k = 1, 2, \dots$ until convergence do **do**
  determine eigenvector $\mathbf{x}_k$ of $\mathbf{Q}(\sigma_{k-1})$ corresponding to its largest eigenvalue
  if $d(\mathbf{x}_k) := (\mathbf{x}_k^H \mathbf{B}\mathbf{x}_k)^2 - 4(\mathbf{x}_k^H \mathbf{A}\mathbf{x}_k)(\mathbf{x}_k^H \mathbf{C}\mathbf{x}_k) < 0$ **then**
    STOP: $\mathbf{Q}(\lambda)$ is not definite
  end if
  determine $\sigma_k = p(\mathbf{x}_k)$
  if $\sigma_k \geqslant \sigma_{k-1}$ **then**
    STOP: $\mathbf{Q}(\lambda)$ is not definite
  end if
  if $\mathbf{Q}(2\sigma_k - \sigma_{k-1})$ is negative definite **then**
    STOP: $\mathbf{Q}(\lambda)$ is definite
  end if
end for

Example 5.1. *Consider the quadratic pencil* (4.1) *of dimension 10 with matrices*

$\mathbf{A} = zeros(10)$
  for $i = 1:7$
    $\mathbf{A}(i,i) = 0,2; \mathbf{A}(i,i+1) = -0,1; \mathbf{A}(i+1,i) = -0,1;$
  end
$\mathbf{A}(8,8) = 0,2;$

$\mathbf{B} = zeros(10)$
  for $i = 5:7$
    for $j = i:9$
      $\mathbf{B}(i,j) = 10 \cdot i;\ \mathbf{B}(j,i) = \mathbf{B}(i,j);$
    end

end
$\mathbf{B}(1,10) = 1;\ \mathbf{B}(10,1) = 1;$

$\mathbf{C} = zeros(10)$
for $i = 1:3$
for $j = i:8$
$\mathbf{C}(i,j) = 0,001\cdot i;\ \mathbf{C}(j,i) = \mathbf{C}(i,j);$
end
end
$\mathbf{C}(2,2) = -\mathbf{C}(2,2);$

The matrix $\mathbf{B}$ *is singular, so we take*

$$\mathbf{x}_0^H = (0,1,0,0,0,0,0,0,0,0)$$

for the initial vector, which corresponds to eigenvalue 0*. Applying the algorithm we obtain the quadratic pencil is not definite after six iterations. Let us point out, in the first and the last iteration we solved one eigenvalue problem and in each of the remaining four iterations we solved two eigenvalue problems. In order to check the quadratic pencil for definiteness, we have to solve ten eigenvalue problems.*

The non-definiteness is much easier obtained from 4.1: The matrix $\mathbf{C}$ *is singular and it has the eigenvector* $\mathbf{x}^H = (0,0,0,0,0,0,0,0,0,1)$ *corresponding to the eigenvalue 0. We have* $\mathbf{x}^H\mathbf{C}\mathbf{x} = 0$ *and* $\mathbf{x}^H\mathbf{B}\mathbf{x} = 0$*, which means the quadratic pencil is not definite according to Theorem 4.1 (i).*

Example 5.2. *Let matrices* $\mathbf{A}$ *and* $\mathbf{C}$ *be as in the previous example and matrix* $\mathbf{B}$ *defined as follows*
$\mathbf{B} = zeros(10)$
for $i = 5:7$
for $j = i:8$
$\mathbf{B}(i,j) = 1;\ \mathbf{B}(j,i) = \mathbf{B}(i,j);$
end
end
$\mathbf{B}(6,6) = 2;\ \mathbf{B}(9,9) = 1;\ \mathbf{B}(10,10) = -0,1.$

We apply the algorithm above using the initial vector

$$\mathbf{x}_0^H = (0,1,0,0,0,0,0,0,0,0).$$

In order to apply this algorithm, it requires four iterations for determining whether a quadratic pencil is definite or not. Therefore, we need to solve four eigenvalue problems. Applying Theorem 4.6, after determining *Rank*$\mathbf{B}$ and *Rank*$\mathbf{C}$, we obtain the quadratic pencil is not definite, because *Rank*$\mathbf{C} = 4$, *Rank*$\mathbf{B} = 5$ hold, and according to Theorem 4.6 *Rank*$\mathbf{B}$ has to be greater than 6.

Example 5.3. *Consider the quadratic pencil* (4.1) *of dimension* 10 *with matrices*

$\mathbf{A} = zeros(10)$
for $i = 1:5$
for $j = i:5$
$\mathbf{A}(i,j) = 1;\ \mathbf{A}(j,i) = 1;$
end
end
for $i = 7:10$
$\mathbf{A}(i,i) = -1;$
end

$\mathbf{B} = zeros(10)$
for $i = 1:9$
$\mathbf{B}(i,i) = 200;\ \mathbf{B}(i,i+1) = -100;\ \mathbf{B}(i+1,1) = -100;$
end
$\mathbf{B}(10,10) = 200$

$\mathbf{C} = zeros(10)$
for $i = 1:8$
$\mathbf{C}(i,i) = 2;\ \mathbf{C}(i,i+1) = -1;\ \mathbf{C}(i+1,1) = -1;$
end
$\mathbf{C}(9,9) = 2;\ \mathbf{C}(10,10) = -7.$

For determination of definiteness, we apply the algorithm, which we described before. For the initial value we use

$$\begin{aligned} x_0 &= 0.9q_1 + 0.1f_n \\ &= (0.012, -0.0231, 0.0322, -0.0388, 0.0422, -0.0422, 0.9388, \\ &\qquad -0.0322, 0.0231, -0.012)^H. \end{aligned} \tag{5.1}$$

We obtain the quadratic pencil is definite, after three iterations, solving six eigenvalue problems. Otherwise,

$$\begin{gathered} \lambda_{1C} = -7 \\ \lambda_{nC} = 3.9021 \\ \lambda_{1A} = -1 \\ \lambda_{nA} = 5 \\ \lambda_{1B} = 8.1014, \end{gathered}$$

hence, we obtain

$$\lambda_{1B}^2 - \max(\lambda_{1A}\lambda_{1C}) = 65.6327 - 19.5105 = 46.1222 > 0.$$

According to Theorem 4.7, we can conclude that the quadratic pencil is definite.

6. Conclusion

In this paper we have studied the impact of the matrix $\mathbf{C}$ from the definite quadratic pencil on the corresponding definite eigenvalue problem. In particular, in the case of a singular matrix $\mathbf{C}$ we have obtained better bounds for parameters ξ and μ. We have also considered definiteness of the matrix $\mathbf{C}$. In the case of a positive or negative definite matrix $\mathbf{C}$, the problem can be reduced to a hyperbolic quadratic eigenvalue problem, where the leading matrix is the matrix $\mathbf{C}$. For the resulting eigenvalue problem we have given the relation between corresponding Rayleigh functionals. We observed also the application of Sylvester's law of inertia on the corresponding quadratic pencil. Further research will continue in the direction of the analysis of the impact of matrices $\mathbf{A}$ and $\mathbf{B}$ from the definite quadratic pencil on the definite eigenvalue problem. Another interesting topic for consideration would be the transformation of the definite quadratic pencil into a quadratic pencil in which the matrices $\mathbf{A}$ and $\mathbf{B}$ remain the same, while the matrix $\mathbf{C}$ will be replaced with a matrix $\mathbf{D} = \mathbf{C} - \alpha\mathbf{I}$, where $\mathbf{D}$ is a singular matrix. We will study the dependence between the starting and resulting eigenvalue problem.

Acknowledgments. The authors would like to thank the reviewers for helpful suggestions.

References

[1] J. Demmel, *Applied Numerical Linear Algebra*, SIAM, Philadelphia, 1997.

[2] R.J. Duffin, *A minimax theory for overdamped networks*, J. Rat. Mech. Anal. 4 (1955), 221–233.

[3] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, SIAM, Phiadelphia, PA, Reprint of the 1982 original (Academic Press, New York), 2009.

[4] G. Golub and C. Van Loan, *Matrix Computations*, 3rd ed., John Hopkins University Press, Baltimore, 1996.

[5] C.-H. Guo, N.J. Higham and F. Tisseur, *Detecting and solving hyperbolic quadratic eigenvalue problems*, SIAM J. Matrix Anal. Appl. 30 (2009), 1593–1613.

[6] C.-H. Guo and P. Lancaster, *Algorithms for hyperbolic quadratic eigenvalue problems*, Math.Comp. 74 (2005), 1777–1791.

[7] N.J. Higham, F. Tisseur and P.M. van Dooren, *Detecting a definite Hermitian pair and a hyperbolic or elliptic quadratic eigenvalue problem, and associated nearness problems*, Linear Algebra Appl. 351–352 (2002), 455–474.

[8] N. Higham, D. Mackey and F. Tisseur, *Definite matrix polynomials and their linearization by definite pencils*, SIAM J. Matrix Anal. Appl. 33 (2009), 478–502.

[9] L. Hogben, *Handbook of linear algebra*, 2nd ed., CRC Press, Boca Raton, 2014.

[10] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[11] A. Kostić, *Eigenvalue Problems, in Applied linear algebra in action*, Vasilios N. katsikis, Ed., Books on Demand. (2016), InTech, Rijeka (2016), 57–83.

[12] A. Kostić and H. Voss, *On Sylvester's law of inertia for nonlinear eigenvalue problems*, Electr. Trans. Numer. Anal. 40 (2013), 82–93.

[13] A. Kostić and Š. Šikalo, *Definite quadratic eigenvalue problems*, Procedia Engineering. 100 (2015), 56–63.

[14] A. Kostić, Z. Aljičević, Š. Šikalo and M. Kustura, *Effective algorithm for determination of the initial vector for the definite quadratic pencil*, Annals of DAAAM & Proceedings. 27 (2016), 10–16.
[15] P. Lancaster, *Lambda-matrices and vibrating systems*, Dover Publications, Mineola, NY, Reprint of the 1966 original, Pergamon Press, New York, 2002.
[16] J. Liesen and C. Mehl, *Nonlinear eigenvalue problems*, Chapter 18 in L. Hogben (ed.), Handbook of Linear Algebra, 2nd ed. CRC Press, Boca Raton, 2014.
[17] D.S. Mackey, N. Mackey, C. Mehl and V. Mehrmann, *Vector spaces of linearization for matrix polynomials*, SIAM J. Matrix Anal. Appl. 28 (2006), 971–1004.
[18] A. Neumaier, *Introduction to Numerical Analysis*, Cambridge University Press, 2001.
[19] V. Niendorf and H. Voss, *Detecting hyperbolic and definite matrix polynomials*, Linear Algebra Appl. 432 (2010), 1017–1035.
[20] B. Parlett, *The Symmetric Eigenvalue Problem*, SIAM, Philadelphia, 1998.
[21] J. Sylvester,*A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by orthogonal substitutions to the form of a sum of positive and negative squares*, Philosophical Magazin. 4 (1852), 138–142.
[22] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Review. 43 (2001), 235–286.
[23] H. Voss, *A minmax principle for nonlinear eigenproblems depending continuously on the eigenparameter*, Numer. Linear Algebra Appl. 16 (2009), 899–913.
[24] H. Voss, *Nonlinear eigenvalue problems*, Chapter 60 in L. Hogben (ed.), Handbook of Linear Algebra, 2nd ed. CRC Press, Boca Raton, 2014.
[25] H. Voss and B. Werner, *A minimax principle for nonlinear eigenvalue problems with applications to nonoverdamped systems*, Math. Meth. Appl. Sci., 4 (1982), 415–424.

(Received: July 31, 2021)
(Revised: October 4, 2021)

Aleksandra Kostić
University of Sarajevo
Faculty of Mechanical Engineering
71 000 Sarajevo, Bosnia and Herzegovina
e-mail: *kostic@mef.unsa.ba*
and
Heinrich Voss
Hamburg University of Technology
Institute of Mathematics
D-21073 Hamburg, Germany
e-mail: *voss@tuhh.de*
and
Valentina Timotić
University of East Sarajevo
Department of Mathematics
71 420 Pale, Bosnia and Herzegovina
e-mail: *valentina.timotic@ffuis.edu.ba*