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THE CAUCHY PROBLEM FOR QUASILINEAR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We use the contracting mapping principle to prove that under some mild restrictions, the Cauchy problem for quasilinear systems of functional differential equations with retarded arguments has a unique solution. As a consequence of this result, we obtain that the Cauchy problem for linear systems of functional differential equations with locally integrable coefficients and with locally measurable retarded arguments has a unique solution. We show that similar results can be obtained for the Cauchy co problem of quasilinear systems of functional differential equations with advanced arguments.

1. INTRODUCTION

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Let $F_k(t,u_{11},\dots,u_{1N},\dots,u_{nN})$, $k=\overline{1,n}$ be real-valued locally integrable in t and continuous in $u_{11},\dots,u_{nN}$ functions on $[t_0,\infty)\times\mathbb{R}^{nN}$, and let $\alpha_{kj}(t)$, $k=\overline{1,n}$, $j=\overline{1,N}$ be real-valued locally measurable functions on $[t_0,\infty)$. Consider the system of functional differential equations

$$\phi_k'(t)=F_k(t,\phi_1(\alpha_{11}(t)),\dots,\phi_1(\alpha_{1N}(t)),\dots,\phi_n(\alpha_{n1}(t)),\dots,\phi_n(\alpha_{nN}(t))),\ \ t\ge t_0, \tag{1.1}$$

$k=\overline{1,n}$. Let $r_k(t)$, $k=\overline{1,n}$ be real-valued continuous functions on $(-\infty,t_0]$. By a Cauchy problem for the system (1.1) we mean to find a real-valued continuous vector function $(\phi_1(t),\dots,\phi_n(t))$ on $\mathbb{R}$, which is absolutely continuous on $[t_0,\infty)$, and which satisfies (1.1) almost everywhere on $[t_0,\infty)$ and the initial conditions

$$\phi_k(t)=r_k(t),\ \ t\le t_0,\ \ k=\overline{1,n}. \tag{1.2}$$

Throughout this paper we will assume that the following conditions are satisfied:
(L) (the Lipshits's condition)

$$|F_k(t,u_{11},\dots,u_{nN})-F_k(t,v_{11},\dots,v_{nN})|\le f_k(t)\sum_{m=1}^{n}\sum_{j=1}^{N}|u_{mj}-v_{mj}|, t\ge t_0, u_{mj},v_{mj}\in\mathbb{R},$$

where $f_k(t)$, $k=\overline{1,n}$ are locally integrable functions on $[t_0,\infty)$;

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(LI) (local integrability)

for any locally measurable functions $\psi_{11}(t),\dots,\psi_{1N}(t),\dots,\psi_{n1}(t),\dots,\psi_{nN}(t)$ on $[t_0,\infty)$ the superpositions

$$F_k(t,\psi_{11}(t),\dots,\psi_{nN}(t)),\qquad k=\overline{1,n}$$

are locally integrable on $[t_0,\infty)$;

(Ret) (the retorsion conditions)

$$\alpha_{kj}(t)\le t,\qquad t\ge t_0,\qquad k=\overline{1,n},\qquad j=\overline{1,N}.$$

Remark 1.1. The condition (LI) is satisfied if in particular

$$F_k(t,u_{11},\dots,u_{nN})=\sum_{m=1}^{M} f_{km}(t)g_{km}(u_{11},\dots,u_{nN}),\ \ t\ge t_0,\ \ u_{11},\dots,u_{nN}\in\mathbb{R}$$

where $f_{km}(t),\ k=\overline{1,n},\ m=\overline{1,M}$ are locally integrable functions on $[t_0,\infty)$, $g_{km}(u_{11},\dots,u_{nN}),\quad k=\overline{1,n},\ m=\overline{1,M}$ are continuous functions on $\mathbb{R}^{nN}$.

Let $G_k(t,u_{11},\dots,u_{1N},\dots,u_{nN}),\ k=\overline{1,n}$ be real-valued locally integrable in t and continuous in $u_{11},\dots,u_{nN}$ functions on $(-\infty,\tau_0]\times\mathbb{R}^{nN}$, and let $\beta_{kj}(t),\ k=\overline{1,n},\ j=\overline{1,N}$ be real-valued locally measurable functions on $(-\infty,\tau_0]$. Consider the system of functional differential equations

$$\phi_k'(t)=G_k(t,\phi_1(\beta_{11}(t)),\dots,\phi_1(\beta_{1N}(t)),\dots,\phi_n(\beta_{n1}(t)),\dots,\phi_n(\beta_{nN}(t))),\ \ t\le\tau_0,\tag{1.3}$$

$k=\overline{1,n}$. Let $s_k(t),\ k=\overline{1,n}$ be real-valued continuous functions on $(-\infty,t_0]$. By a Cauchy co problem for the system (1.3) we mean to find a real-valued continuous vector function $(\phi_1(t),\dots,\phi_n(t))$ on $\mathbb{R}$, which is absolutely continuous on $(-\infty,\tau_0]$, and which satisfies (1.3) almost everywhere on $(-\infty,\tau_0]$ and the terminal conditions

$$\phi_k(t)=s_k(t),\ \ t\ge\tau_0,\ \ k=\overline{1,n}.\tag{1.4}$$

Throughout this paper we will assume that the following conditions are satisfied:

(L^*) (the Lipshits's condition)

$$|G_k(t,u_{11},\dots,u_{nN})-G_k(t,v_{11},\dots,v_{nN})|\le h_k(t)\sum_{m=1}^{n}\sum_{j=1}^{N}|u_{mj}-v_{mj}|,t\ge t_0,u_{mj},v_{mj}\in\mathbb{R},$$

where $h_k(t),\ k=\overline{1,n}$ are locally integrable functions on $(-\infty,\tau_0]$;

(LI^*) (local integrability)

for any locally measurable functions $\psi_{11}(t),\dots,\psi_{1N}(t),\dots,\psi_{n1}(t),\dots,\psi_{nN}(t)$ on $(-\infty,\tau_0]$ the superpositions

$$G_k(t,\psi_{11}(t),\dots,\psi_{nN}(t)),\qquad k=\overline{1,n}$$

are locally integrable on $(-\infty,\tau_0]$;

(Adv) (the advance conditions)

$$\beta_{kj}(t)\ge t,\qquad t\le\tau_0,\qquad k=\overline{1,n},\qquad j=\overline{1,N}.$$

Functional differential equations and systems of such equations appear in various areas of natural science, such as in Economics (see, e.g. [1-6]), in probability theory (see, e.g., [7-9]), in Biology (see, e.g., the predator-prey model of Volterra [11, p. 3], the model of circumnutation of plants [11, p.3], the model of dynamics of individuals, infected by gonorrhea [11, p. 4]), in electrodynamics [11, p. 7], so on. Therefore the qualitative study of functional differential equations is very actual. For a qualitative study of the solutions of these equations and systems of such equations the main focus is on the study of the case when the studying solution is continuable on the whole semi axis. Therefore, the study of the Cauchy problem (1.1), (1.2) (the Cauchy co problem (1.3), (1.4)) is very actual as well. Probably the Cauchy problem is studied systematically and solved for linear systems of functional differential equations with coefficients from wide classes of functions but with specified deviations of the argument of the form $t+\xi_k,\ \ \xi_k = const,\ \ k = 1,2,\dots$ (see, e.g., [11, 12]) and for some particular classes of equations with deviations of the argument from wide classes of functions (see, e.g., [12-14]). However, probably, in most of cases the global existence of solutions of studying equations and (or) systems of equations is only assumed, consequently making this way their results conditional (see, e.g., [15-25]).

In this way we show that the Cauchy problem (1.1), (1.2) (as well as the Cauchy co problem (1.3), (1.4)) under some mild restrictions has a unique solution, which weakens (to some extent) the conditionality of some results, in which the global existence of solutions of studying equations and systems of equations is only assumed.

2. Auxiliary propositions

Let $t_0 \le t_1 < t_2 < \infty$ and let $\psi_1(t),\dots,\psi_n(t)$ be real-valued continuous functions on $(-\infty,t_2]$. Denote by $AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$ the set of all real-valued continuous vector functions $x(t) \equiv (x_1(t),\dots,x_n(t))$ on $(-\infty,t_2]$ such that $x(t)$ is absolutely continuous on $[t_1,t_2]$ and $x_k(t) = \psi_k(t),\ \ t \le t_1,\ \ k = \overline{1,n}$. Obviously, $AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$ is a closed (full) metric space with the distance $\rho(x,t) \equiv \sum\limits_{k=1}^{n} \max\limits_{t\in[t_1,t_2]} |x_k(t) - y_k(t)|$ between its elements $x(t) \equiv (x_1(t),\dots,x_n(t)),\ \ y(t) \equiv (y_1(t),\dots,y_n(t))$. For any $x(t) \equiv (x_1(t),\dots,x_n(t)) \in AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$ set

$$(I_{\psi_1,\dots,\psi_n,t_1,t_2}x)(t) \equiv \begin{cases} x(t), & t \le t_1 \\ ((I_{1,\psi_1,\dots,\psi_n,t_1,t_2}x)(t),\dots,(I_{n,\psi_1,\dots,\psi_n,t_1,t_2}x)(t)), & t \in [t_1,t_2], \end{cases}$$

where $(I_{k,\psi_1,\dots,\psi_n,t_1,t_2}x)(t) \equiv \psi_k(t_1)+$

$$+\int_{t_1}^{t} F_k(\tau, x_1(\alpha_{11}(\tau)),\dots,x_1(\alpha_{1N}(\tau)),\dots,x_n(\alpha_{n1}(\tau)),\dots,x_n(\alpha_{nN}(\tau)))d\tau,\ \ t \in [t_1,t_2],$$

$k = \overline{1,n}$. Obviously under the restrictions (LI) and (Ret), the operator $I_{\psi_1,\dots,\psi_n,t_1,t_2}$ is a mapping from $AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$ into itself.

Theorem 2.1. *Let the conditions (L), (LI) and (Ret) be satisfied. If* $N\sum\limits_{k=1}^{n}\int\limits_{t_1}^{t_2} f_k(\tau)d\tau < 1$, *then* $I_{\psi_1,\dots,\psi_n,t_1,t_2}$ *is a contracting mapping in* $AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$.

Proof. It follows from (L) and (LI) that for every $x(t) \equiv (x_1(t),\dots,x_n(t))$, $y(t) \equiv (y_1(t),\dots,y_n(t)) \in AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$ the following chain of relations is valid.

$$\rho(I_{\psi_1,\dots,\psi_n,t_1,t_2}x, I_{\psi_1,\dots,\psi_n,t_1,t_2}y) = \sum_{k=1}^{n}\max_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} F_k(\tau,x_1(\alpha_{11}(\tau)),\dots,x_n(\alpha_{nN}(\tau)))d\tau - \right.$$
$$\left. - \int_{t_1}^{t} F_k(\tau,y_1(\alpha_{11}(\tau)),\dots,y_n(\alpha_{nN}(\tau)))d\tau\right| \le \sum_{k=1}^{n}\max_{t\in[t_1,t_2]}\int_{t_1}^{t} f_k(\tau)\times$$
$$\times\left\{\sum_{m=1}^{n}\sum_{j=1}^{N}|x_m(\alpha_{mj}(\tau)) - y_m(\alpha_{mj}(\tau))|\right\}d\tau \le \sum_{k=1}^{n}\int_{t_1}^{t_2} f_k(\tau)d\tau D(x,y), \tag{2.1}$$

where $D(x,y) \equiv \max\limits_{\tau\in[t_1,t_2]}\sum\limits_{m=1}^{n}\sum\limits_{j=1}^{N}|x_m(\alpha_{mj}(\tau)) - y_m(\alpha_{mj}(\tau))|$. From (Ret) it follows that $D(x,y) \le N\rho(x,y)$. This together with (2.1) implies that $\rho(I_{\psi_1,\dots,\psi_n,t_1,t_2}x, I_{\psi_1,\dots,\psi_n,t_1,t_2}y) \le N\sum\limits_{k=1}^{n}\int\limits_{t_1}^{t_2} f_k(\tau)d\tau\rho(x,y)$. Therefore, if $N\sum\limits_{k=1}^{n}\int\limits_{t_1}^{t_2} f_k(\tau)d\tau < 1$, then $I_{\psi_1,\dots,\psi_n,t_1,t_2}$ is a contracting mapping in $AC^n_{\psi_1,\dots,\psi_n}[t_1,t_2]$. The theorem is proved. □

Let $-\infty < t_1 < t_2 \le \tau_0$ and let $\chi_1(t),\dots,\chi_n(t)$ be real-valued continuous functions on $[t_2,+\infty)$. Denote by $AC^{n*}_{\chi_1,\dots,\chi_n}[t_1,t_2]$ the set of all real-valued continuous vector functions $x(t) \equiv (x_1(t),\dots,x_n(t))$ on $[t_1,\infty)$ such that $x(t)$ is absolutely continuous on $[t_1,t_2]$ and $x_k(t) = \chi_k(t)$, $t \ge t_2$, $k = \overline{1,n}$. Obviously, $AC^{n*}_{\chi_1,\dots,\chi_n}[t_1,t_2]$ is a closed (full) metric space with the distance $\rho(x,t) \equiv \sum\limits_{k=1}^{n}\max\limits_{t\in[t_1,t_2]}|x_k(t) - y_k(t)|$ between its elements $x(t) \equiv (x_1(t),\dots,x_n(t))$, $y(t) \equiv (y_1(t),\dots,y_n(t))$. For any $x(t) \equiv (x_1(t),\dots,x_n(t)) \in AC^{n*}_{\psi_1,\dots,\psi_n}[t_1,t_2]$ set

$$(J_{\chi_1,\dots,\chi_n,t_1,t_2}x)(t) \equiv \begin{cases} x(t), & t \ge t_2, \\ ((J_{1,\chi_1,\dots,\chi_n,t_1,t_2}x)(t),\dots,(J_{n,\chi_1,\dots,\chi_n,t_1,t_2}x)(t)), & t \in [t_1,t_2], \end{cases}$$

where $(J_{k,\chi_1,\dots,\chi_n,t_1,t_2}x)(t) \equiv \chi_k(t_2) +$
$+\int\limits_{t}^{t_2} G_k(\tau,x_1(\beta_{11}(\tau)),\dots,x_1(\beta_{1N}(\tau)),\dots,x_n(\beta_{n1}(\tau)),\dots,x_n(\beta_{nN}(\tau)))d\tau$, $t \in [t_1,t_2]$, $k = \overline{1,n}$. Obviously under the restrictions (LI^*) and (Adv) the operator $J_{\chi_1,\dots,\chi_n,t_1,t_2}$ is a mapping from $AC^{n*}_{\psi_1,\dots,\psi_n}[t_1,t_2]$ into itself. By analogy to the proof of Theorem 2.1 we can prove the following theorem.

Theorem 2.2. *Let the conditions (L^*), (LI^*) and (Adv) be satisfied. If $N \sum_{k=1}^{n} \int_{t_1}^{t_2} h_k(\tau)d\tau < 1$, then $J_{\chi_1,\ldots,\chi_n,t_1,t_2}$ is a contracting mapping in $AC^{n*}_{\psi_1,\ldots,\psi_n}[t_1,t_2]$.*

3. Main results.

By a solution of the system (1.1) on $[t_1,t_2)$ $(t_0 \le t_1 < t_2 \le \infty)$ we mean a real-valued continuous vector function $(\phi_1(t),\ldots,\phi_n(t))$ on $(-\infty,t_2)$, which is absolutely continuous on $[t_1,t_2)$ and satisfies (1.1) almost everywhere on $[t_1,t_2)$.

Definition 3.1. *An interval $[t_0,T)$ $(t_0 < T \le \infty)$ is called the maximum existence interval for a solution $\Phi(t)$ of the system (1.1), if $\Phi(t)$ exists on $[t_0,T)$ and cannot be continued to the right from T as a solution of (1.1).*

Theorem 3.1. *Let the conditions (L), (LI) and (Ret) be satisfied. Then the Cauchy problem (1.1), (1.2) has a unique solution.*

Proof. Since $f_k(t)$, $k=\overline{1,n}$ are locally integrable chose $t_1 > t_0$ so close to t_0 that

$$N\sum_{k=1}^{n}\int_{t_0}^{t_1} f_k(\tau)d\tau < 1.$$

Then by Theorem 2.1 it follows from (L), (LI) and (Ret) that the operator $I_{r_1,\ldots,r_n,t_0,t_1}$ is a contracting mapping in $AC^n_{r_1,\ldots,r_n}[t_0,t_1]$. Therefore, according to the contracting mapping principle, the operator $I_{r_1,\ldots,r_n,t_0,t_1}$ has a unique fixed point $\Phi(t) \equiv (\phi_1(t),\ldots,\phi_n(t))$ in $AC^n_{r_1,\ldots,r_n}[t_0,t_1]$, which is a solution of the system (1.1) on $[t_0,t_1)$, satisfying the initial conditions $\phi_k(t) = r_k(t)$, $t \le t_0$, $k=\overline{1,n}$. It follows from here that $\Phi(t)$ has a maximum existence interval. Let $[t_0,T)$ be that interval. We will show that

$$T=\infty. \tag{3.1}$$

Suppose

$$T<\infty. \tag{3.2}$$

Chose $\varepsilon > 0$ so small that

$$N\sum_{k=1}^{n}\int_{T-\varepsilon}^{T+\varepsilon} f_k(\tau)d\tau < 1.$$

Then by Theorem 2.1 it follows from (L), (LI) and (Ret) that $I_{\phi_1,\ldots,\phi_n,T-\varepsilon,T+\varepsilon}$ is a contracting mapping in $AC^n_{r_1,\ldots,r_n}[T-\varepsilon,T+\varepsilon]$, and, according to the contracting mapping principle, has a unique fixed point $\widetilde{\Phi}(t) \equiv (\widetilde{\phi}_1(t),\ldots,\widetilde{\phi}_n(t))$ in $AC^n_{r_1,\ldots,r_n}[T-\varepsilon,T+\varepsilon]$, which is a solution of the system (1.1) on $[T-\varepsilon,T+\varepsilon)$, satisfying the initial conditions

$$\widetilde{\phi}_k(t) = \phi_k(t),\ \ t \le T-\varepsilon,\ \ k=\overline{1,n}.$$

This means (since $\widetilde{\Phi}(t)$ is the unique) that $\widetilde{\Phi}(t)$ coincides with $\Phi(t)$ on $(-\infty,T]$ and is a continuation of $\Phi(t)$ on $(-\infty,T+\varepsilon]$ as a solution of the system (1.1) on

$[t_0, T+\varepsilon)$, which contradicts (3.2). The obtained contradiction proves (3.1). So, to complete the proof of the theorem it remains to show the uniqueness of $\Phi(t)$. Suppose there exists another solution $\Psi(t) \equiv (\psi_1(t), \dots, \psi_n(t))$ to the problem (1.1), (1.2), different from $\Phi(t)$. Then there exist $t_0 \le T_1 < T_2 < \infty$ such that

$$\Phi(t) = \Psi(t), \;\; t \le T_1, \;\; \Phi(t) \ne \Psi(t), \;\; t \in (T_1, t_2). \tag{3.3}$$

Without loss of generality we may assume that

$$N \sum_{k=1}^{n} \int_{T_1}^{T_2} f_k(\tau) d\tau < 1.$$

Then by virtue of Theorem 2.1 and the contracting mapping principle the operator $I_{\phi_1,\dots,\phi_n,T_1,T_2}$ has the unique fixed point $(v_1(t), \dots, v_n(t))$ in $AC^n_{r_1,\dots,r_n}[T_1, T_2]$, which is a unique solution of the system (1.1) on $[T_1, T_2)$, satisfying the initial conditions

$$v_k(t) = \phi_k(t), \;\; t \le T_1, \;\; k = \overline{1,n}.$$

We obtain a contradiction with (3.3). The obtained contradiction completes the proof of the theorem.

Let $a_{kjm}(t), \;\; b_k(t), \;\; k, j = \overline{1,n}, \;\; m = \overline{1,N}$ be real-valued locally integrable functions on $[t_0, \infty)$. Consider the linear system of functional differential equations

$$\phi_k'(t) = \sum_{j=1}^{n} \sum_{m=1}^{N} a_{kjm}(t) \phi_j(\alpha_{jm}(t)) + b_k(t), \qquad t \ge t_0, \;\; k = \overline{1,n}. \tag{3.4}$$

This system is a particular case of (1.1), for which the conditions (L) and (LI) are satisfied. Then from Theorem 3.1 we immediately obtain

Corollary 3.1. *Let (Ret) be satisfied. Then the Cauchy problem (3.4), (1.2) has a unique solution.*

Using Theorem 2.2 instead of Theorem 2.1 by analogy to the proof of Theorem 3.1 we can prove the following theorem.

Theorem 3.2. *Let the conditions (L^*), (LI^*) and (Adv) be satisfied. Then the Cauchy co problem (1.3), (1.4) has a unique solution.*

Let $c_{kjm}(t), \;\; d_k(t), \;\; k, j = \overline{1,n}, \;\; m = \overline{1,N}$ be real-valued locally integrable functions on $(-\infty, \tau_0]$. Consider the linear system of functional differential equations

$$\phi_k'(t) = \sum_{j=1}^{n} \sum_{m=1}^{N} c_{kjm}(t) \phi_j(\beta_{jm}(t)) + d_k(t), \qquad t \le \tau_0, \;\; k = \overline{1,n}. \tag{3.5}$$

This system is a particular case of (1.3), for which the conditions (L^*) and (LI^*) are satisfied. Then from Theorem 3.2 we immediately obtain the following corollary.

Corollary 3.2. *Let (Adv) be satisfied. Then the Cauchy co problem (3.5), (1.4) has a unique solution.*

REFERENCES

[1] R. Frisch and H. Holme, *The Characteristic Solutions of a Mixed Difference and Differential Equations Occurring in Economic Dynamics. Economica,* III (1935) pp. 225–239.

[2] M. Kalecki, *A Macrodinamic Theory of Business Cycles.* Economica, III (1935) pp. 327–344.

[3] A. Callender, D. R. Hertree, A. Porter, *Time-Lag in a Control System, Philos. Trans.* Roy. Soc. London (A), 235 (766) (1936) pp. 415–444.

[4] D. R. Hartree, A. Porter, A. Callender, A. B. Stevenson, *Time Lag in a Control System,* II. Proc. Roy. Soc. London (A) 161 (907) (1937) pp. 460–476.

[5] N. Minorsky, *Control Problems,* II, J. Frankl. Inst., 232: 6 (1941) pp. 519–551.

[6] H. Bateman, *The Control of an Elastic Fluid.* Bull. Amer. Math. Soc., 51 (1945) pp. 601–641.

[7] L. Silberstein, *On a Hystero-Differential Equation Arising in a Probability Problem.* Lond. Edinb. Dublin philos. Mag. (t), 29: 192 (1940) pp. 75–84.

[8] N. Hale, *On the Statistical Treatment of Counting Experiments in Nuclear Physics.* Ark. Mat. Astr. Fys., 33A: 11 (1946) pp 1–11.

[9] N. Hole, *On the Distribution on Counts in a Counting Apparatus.* Ark. Mat. Astr. Fys., 33:3 (1947) B:8, pp. 1–8.

[10] N. Minorsky, *Self-Exited Mechanical Oscillations.* J. Appl. Phys, 19 (1948) pp. 332–338.

[11] J. Hale, *Theory of Functional Differential Equations.* Applied Mathematical Sciences Vol. 3, Springer-Verlag, New York, Heidelberg, Berlin, 1977, 366 pages.

[12] R. Bellman, K. Cooke, *Differential-Difference Equations.* Academic Press, New York, London, 1963, 465 pages.

[13] L. Berezansky and E. Braverman, *Some Oscillation Problems for Second Order Linear Delay Differential Equations.* J. Math. Anal., Appl., vpl. 220, pp. 719–740 (1998).

[14] G. A. Grigorian, *Oscillation Criteria for the Second Order Linear Functional-Differential Equations with Locally Integrable Coefficients.* Sarajevo J. Math. Vol.14 (27), No.1, (2018), pp. 71–86.

[15] J. Džurina, *Oscillation theorems for Second Order Advanced Neutral Differential Equations.* Tatra Mt. Math. Publ. 48 (2011) pp. 61–71.

[16] R. Guo, Q. Hung and Q. Liu, *Some New Oscillation Criteria of Even-Order Quasi-linear Delay Differential Equations with Neutral Term,* Mathematics, 2021, 9, 2074, pp. 1–11.

[17] M. Pašić, *Parametric Excited Oscillation of Second-Order Functional-Differential Equations and Application to Duffing Equations with Time Delay Feedback.* Discrete Dyn. Nat. Soc., 2014, pp. 1–17 (875020).

[18] M. M. El-sheikh, R. Sallam and N. Mohamady. *On the Oscillation of Third Order Neutral Delay Differential Equations.* Appl. Math. Inf. Sci. Lett., 1, No. 3, (2-13) pp. 77–80.

[19] X. Lin, *Oscillation of Second-Order Neutral Differential Equations.* J. Math. Anal. Appl. 309 (2005) pp. 442–452.

[20] H.-J. Li, Ch.-Ch. Yen, *Oscillation of Nonlinear Functional Differential Equations of the Second Order.* Appl. Math. Lett., vol. 11, No. 1, (1998) pp. 71–77.

[21] Sh. Tang, T. Li, E. Thandapani, *Oscillation of Higher-Order Half-Linear Neutral Differential Equations.* Demonstratio Math., Vol. 46, No. 1, 2013, pp. 101–109.

[22] A. A. Soliman, R. A. Sallam, A. M. Hassan, *Oscillation Criteria of Second Order Nonlinear Neutral Differential Equations.* Int. J. Appl. Math. Research, 1 (3) (2012) pp. 314–322.

[23] J. R. Graef, S. Murugadas, E. Thandapani, *Oscillation Criteria for Second Order Neutral Delay Differential Equations with Mixed Nonlinearities.* International Electronic J. Pure and Appl. Math., Vol. 2. No. 1, 2010, pp. 85–99.

[24] P. Wang, Zh. Xu, *On the Oscillation of a Two-Dimensional Delay Differential System.* Int. J. Qualitative Theory of Diff. Eq. and Appl., vol. 2., No. 1 (2008) pp. 38–46.

[25] R. P. Agarwal, M. Bohner, T. Li, Ch. Zheng, *Oscillation of Second-Order Emden-Fowler Neutral Delay Differential Equations.* Ann. Mat. Pura Appl. (4) 193 (2014) pp. 1861–1875.

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