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ON DECOMPOSITION OF THE DIRICHLET KERNEL ON VILENKIN GROUPS

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ABSTRACT. We give a useful decomposition of the Dirichlet kernel on Vilenkin groups.

1. INTRODUCTION

The Dirichlet kernel is an important concept in harmonic analysis on Vilenkin groups. Indeed, it is well known (see [1]) that for a function $f \in L^1(G)$ we have

$$S_n(f,x) = \int_G f(x-u) \cdot D_n(u) d\mu(u), x \in G, \tag{1.1}$$

where G is a Vilenkin group, μ is the normalized Haar measure on G, D_n is the Dirichlet kernel on G, and $S_n(f,x)$ is the $n-$th partial sum of the Fourier-Vilenkin series of the function f. So, (1.1) shows that the properties of the Dirichlet kernel affect the properties of the sequence $(S_n(f,x))_{n=0}^{\infty}$. But, one of the main questions in harmonic analysis on Vilenkin groups is whether $(S_n(f,x))_{n=0}^{\infty}$ converges in some sense to f (and under which conditions on f).

Some properties of the Dirichlet kernel on the dyadic group are given in [5], [9], and on Vilenkin groups are given in [1], [3], [8], [10]. In [4] the author introduced the Dirichlet kernel of the Vilenkin-like orthonormal system, which is a generalization of the Vilenkin system. In [7] we studied the Dirichlet kernel on the group of 2-adic integers (which is an example of a Vilenkin group).

2. PRELIMINARIES

Let us denote with $\mathbb{N}$ the set of nonnegative integers.

For a positive integer n define

$$\mathbb{Z}_n := \{0, 1, \dots, n-1\}.$$

We shall endow $\mathbb{Z}_n$ with the discrete topology as well with the operation of addition modulo n.

Let $(m_i)_{i=0}^{\infty}$ be a sequence of positive integers which satisfies $m_i \geq 2, \forall i \in \mathbb{N}$.

Define

$$G := \prod_{i=0}^{\infty} \mathbb{Z}_{m_i}.$$

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We shall endow G with the product topology and with the component-wise addition.

Then (see [1]) G is a Vilenkin group.

In particular, if $m_i = 2$ for all $i \in \mathbb{N}$, we call G the dyadic group.

Define the sequence of integers $(M_i)_{i=0}^{\infty}$ by

$$M_0 := 1, M_{i+1} := m_i \cdot M_i, i \geq 0. \tag{2.1}$$

It is known (see [1]) that each $n \in \mathbb{N}$ can be expressed as

$$n = \sum_{i=0}^{\infty} n_i \cdot M_i, n_i \in \{0, 1, \ldots, m_i - 1\} \tag{2.2}$$

in a unique way. We'll call (2.2) the representation of n.

Define

$$I_0 := G, I_n := \{(x_i)_{i=0}^{\infty} \in G \,|\, x_0 = \cdots = x_{n-1} = 0\}. \tag{2.3}$$

Then $\{I_n | n \in \mathbb{N}\}$ is a family of both open and closed sets which satisfies

$$I_0 \supseteq I_1 \supseteq \ldots I_n \supseteq \ldots. \tag{2.4}$$

It is known (see [1], [6]) that there is a unique Haar measure μ on G such that $\mu(G) = 1$.

For $k \in \mathbb{N}$ define

$$r_k(x) := e^{\frac{2\pi i x_k}{m_k}}, x = (x_i)_{i=0}^{\infty} \in G. \tag{2.5}$$

For $n \in \mathbb{N}$ define

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), x \in G.$$

It is known (see [1]) that $(\psi_n)_{n=0}^{\infty}$ is an orthonormal system on G, i.e. it satisfies

$$\int_G \psi_n(x) \overline{\psi_m(x)} d\mu(x) = \delta_{m,n}, m, n \in \mathbb{N},$$

where $\delta_{m,n}$ is the Kronecker symbol. The system $(\psi_n)_{n=0}^{\infty}$ is called a Vilenkin system.

The Dirichlet kernel is defined by

$$D_0 := (x) = 0, D_n(x) := \sum_{i=0}^{n-1} \psi_i(x), n \geq 1.$$

It is known (see [1], [2]) that for each $n \in \mathbb{N}$

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases} \tag{2.6}$$

It is also known (see [2]) that

$$D_n(x) = \psi_n(x) \cdot \left(\sum_{k=0}^{\infty} D_{M_k}(x) \cdot \sum_{s=m_k-n_k}^{m_k-1} {r_k}^s(x) \right), x \in G, n \in \mathbb{N}, \tag{2.7}$$

where $n = \sum_{k=0}^{\infty} n_k \cdot M_k$ is the representation of n.

The sequence $(e_i)_{i=0}^{\infty}$ of elements in G is defined as

$$e_i := (\delta_{j,i})_{j=0}^{\infty}, i \in \mathbb{N}, \tag{2.8}$$

where $\delta_{j,i}$ denotes the Kronecker symbol.

The characteristic function of the set A will be denoted with χ_A.

The support of a function $f : G \to \mathbb{C}$ will be defined by

$$supp(f) := \overline{\{x \in G : f(x) \neq 0\}}.$$

3. Results

For $j \in \mathbb{N}$ define

$$y_j := \sum_{i=0}^{\infty} j_i \cdot e_i,$$

where $j = \sum_{i=0}^{\infty} j_i \cdot M_i$ is the representation of j.

Lemma 3.1. *Let $j \in \mathbb{N}$. Suppose*

$$j = \sum_{i=0}^{\infty} j_i \cdot M_i, j_i \in \{0, 1, \ldots, m_i - 1\} \tag{3.1}$$

is the representation of j. Then:

i) For each positive integer k

$$j \leq M_k - 1 \Leftrightarrow y_j = j_0 \cdot e_0 + j_1 \cdot e_1 + \ldots + j_{k-1} \cdot e_{k-1}. \tag{3.2}$$

ii) For each positive integer k, the function

$$f_k : \{0, 1, \ldots, M_k - 1\} \to \{\alpha_0 \cdot e_0 + \ldots \alpha_{k-1} \cdot e_{k-1} : \alpha_i \in \{0, 1, \ldots, m_i - 1\}, i = \overline{0, k-1}\}$$

defined by

$$f_k(j) = \sum_{i=0}^{k-1} j_i \cdot e_i, \forall j = \sum_{i=0}^{\infty} j_i M_i \in \{0, 1, \ldots, M_k - 1\} \tag{3.3}$$

is bijective.

Notice that (3.3) is equivalent to $f_k(j) = y_j$, $j = \overline{0, M_k - 1}$.

iii) For each positive integer k

$$G = \biguplus_{j=0}^{M_k - 1} (y_j + I_k). \tag{3.4}$$

iv) For each $k, j \in \mathbb{N}$

$$y_j \in I_k \Leftrightarrow j \equiv 0 \, (mod \, M_k). \tag{3.5}$$

In particular,

$$y_j \in I_k \backslash I_{k+1} \Leftrightarrow j \equiv 0 \, (mod \, M_k) \wedge j \not\equiv 0 \, (mod \, M_{k+1}).$$

Proof.

i) If $j \leq M_k - 1$, then obviously $j_i = 0$ for $i \geq k$. Conversely, if $y_j = j_0 \cdot e_0 + \ldots + j_{k-1} \cdot e_{k-1}$, we get

$$j=\sum_{i=0}^{k-1} j_iM_i \leq \sum_{i=0}^{k-1}(m_i-1)M_i=\sum_{i=0}^{k-1}(M_{i+1}-M_i)=M_k-1,$$

where we used (2.1).

ii) The function f_k is well defined because of (3.1) and (3.2). Obviously, f_k is bijective.

iii) Using (2.3) we get

$$G=\biguplus_{\alpha_0=0}^{m_0-1}\cdots\biguplus_{\alpha_{k-1}=0}^{m_{k-1}-1}(\alpha_0\cdot e_0+\cdots+\alpha_{k-1}\cdot e_{k-1}+I_k). \tag{3.6}$$

Now, (3.4) follows from (3.6) and claim ii) of this Lemma.

iv) If $y_j \in I_k$, then (2.3) implies $j_0=\cdots=j_{k-1}=0$. Therefore,

$$j=\sum_{i=k}^{\infty} j_i\cdot M_i \equiv 0\,(mod\,M_k).$$

Conversely, if $j\equiv 0\,(mod\,M_k)$, then

$$\sum_{i=0}^{k-1} j_iM_i \equiv \sum_{i=0}^{\infty} j_iM_i = j \equiv 0\,(mod\,M_k).$$

But, since

$$0\leq \sum_{i=0}^{k-1} j_iM_i \leq \sum_{i=0}^{k-1}(m_i-1)M_i=M_k-1,$$

we get $j_0=\cdots=j_{k-1}=0$. Therefore, (3.5) holds. □

Lemma 3.2. *For each $n,k\in\mathbb{N}$*

$$D_{M_n}=M_n\cdot\sum_{s_0=0}^{m_n-1}\sum_{s_1=0}^{m_{n+1}-1}\cdots\sum_{s_k=0}^{m_{n+k}-1}\chi_{s_0e_n+s_1e_{n+1}+\ldots+s_ke_{n+k}+I_{n+k+1}}, \tag{3.7}$$

where the supports of characteristic functions appearing on the right side of (3.7) are pairwise disjoint.

Proof. By applying (2.3) and (2.8) we get

$$I_n=\biguplus_{s_0=0}^{m_n-1}\biguplus_{s_1=0}^{m_{n+1}-1}\cdots\biguplus_{s_k=0}^{m_{n+k}-1}(s_0e_n+s_1e_{n+1}+\ldots+s_ke_{n+k}+I_{n+k+1}) \tag{3.8}$$

for all $n,k\in\mathbb{N}$.

Therefore, using (2.6) we have

$$D_{M_n}=M_n\cdot\chi_{I_n}=M_n\cdot\sum_{s_0=0}^{m_n-1}\sum_{s_1=0}^{m_{n+1}-1}\cdots\sum_{s_k=0}^{m_{n+k}-1}\chi_{s_0e_n+s_1e_{n+1}+\ldots+s_ke_{n+k}+I_{n+k+1}}.$$

Disjointness of supports of the functions appearing on the right side of (3.7) follows from (3.8). □

Lemma 3.3. *For each $n,l,k \in \mathbb{N}$ and each $x \in G$*

$$D_{M_n}(x) \cdot r_n^l(x) = \sum_{j=0}^{M_{n+k+1}-1} a_j(n,l,k) \cdot D_{M_{n+k+1}}(x - y_j),$$

where $a_j(n,l,k)$, $j = \overline{0, M_{n+k+1}-1}$ are constants.

Proof. Using Lemma 3.2 we get

$$D_{M_n}(x) r_n^l(x) = M_n \cdot \sum_{s_0=0}^{m_n-1} \sum_{s_1=0}^{m_{n+1}-1} \cdots \sum_{s_k=0}^{m_{n+k}-1} r_n^l(x) \chi_{s_0 e_n + s_1 e_{n+1} + \ldots + s_k e_{n+k} + I_{n+k+1}}(x), \tag{3.9}$$

for each $x \in G$ and each $n,l,k \in \mathbb{N}$.
By applying (2.5) we have

$$r_n^l(x) \cdot \chi_{s_0 e_n + \ldots + s_k e_{n+k} + I_{n+k+1}}(x) = e^{\frac{2\pi i s_0 l}{m_n}} \chi_{s_0 e_n + \ldots + s_k e_{n+k} + I_{n+k+1}}(x) \tag{3.10}$$

for all $x \in G$.
Besides, using (2.6) we get

$$\chi_{s_0 e_n + \ldots + s_k e_{n+k} + I_{n+k+1}}(x) = \frac{1}{M_{n+k+1}} D_{M_{n+k+1}}(x - (s_0 e_n + \ldots + s_k e_{n+k})) \tag{3.11}$$

for all $x \in G$ and all $s_i \in \{0, 1, \ldots, m_{n+i} - 1\}$, $i = \overline{0,k}$.
Now, combining (3.9), (3.10) and (3.11) gives us

$$D_{M_n}(x) r_n^l(x) = \sum_{s_0=0}^{m_n-1} \sum_{s_1=0}^{m_{n+1}-1} \cdots \sum_{s_k=0}^{m_{n+k}-1} \frac{M_n \cdot e^{\frac{2\pi s_0 l}{m_n}}}{M_{n+k+1}} D_{M_{n+k+1}}(x - (s_0 e_n + \ldots + s_k e_{n+k})).$$

Finally, this and claim i) of Lemma 3.1 gives us the claim of this Lemma. □

Theorem 3.1. *Let n be a positive integer and*

$$n = n_{k_1} M_{k_1} + \ldots + n_{k_l} M_{k_l}$$

be the representation of n, where $0 \le k_1$, $k_i < k_j$ for $i < j$, and $n_{k_i} \in \{1, \ldots, m_{k_i} - 1\}$, $i = \overline{1,l}$. Then

$$D_n(x) = \psi_n(x) \left(\sum_{j=0}^{M_{k_l}-1} a_{n,j} \cdot D_{M_{k_l}}(x - y_j) \right), x \in G, \tag{3.12}$$

where $a_{n,j}$, $j = \overline{1, M_{k_l} - 1}$ are constants, and $a_{n,0}$ is a function on G. Moreover, we have:

i) If $j \not\equiv 0 \,(mod\, M_{k_1})$, then $a_{n,j} = 0$,
ii) If $j \equiv 0 \,(mod\, M_{k_p})$, $j \not\equiv 0 \,(mod\, M_{k_{p+1}})$ for some $p \in \{1, \ldots, l-1\}$, then

$$a_{n,j} = \frac{1}{M_{k_l}} \sum_{i=1}^{p-1} n_{k_i} M_{k_i} + \frac{M_{k_p}}{M_{k_l}} \sum_{s=m_{k_p}-n_{k_p}}^{m_{k_p}-1} e^{\frac{2\pi i s a}{m_{k_p}}},$$

where $a = \frac{j\,(mod\, M_{k_p+1})}{M_{k_p}}$. *In particular, if* $j \equiv 0\,(mod\, M_{k_p+1})$, *then*

$$a_{n,j} = \frac{1}{M_{k_l}} \sum_{i=1}^{p} n_{k_i} M_{k_i},$$

iii)

$$a_{n,0}(x) = \frac{1}{M_{k_l}} \sum_{i=1}^{l-1} n_{k_i} M_{k_i} + \sum_{s=m_{k_l}-n_{k_l}}^{m_{k_l}-1} e^{\frac{2\pi i x_{k_l} s}{m_{k_l}}}, x \in I_{k_l}.$$

Proof. From (2.7) we have

$$D_n(x) = \psi_n(x) \left(\sum_{i=1}^{l} D_{M_{k_i}}(x) \cdot \sum_{s=m_{k_i}-n_{k_i}}^{m_{k_i}-1} r_{k_i}^s(x) \right).$$

Applying Lemma 3.3 to each of the functions $D_{M_{k_i}}(x) \cdot r_{k_i}^s(x)$, $i \in \{1, \ldots, l-1\}$, $s \in \{m_{k_i} - n_{k_i}, \ldots, m_{k_i} - 1\}$ gives us for each $x \in G$

$$\sum_{i=1}^{l-1} D_{M_{k_i}}(x) \cdot \sum_{s=m_{k_i}-n_{k_i}}^{m_{k_i}-1} r_{k_i}^s(x) = b_0 \cdot D_{M_{k_l}}(x - y_0) + \sum_{j=1}^{M_{k_l}-1} a_{n,j} \cdot D_{M_{k_l}}(x - y_j),$$

where $b_0, a_{n,j}, j = \overline{1, M_{k_l} - 1}$ are constants. Therefore,

$$\sum_{i=1}^{l} D_{M_{k_i}}(x) \sum_{s=m_{k_i}-n_{k_i}}^{m_{k_i}-1} r_{k_i}^s(x) = a_{n,0}(x) \cdot D_{M_{k_l}}(x - y_0) + \sum_{p=1}^{M_{k_l}-1} a_{n,p} \cdot D_{M_{k_l}}(x - y_p), \tag{3.13}$$

for all $x \in G$, where $a_{n,j}, j = \overline{1, M_{k_l} - 1}$ are constants.

On the other hand, using (2.6) and the fact that $y_j + I_{k_l}$ is closed we conclude that the support of the function $D_{M_{k_l}}(x - y_j)$ satisfies

$$supp(D_{M_{k_l}}(\cdot - y_j)) = y_j + I_{k_l}, j \in \{0, 1, \ldots, M_{k_l}\}. \tag{3.14}$$

Besides, using claim iii) of Lemma 3.1 we have

$$G = \biguplus_{j=0}^{M_{k_l}-1} (y_j + I_{k_l}). \tag{3.15}$$

Let's take $j \in \{1, 2, \ldots, M_{k_l} - 1\}$ arbitrary and fix it. If we put $x = y_j$ into (3.13) and take into account (3.14) and (3.15), we get

$$a_{n,j} \cdot M_{k_l} = \sum_{i=1}^{l} D_{M_{k_i}}(y_j) \cdot \sum_{s=m_{k_i}-n_{k_i}}^{m_{k_i}-1} r_{k_i}^s(y_j). \tag{3.16}$$

Now, we consider two cases:

i) $j \not\equiv 0\,(\,mod\, M_{k_1}\,)$.

Then, using (2.4) and claim iv) of Lemma 3.1 gives us

$$y_j \notin I_{k_1} \supseteq I_{k_2} \ldots \supseteq I_{k_l}.$$

This together with (2.6) implies

$$D_{M_{k_1}}(y_j) = \cdots = D_{M_{k_l}}(y_j) = 0. \tag{3.17}$$

Now, combining (3.17) with (3.16) gives us $a_{n,j} = 0$.

ii) $j \equiv 0\,(\,mod\, M_{k_p}\,),\, j \not\equiv 0\,(\,mod\, M_{k_{p+1}}\,)$ for some $1 \le p \le l-1$.

Therefore, the representation of j is

$$j = \sum_{i=k_p}^{\infty} j_i \cdot M_i,\, j_i \in \{0,1,\ldots,m_i-1\}, \forall i \ge k_p.$$

Put $a := j_{k_p}$. Notice that $a = \frac{j\,(mod\, M_{k_p+1})}{M_{k_p}}$. Using claim iv) of Lemma 3.1 we conclude $y_j \in I_{k_p},\, y_j \notin I_{k_{p+1}}$. Applying (2.6) gives us

$$D_{M_{k_i}}(y_j) = M_{k_i}, \forall i \in \{1,\ldots,p\}, D_{M_{k_i}}(y_j) = 0, \forall i \in \{p+1,\ldots,l\}. \tag{3.18}$$

Combining (3.18) with (3.16) gives us

$$a_{n,j} \cdot M_{k_l} = \sum_{i=1}^{p} M_{k_i} \cdot \sum_{s=m_{k_i}-n_{k_i}}^{m_{k_i}-1} r_{k_i}^{s}(y_j). \tag{3.19}$$

On the other hand, using (2.5) and the fact $y_j \in I_{k_p}$, we get

$$r_{k_1}(y_j) = \cdots = r_{k_{p-1}}(y_j) = 1, \text{ a } r_{k_p}(y_j) = e^{\frac{2\pi i a}{m_{k_p}}}.$$

This together with (3.19) gives us

$$a_{n,j} = \frac{1}{M_{k_l}} \sum_{i=1}^{p-1} n_{k_i} M_{k_i} + \frac{M_{k_p}}{M_{k_l}} \sum_{s=m_{k_p}-n_{k_p}}^{m_{k_p}-1} e^{\frac{2\pi i s a}{m_{k_p}}}.$$

Now, let's assume $x \in I_{k_l}$. Since $y_0 = (0)_{n=0}^{\infty}$, we have

$$x \in y_0 + I_{k_l}. \tag{3.20}$$

From (3.20) and (3.15) we get

$$x \notin y_j + I_{k_l}, \forall j \in \{1,2,\ldots,M_{k_l}-1\}. \tag{3.21}$$

Finally, (3.21), (3.13) and (3.14) imply

$$\begin{aligned} a_{n,0}(x) \cdot M_{k_l} &= \sum_{i=1}^{l} D_{M_{k_i}}(x) \cdot \sum_{s=m_{k_i}-n_{k_i}}^{m_{k_i}-1} r_{k_i}^{s}(x) \\ &= \sum_{i=1}^{l-1} n_{k_i} M_{k_i} + M_{k_l} \sum_{s=m_{k_l}-n_{k_l}}^{m_{k_l}-1} e^{\frac{2\pi i x_{k_l} s}{m_{k_l}}}, \end{aligned}$$

where we have used $D_{M_{k_i}}(x) = M_{k_i}, i = \overline{1,l}$, and

$$r_{k_1}(x) = \cdots = r_{k_{l-1}}(x) = 1, r_{k_l}(x) = e^{\frac{2\pi i x_{k_l} s}{m_{k_l}}} \text{ (since } x \in I_{k_l}). \qquad \square$$

Remark 3.1. From (3.14) and (3.15) we see that (3.12) gives us a decomposition of the Dirichlet kernel D_n into the sum of functions with disjoint supports.

Remark 3.2. We can define the function $a_{n,0}$ in an arbitrary way outside the set I_{k_l}, because of (3.12) and the fact that $D_{M_{k_l}}$ vanishes outside the set I_{k_l}.

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