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UNIQUENESS OF THE L-FUNCTION AND MEROMORPHIC FUNCTION CONCERNING WEAKLY WEIGHTED SHARING

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ABSTRACT. We introduce homogeneous differential polynomials of a L-function and of a meromorphic function and investigate the uniqueness results using the concept of weakly weighted sharing.

1. INTRODUCTION, DEFINITIONS AND PREVIOUS RESULTS

Since a lot of works are done on the general meromorphic function, we draw our attention to the L-function. L-function $\mathcal{L}$ and ξ are non-constant meromorphic functions are defined in $\mathbb{C}$. We adopt the standard results of Nevanlinna's value distribution theory (see [2, 12, 13]) and of L-function(see [8]). The Nevanlinna's characteristic function is denoted by $T(r,\xi)$ and $S(r,\xi)$ is a small quantity defined by $o(T(r,\xi)) = S(r,\xi)$, with $r \to \infty$ and $r \notin E$ where $E \subseteq \mathbb{R}^+$ and the measure of E is finite.
Since the L-function with Reimann Zeta function as a prototype, was mainly studied in Number theory and as L-function is a meromorphic function, it is interesting to study the distribution of values of the function.
In this article we work on the Selberg class L-function and it will be denoted by $\mathcal{L}$. L-function, $\mathcal{L}$ include essentially those dirichlet series that satisfy the Reimann hypothesis and also include Reimann Zeta function, $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$. L-function, $\mathcal{L}$ is taken as $\mathcal{L} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}$ where $s \in \mathbb{C}$ and $\mathcal{L}$ satisfy the following axioms:
(i) Ramanujan Hypothesis: For every $\varepsilon(>0)$, $a(s) << m^{\varepsilon}$.
(ii) Analytic Continuation: There exist a non-negative integer η such that $(s-1)^{\eta}\mathcal{L}$ is an entire function of finite order.
(iii) Functional Equation: $\mathcal{L}$ satisfies a functional equation
$\chi_{\mathcal{L}}(s) = \omega\overline{\chi_{\mathcal{L}}(1-\overline{s})}$ where $\chi_{\mathcal{L}} = \mathcal{L}\rho^s \prod_{j=1}^{\tau} \Gamma(\lambda_j s + \nu_j)$ with $\rho \in \mathbb{R}^+$, $\nu_j, \omega \in \mathbb{C}$, and $Re(\nu_j) \geq 0$ and $|\omega| = 1$.
(iv) Euler Production Hypothesis: $\mathcal{L} = \prod_q \exp(\sum_{\tau=1}^{\infty}) \frac{b(q^{\tau})}{q^{\tau s}}$, with suitable coefficients $b(q^{\tau})$ that satisfy $b(q^{\tau}) << q^{\tau\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers q.

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Suppose $S(\xi)$ is a collection of small functions of ξ and hence $\mathbb{C}\cup\{\infty\}\subseteq S(\xi)$. Generally, we discuss the distribution of the zeros of L-functions. In general we discuss distribution of roots of the equation $\mathcal{L}(s)=\rho$ where $\rho\in\mathbb{C}$ or the values of the pre-image set $\mathcal{L}^{-1}=\{s\in\mathbb{C}:\mathcal{L}(s)=\rho\}$.
Suppose $\xi(z)$ and $\xi_1(z)$ are two meromorphic functions in the complex plane and We say that $\xi(z)$ and $\xi_1(z)$ share ρ CM(Counting Multiplicities) if they share the value ρ and if the zeros of the equations $\xi(z)-\rho=0$ and $\xi_1(z)-\rho=0$ have the same multiplicity. Again we say ξ and ξ_1 share a value $\rho\in\mathbb{C}\cup\{\infty\}$ IM(Ignoring Multiplicities) if $\xi^{-1}(\rho)=\xi_1^{-1}(\rho)$ as two sets in $\mathbb{C}$. We denote the notion that ξ and ξ_1 share ρ with weight τ by (ρ,τ). Hence $(\rho,0)$ and (ρ,∞) assert that ξ and ξ_1 share ρ IM and CM accordingly.
We defined deficiency $\delta(\rho,\xi)$ and ramification index $\Theta(\rho,\xi)$ of ρ for the function ξ by,

$$\delta(\rho,\xi)=1-\limsup_{r\to\infty}\frac{N(r,\rho;\xi)}{T(r,\xi)},$$
$$\Theta(\rho,\xi)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\rho;\xi)}{T(r,\xi)},$$

accordingly. The set of all ρ-points of $\xi(z)$ where an ρ point with multiplicity m is counted m times if $m\leq\tau$ and $\tau+1$ times if $m>\tau$ is denoted by $E(\rho,\xi)(\tau)$ and if $E(\rho,\xi)(\tau)=E(\rho,\xi_1)(\tau)$, then we say that $\xi(z)$ and $\xi_1(z)$ share the value ρ with weight τ.

Definition 1.1. [3] *Let $\tau\in\mathbb{N}\cup\{\infty\}$ and $N(r,\rho;\xi)(\leq\tau)$ denote the counting function for the zeros of $\xi-\rho$ with multiplicity $\leq\tau$ and $N(r,\rho;\xi)(\geq\tau)$ denote the counting function for the zeros of $\xi-\rho$ with multiplicity $\geq\tau$ (corresponding reduced counting functions are denoted by $\overline{N}(r,\rho;\xi)(\leq\tau)$ and $\overline{N}(r,\rho;\xi)(\geq\tau)$ accordingly). Let $N(r,\rho;\xi)(\tau)$ denote the counting function for the zeros of $\xi-\rho$, where multiplicity m is counted m times if $m\leq\tau$ and τ times if $m>\tau$ and*

$$N(r,\rho;\xi)(\tau)=\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\xi)(\geq 2)+\ldots+\overline{N}(r,\rho;\xi)(\geq\tau).$$

We define the quantity $\delta(\sigma,\xi)(\tau)$ by

$$\delta(\rho,\xi)(\tau)=1-\limsup_{r\to\infty}\frac{N(r,\rho;\xi)(\tau)}{T(r,\xi)},$$

and hence $\delta(\rho,\xi)(\tau)\geq\delta(\rho,\xi)$.

Definition 1.2. [6] *Let ξ and ξ_1 be two non-constant meromorphic functions and $\rho\in\mathbb{C}$. The counting function of all common zeros with the same multiplicities of $\xi-\rho=0$ and $\xi_1-\rho=0$ is denoted by $N(r,\rho)(E)$ and the counting function of all common zeros in ignorance of multiplicities is denoted by $N(r,\rho)(0)$ ($\overline{N}(r,\rho)(E)$ and $\overline{N}(r,\rho)(0)$ are corresponding reduce counting functions). We say that ξ and ξ_1 share ρ CM weakly, if,*

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\xi_1)-2\overline{N}(r,\rho)(E)=S(r,\xi)+S(r,\xi_1),$$

and say ξ *and* ξ_1 *share* ρ *IM weakly, if,*

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\xi_1)-2\overline{N}(r,\rho)(0)=S(r,\xi)+S(r,\xi_1).$$

In 2006, S. Lin and W. Lin [6] introduced the concept of weakly weighted sharing:

Definition 1.3. [6] *Let* ξ *and* ξ_1 *be two non-constant meromorphic functions and* $\rho\in S(\xi)\cap S(\xi_1)$, $\tau\in\mathbb{Z}^+\cup\{\infty\}$. *If*

$$\overline{N}(r,\rho;\xi)(\leq\tau)+\overline{N}(r,\rho;\xi_1)(\leq\tau)-2\overline{N}(r,\rho)(E(\leq\tau))=S(r,\xi)+S(r,\xi_1),$$

$$\begin{aligned}\overline{N}(r,\rho;\xi)(\geq\tau+1)+\overline{N}(r,\rho;\xi_1)(\geq\tau+1)-2\overline{N}(r,\rho)(0(\geq\tau+1))\\=S(r,\xi)+S(r,\xi_1),\end{aligned}$$

or, if $\tau=0$ *then,*

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\xi_1)-2\overline{N}(r,\rho)(0)=S(r,\xi)+S(r,\xi_1),$$

then we say that ξ *and* ξ_1 *weakly share* ρ *with weight* τ *and the notion will be denoted by* $\omega(\rho,\tau)$.

Let ϕ and ψ share 1 IM weakly. Then the counting function of 1 points of ϕ with multiplicities greater than of 1 points of ψ is denoted by $\overline{N}(r,1;\phi)(L)$. $\overline{N}(r,1;\psi)(L)$ is similarly defined.

In 2017, F. Liu, X.M. Li and H.X. Yi [7] consider a L-function and a meromorphic function and established following relation when differential polynomial of a L-function and a meromorphic function share a value:

Theorem 1.1. [7] *Let* $\xi(z)$ *be a non-constant meromorphic function and* $\mathcal{L}$ *be a L-function such that* $[\xi^n]^{(k)}$ *and* $[\mathcal{L}^n]^{(k)}$ *share* 1 *CM, where* $n,k\in\mathbb{Z}^+$. *If* $n>3k+6$ *then,* $\xi\equiv\kappa\mathcal{L}$ *where* κ *is a constant and* $\kappa^n=1$.

Again, X.M. Li, F. Liu and H.X. Yi [5] improve their own result in theorem 1.1 in following manner:

Theorem 1.2. [5] *Let* $\xi(z)$ *be a non-constant meromorphic function and* $\mathcal{L}$ *be a L-function such that* $[\xi^n(\xi-1)]^{(k)}$ *and* $[\mathcal{L}^n(\mathcal{L}-1)]^{(k)}$ *share* 1 *CM, where* $n,k\in\mathbb{Z}^+$. *If* $n>3k+9$ *and* $k\geq 2$, *then,* $\xi\equiv\mathcal{L}$.

In 2018, W. J. Wao and J. F. Chen [1], generalized the result of X.M. Li, F. Liu and H.X. Yi [5] for more general differential polynomial and obtained the following uniqueness results for the L-function:

Theorem 1.3. [1] *Let* $\xi(z)$ *be a non-constant meromorphic function and* $\mathcal{L}$ *be a L-function such that* $[\xi^n(\xi-1)^p]^{(k)}$ *and* $[\mathcal{L}^n(\mathcal{L}-1)^p]^{(k)}$ *share* 1 *CM, where* $n,p,k\in\mathbb{Z}^+$. *If* $n>p+3k+6$ *and* $k\geq 2$, *then,* $\xi\equiv\mathcal{L}$ *or,* $\xi^n(\xi-1)^p=\mathcal{L}^n(\mathcal{L}-1)^p$.

Theorem 1.4. [1] *Let* $\xi(z)$ *be a non-constant meromorphic function and* $\mathcal{L}$ *be a L-function such that* $[\xi^n(\xi-1)^p]^{(k)}$ *and* $[\mathcal{L}^n(\mathcal{L}-1)^p]^{(k)}$ *share* 1 *IM, where* $n,p,k \in \mathbb{Z}^+$. *If* $n > 4p+7k+11$ *and* $k \geq 2$, *then,* $\xi \equiv \mathcal{L}$ *or,* $\xi^n(\xi-1)^p = \mathcal{L}^n(\mathcal{L}-1)^p$.

In 2018, H. P. Waghamore and S.H. Naveenkumar [10] proved the result on the weighted share of a L-function and a meromorphic function as follows:

Theorem 1.5. [10] *Let* $\xi(z)$ *be a non-constant meromorphic function and* $\mathcal{L}$ *be a L-function and* $n,\tau \in \mathbb{Z}^+$. *Suppose* $(\xi^n)^{(k)}$ *and* $(\mathcal{L}^n)^{(k)}$ *share* $(q(z),\tau)$, *where* $q(z)$ *is a non-zero polynomial with* $deg(q) = d_q$ *and* ξ *and* $\mathcal{L}$ *share* ∞ *IM. If one of following conditions holds:*

(i) $\tau \geq 3$ *and* $n > 3k+4$;

(ii) $\tau = 2$ *and* $n > 3k+6$;

(iii) $\tau = 1$ *and* $n > 3k+7$;

(iv) $\tau = 0$ *and* $n > 7k+11$;

then $\xi = \kappa\mathcal{L}$ *where* κ *is a constant and* $\kappa^n = 1$.

We introduce homogenous differential polynomials of L-function and of meromorphic function and discuss the value distribution of such polynomial functions through a different approach and technique and investigate a uniqueness result in view of weakly weighted sharing.

Definition 1.4. *We define a homogenous differential polynomial as*

$$Q(z) = \sum_{r=1}^{n} a_r \prod_{s=0}^{p} (z^{(s)})^{t_{rs}},$$

where $n(\geq 1), p(\geq 0), r,s,t \in \mathbb{Z}^+ \cup \{0\}$ *and the degree of* $Q(z)$ *is* d_Q *where* $d_Q = \sum_{s=0}^{p} t_{rs}$. *We define* D *by*

$$D = \max_{1\leq r\leq n} \sum_{s=0}^{p} s t_{rs}.$$

We assert our main result on homogeneous differential polynomial in the following section:

2. Main Results

Theorem 2.1. *Let* ξ *be a non-constant meromorphic function and* $\mathcal{L}$ *be a L-function, and* $\tau \in \mathbb{Z}$, $\rho \in S(\xi) \cap S(\mathcal{L})$. *Suppose* $Q(\xi)$ *and* $Q(\mathcal{L})$ *share* $\omega(\rho,\tau)$. *If one of the following conditions holds:*

(i) $2\delta(0,\xi) + \frac{D+4}{d_Q}\Theta(\infty,\xi) > \frac{D+d_Q+4}{d_Q}$ *and* $2\delta(0,\mathcal{L}) > 1$ *when* $\tau \geq 2$;

(ii) $\frac{5}{2}\delta(0,\xi) + \frac{3D+9}{2d_Q}\Theta(\infty,\xi) > \frac{3D+3d_Q+9}{2d_Q}$ *and* $\frac{5}{2}\delta(0,\mathcal{L}) > 1$ *when* $\tau = 1$;

(iii) $5\delta(0,\xi) + \frac{4D+7}{d_Q}\Theta(\infty,\xi) > \frac{4D+4d_Q+7}{d_Q}$ *and* $5\delta(0,\mathcal{L}) > 1$ *when* $\tau = 0$

then, either $Q(\xi) = Q(\mathcal{L})$ *or,* $Q(\xi)Q(\mathcal{L}) = \rho^2$.

Corollary 2.1. *Taking a meromorphic function with a finite number of poles except the non-constant meromorphic function in theorem 2.1, and using the notion* $N(r,\infty;\xi)=S(r,\xi)$ *we also can establish another result.*

Remark 2.1. We also can introduce an entire function other than a meromorphic function with a finite number of poles in the corollary.

3. Lemmas

In this section we state some lemmas which play an important role in proving our theorems.

Lemma 3.1. [11] *Let* ξ *be a non-constant meromorphic function,* $a_0,a_1,a_2,\ldots,a_n(\neq 0)$ *complex constants and* $n,i\in\mathbb{Z}^+$. *Then* $T(r,\sum_{i=0}^{n}a_i\xi^i)=nT(r,\xi)+S(r,\xi)$.

Lemma 3.2. [4] *Let* ξ *be a non-constant meromorphic function and* $Q(\xi)$ *defined as above. Then,*

(*i*) $T(r,Q)=d_QT(r,\xi)+D\overline{N}(r,\infty;\xi)+S(r,\xi)$;
(*ii*) $N(r,0;Q)\leq T(r,Q)-d_QT(r,\xi)+d_QN(r,0;\xi)+S(r,\xi)\leq D\overline{N}(r,\infty;\xi)+d_QN(r,0;\xi)+S(r,\xi)$.

Lemma 3.3. [6] *Let* ϕ *and* ψ *be two non-constant meromorphic functions sharing* $\omega(1,\tau)$ *where* $\tau\in\mathbb{Z}^+\cup\{0\}\cup\{\infty\}$ *and let,*

$$\Omega=(\frac{\phi^{(2)}}{\phi^{(1)}}-\frac{2\phi^{(1)}}{\phi-1})-(\frac{\psi^{(2)}}{\psi^{(1)}}-\frac{2\psi^{(1)}}{\psi-1}).$$

If $\Omega\not\equiv 0$, *then,*

(*i*) $T(r,\phi)\leq N(r,\infty;\phi)(2)+N(r,\infty;\psi)(2)+N(r,0;\phi)(2)+N(r,0;\psi)(2)+S(r,\phi)+S(r,\psi)$ *when* $2\leq\tau\leq\infty$;
(*ii*) $T(r,\phi)\leq N(r,\infty;\phi)(2)+N(r,\infty;\psi)(2)+N(r,0;\phi)(2)+N(r,0;\psi)(2)+\overline{N}(r,1;\phi)(L)+S(r,\phi)+S(r,\psi)$ *when* $\tau=1$;
(*iii*) $T(r,\phi)\leq N(r,\infty;\phi)(2)+N(r,\infty;\psi)(2)+N(r,0;\phi)(2)+N(r,0;\psi)(2)+2\overline{N}(r,1;\phi)(L)+\overline{N}(r,1;\psi)(L)+S(r,\phi)+S(r,\psi)$ *when* $\tau=0$;

and the same inequality holds for $T(r,\psi)$.

Lemma 3.4. [9] *Let* ϕ *and* ψ *be non-constant meromorphic functions sharing* $\omega(1,1)$. *Then,*

$$\overline{N}(r,1;\phi)(L)\leq\frac{1}{2}\overline{N}(r,0;\phi)+\frac{1}{2}\overline{N}(r,\infty;\phi)+S(r,\phi).$$

Lemma 3.5. [9] *Let* ϕ *and* ψ *be non-constant meromorphic functions sharing* $\omega(1,0)$. *Then,*

$$\overline{N}(r,1;\phi)(L)\leq\overline{N}(r,0;\phi)+\overline{N}(r,\infty;\phi)+S(r,\phi).$$

Lemma 3.6. [8] *Let $\mathcal{L}$ be an L-function with degree $d_{\mathcal{L}}$. Then,*

$$T(r,\mathcal{L}) = \frac{d_{\mathcal{L}}}{\Pi} r\log r + O(r).$$

Lemma 3.7. *If $\mathcal{L}$ is an L-function, then $N(r,\infty;\mathcal{L}) = \overline{N}(r,\infty;\mathcal{L}) = S(r,\mathcal{L})$.*

Proof. From the definition of L-function, it has at most one pole in $\mathbb{C}$. Then obviously $N(r,\infty;\mathcal{L}) = \overline{N}(r,\infty;\mathcal{L}) = O(\log r)$. Therefore from Lemma 3.6, $N(r,\infty;\mathcal{L}) = \overline{N}(r,\infty;\mathcal{L}) = S(r,\mathcal{L})$. Hence the lemma follows. □

4. Proof of The Theorem

Proof. (proof of theorem (2.1)) First we assume that $d_{\mathcal{L}}$ is the degree of $\mathcal{L}$ and by applying [8], $d_{\mathcal{L}} = 2\sum_{j=1}^{k}\lambda_j$ where $k \in \mathbb{Z}^+$ and $\lambda_j \in \mathbb{R}^+$ is as defined in the axiom (iii) of the definition of L-function. From Lemma 3.6,

$$T(r,\mathcal{L}) = \frac{d_{\mathcal{L}}}{\Pi} r\log r + O(r).$$

Then f and $\mathcal{L}$ are transcendental meromorphic functions and $\mathcal{L}$ has only one pole at $z = 1$ in $\mathbb{C}$.

Let us consider $\phi = \frac{Q(\xi)}{\rho}$ and $\psi = \frac{Q(\mathcal{L})}{\rho}$. Since $Q(\xi)$ and $Q(\mathcal{L})$ share $\omega(\rho,\tau)$, then it immediately follows that ϕ and ψ share $\omega(1,\tau)$, except at the poles and zeros of ρ. We prove the result through the following cases,

Case 1. $\Omega \not\equiv 0$.

Now we discuss following three subcases

Subcase 1.1. $2 \le \tau \le \infty$.

We deduce from Lemma 3.3,

$$\begin{aligned} T(r,\phi) &\le N(r,\infty;\phi)(2) + N(r,\infty;\psi)(2) + N(r,0;\phi)(2) + N(r,0;\psi)(2) \\ &\quad + S(r,\phi) + S(r,\psi) \\ &\le 2\overline{N}(r,\infty;\phi) + 2\overline{N}(r,\infty;\psi) + N(r,0;\phi) + N(r,0;\psi) \\ &\quad + S(r,\phi) + S(r,\psi). \end{aligned}$$

With the help of Lemma 3.2 we deduce from the above inequality,

$$\begin{aligned} T(r,\phi) \le\ & 2\overline{N}(r,\infty;\phi) + 2\overline{N}(r,\infty;\psi) + T(r,\phi) - d_Q T(r,\xi) + d_Q N(r,0;\xi) \\ & + D\overline{N}(r,\infty;\mathcal{L}) + d_Q N(r,0;\mathcal{L}) + S(r,\xi) + S(r,\mathcal{L}), \end{aligned}$$

hence,

$$\begin{aligned} d_Q T(r,\xi) \le\ & 2\overline{N}(r,\infty;\xi) + (D+2)\overline{N}(r,\infty;\mathcal{L}) + d_Q N(r,0;\xi) \\ & + d_Q N(r,0;\mathcal{L}) + S(r,\xi) + S(r,\mathcal{L}). \end{aligned} \tag{4.1}$$

Similarly we obtain,

$$d_Q T(r,\mathcal{L}) \le 2\overline{N}(r,\infty;\mathcal{L}) + (D+2)\overline{N}(r,\infty;\xi) + d_Q N(r,0;\mathcal{L}) + d_Q N(r,0;\xi) + S(r,\xi) + S(r,\mathcal{L}). \tag{4.2}$$

Combining (4.1) and (4.2),

$$T(r,\xi) + T(r,\mathcal{L}) \le 2N(r,0;\xi) + \frac{D+4}{d_Q}\overline{N}(r,\infty;\xi) + 2N(r,0;\mathcal{L}) + \frac{D+4}{d_Q}\overline{N}(r,\infty;\mathcal{L}) + S(r,\xi) + S(r,\mathcal{L}).$$

From Lemma 3.7 we have, $N(r,\infty;\mathcal{L}) = \overline{N}(r,\infty;\mathcal{L}) = S(r,\mathcal{L})$ and hence $\Theta(\infty,\mathcal{L}) = 1$, then we deduce from the above inequality,

$$[2\delta(0,\xi) + \frac{D+4}{d_Q}\Theta(\infty,\xi) - \frac{D+d_Q+4}{d_Q}]T(r,\xi) + [2\delta(0,\mathcal{L}) - 1]T(r,\mathcal{L}) \le S(r,\xi) + S(r,\mathcal{L}).$$

This is a contradiction to our assumption.

Subcase 1.2. $\tau = 1$.
We deduce from Lemma 3.3,

$$\begin{aligned} T(r,\phi) &\le N(r,\infty;\phi)(2) + N(r,\infty;\psi)(2) + N(r,0;\phi)(2) + N(r,0;\psi)(2) \\ &\quad + \overline{N}(r,1;\phi)(L) + S(r,\phi) + S(r,\psi) \\ &\le 2\overline{N}(r,\infty;\phi) + 2\overline{N}(r,\infty;\psi) + N(r,0;\phi) + N(r,0;\psi) \\ &\quad + \overline{N}(r,1;\phi)(L) + S(r,\phi) + S(r,\psi). \end{aligned}$$

With the help of Lemma 3.2 we deduce from the above inequality,

$$T(r,\phi) \le 2\overline{N}(r,\infty;\phi) + 2\overline{N}(r,\infty;\psi) + T(r,\phi) - d_Q T(r,\xi) + d_Q N(r,0;\xi) + D\overline{N}(r,\infty;\mathcal{L}) + d_Q N(r,0;\mathcal{L}) + \overline{N}(r,1;\phi)(L) + S(r,\phi) + S(r,\psi),$$

hence from Lemma 3.4,

$$\begin{aligned} d_Q T(r,\xi) &\le 2\overline{N}(r,\infty;\xi) + (D+2)\overline{N}(r,\infty;\mathcal{L}) + d_Q N(r,0;\xi) \\ &\quad + d_Q N(r,0;\mathcal{L}) + \frac{1}{2}d_Q N(r,0;\xi) + \frac{1}{2}\overline{N}(r,\infty;\xi) \\ &\quad + \frac{1}{2}DN(r,\infty;\xi) + S(r,\xi) + S(r,\mathcal{L}) \\ &\le \frac{D+5}{2}\overline{N}(r,\infty;\xi) + (D+2)\overline{N}(r,\infty;\mathcal{L}) + \frac{3}{2}d_Q N(r,0;\xi) \\ &\quad + d_Q N(r,0;\mathcal{L}) + S(r,\xi) + S(r,\mathcal{L}). \end{aligned} \tag{4.3}$$

Similarly we obtain,

$$d_Q T(r,\mathcal{L}) \le \frac{D+5}{2}\overline{N}(r,\infty;\mathcal{L}) + (D+2)\overline{N}(r,\infty;\xi) + \frac{3}{2}d_Q\overline{N}(r,0;\mathcal{L}) + d_Q N(r,0;\xi) + S(r,\xi) + S(r,\mathcal{L}). \tag{4.4}$$

Combining (4.3) and (4.4),

$$
\begin{aligned}
T(r,\xi)+T(r,\mathcal{L}) \leq{}& \frac{5}{2}N(r,0;\xi)+\frac{3D+9}{2d_Q}\overline{N}(r,\infty;\xi)+\frac{5}{2}N(r,0;\mathcal{L})\\
&+\frac{3D+9}{2d_Q}\overline{N}(r,\infty;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}).
\end{aligned}
$$

From Lemma 3.7 we have, $N(r,\infty;\mathcal{L})=\overline{N}(r,\infty;\mathcal{L})=S(r,\mathcal{L})$ and hence $\Theta(\infty,\mathcal{L})=1$, then we deduce from the above inequality,

$$
\begin{aligned}
&[\frac{5}{2}\delta(0,\xi)+\frac{3D+9}{2d_Q}\Theta(\infty,\xi)-\frac{3D+3d_Q+9}{2d_Q}]T(r,\xi)\\
&+[\frac{5}{2}\delta(0,\mathcal{L})-1]T(r,\mathcal{L})\leq S(r,\xi)+S(r,\mathcal{L}).
\end{aligned}
$$

This is a contradiction to our assumption.

Subcase 1.3. $\tau=0$.

We deduce from Lemma 3.3,

$$
\begin{aligned}
T(r,\phi) \leq{}& N(r,\infty;\phi)(2)+N(r,\infty;\psi)(2)+N(r,0;\phi)(2)+N(r,0;\psi)(2)\\
&+2\overline{N}(r,1;\phi)(L)+\overline{N}(r,1;\psi)(L)+S(r,\phi)+S(r,\psi)\\
\leq{}& 2\overline{N}(r,\infty;\phi)+2\overline{N}(r,\infty;\psi)+N(r,0;\phi)+N(r,0;\psi)\\
&+2\overline{N}(r,1;\phi)(L)+\overline{N}(r,1;\psi)(L)+S(r,\phi)+S(r,\psi).
\end{aligned}
$$

With the help of Lemma 3.2 we deduce from the above inequality,

$$
\begin{aligned}
T(r,\phi) \leq{}& 2\overline{N}(r,\infty;\phi)+2\overline{N}(r,\infty;\psi)+T(r,\phi)-d_QT(r,\xi)+d_QN(r,0;\xi)\\
&+D\overline{N}(r,\infty;\mathcal{L})+d_QN(r,0;\mathcal{L})+2\overline{N}(r,1;\phi)(L)\\
&+\overline{N}(r,1;\psi)(L)+S(r,\phi)+S(r,\psi),
\end{aligned}
$$

hence from Lemma 3.5,

$$
\begin{aligned}
d_QT(r,\xi) \leq{}& 2\overline{N}(r,\infty;\xi)+(D+2)\overline{N}(r,\infty;\mathcal{L})+d_QN(r,0;\xi)+d_QN(r,0;\mathcal{L})\\
&+2d_QN(r,0;\xi)+2D\overline{N}(r,\infty;\xi)+d_QN(r,0;\mathcal{L})\\
&+D\overline{N}(r,\infty;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L})\\
\leq{}& (2D+4)\overline{N}(r,\infty;\xi)+(2D+3)\overline{N}(r,\infty;\mathcal{L})+3d_QN(r,0;\xi)\\
&+2d_QN(r,0;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}). \qquad (4.5)
\end{aligned}
$$

Similarly we obtain,

$$
\begin{aligned}
d_QT(r,\mathcal{L}) \leq{}& (2D+4)\overline{N}(r,\infty;\mathcal{L})+(2D+3)\overline{N}(r,\infty;\xi)+3d_Q\overline{N}(r,0;\mathcal{L})\\
&+2d_QN(r,0;\xi)+S(r,\xi)+S(r,\mathcal{L}). \qquad (4.6)
\end{aligned}
$$

Combining (4.5) and (4.6),

$$
\begin{aligned}
T(r,\xi)+T(r,\mathcal{L}) \leq{}& 5N(r,0;\xi)+\frac{4D+7}{d_Q}\overline{N}(r,\infty;\xi)+5N(r,0;\mathcal{L})\\
&+\frac{4D+7}{d_Q}\overline{N}(r,\infty;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}).
\end{aligned}
$$

From Lemma 3.7 we have, $N(r,\infty;\mathcal{L})=\overline{N}(r,\infty;\mathcal{L})=S(r,\mathcal{L})$ and hence $\Theta(\infty,\mathcal{L})=1$, then we deduce from the above inequality,

$$\begin{aligned}&[5\delta(0,\xi)+\frac{4D+7}{d_Q}\Theta(\infty,\xi)-\frac{4D+4d_Q+7}{d_Q}]T(r,\xi)\\&+[5\delta(0,\mathcal{L})-1]T(r,\mathcal{L})\leq S(r,\xi)+S(r,\mathcal{L}).\end{aligned}$$

This is a contradiction to our assumption.

Case 2. $\Omega\equiv 0$.

Now integrating twice we find,

$$\frac{1}{\psi-1}=\frac{U}{\phi-1}+V,$$

where $U(\neq 0)$ and V are two complex constants. Which implies that,

$$\psi=\frac{(V+1)\phi+(U-V-1)}{V\phi+(U-V)} \tag{4.7}$$

and

$$\phi=\frac{(V-U)\psi+(U-V-1)}{V\psi-(V+1)}. \tag{4.8}$$

Now we discuss the following subcases:

Subcase 2.1.

Let $V\neq 0,-1$.

We obtain from (4.8), $\overline{N}(r,\frac{V+1}{V};\psi)=\overline{N}(r,\infty;\phi)$. Using (ii) of Lemma 3.2 on Nevanlinna's 2^{nd} fundamental theorem we have,

$$\begin{aligned}T(r,\psi)&\leq\overline{N}(r,\infty;\psi)+\overline{N}(r,0;\psi)+\overline{N}(r,\frac{V+1}{V};\psi)+S(r,\psi)\\&\leq\overline{N}(r,\infty;\psi)+\overline{N}(r,0;\psi)+\overline{N}(r,\infty;\phi)+S(r,\psi)\\&\leq\overline{N}(r,\infty;\psi)+T(r,\psi)-d_QT(r,\mathcal{L})+d_QN(r,0;\mathcal{L})\\&\quad+\overline{N}(r,\infty;\phi)+S(r,\xi),\end{aligned}$$

hence,

$$d_QT(r,\mathcal{L})\leq\overline{N}(r,\infty;\mathcal{L})+d_QN(r,0;\mathcal{L})+\overline{N}(r,\infty;\xi)+S(r,\xi)+S(r,\mathcal{L}). \tag{4.9}$$

We assume that $U-V-1\neq 0$, then it follows from (4.7) that $N(r,\frac{-U+V-1}{V+1};\phi)=N(r,0;\psi)$. Using (ii) from Lemma 3.2 on Nevanlinna's 2^{nd} fundamental theorem we have,

$$\begin{aligned}T(r,\phi)&\leq\overline{N}(r,\infty;\phi)+\overline{N}(r,0;\phi)+\overline{N}(r,\frac{-U+V-1}{V+1};\phi)+S(r,\phi)\\&\leq\overline{N}(r,\infty;\phi)+T(r,\phi)-d_QT(r,\xi)+d_QN(r,0;\xi)\\&\quad+N(r,0;\psi)+S(r,\phi)+S(r,\psi)\\&\leq\overline{N}(r,\infty;\xi)+T(r,\phi)-d_QT(r,\xi)+d_QN(r,0;\xi)+D\overline{N}(r,\infty;\mathcal{L})\\&\quad+d_QN(r,0;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}),\end{aligned}$$

hence,

$$\begin{aligned}d_QT(r,\xi)\leq&\ \overline{N}(r,\infty;\xi)+d_QN(r,0;\xi)+D\overline{N}(r,\infty;\mathcal{L})\\&+d_QN(r,0;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}).\end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) and using Lemma 3.7, that is, $N(r,\infty;\mathcal{L})=\overline{N}(r,\infty;\mathcal{L})=S(r,\mathcal{L})$, we deduce that,

$$T(r,\xi)+T(r,\mathcal{L})\leq\frac{2}{d_Q}\overline{N}(r,\infty;\xi)+N(r,0;\xi)+2N(r,0;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}),$$

which implies a contradiction.
Therefore we assume $U-V-1=0$, then it follows from (4.7) that, $\overline{N}(r,\frac{-1}{V};\phi)=\overline{N}(r,\infty;\psi)$. Using (ii) from Lemma 3.2 on Nevanlinna's 2^{nd} fundamental theorem we have,

$$\begin{aligned}T(r,\phi) &\leq \overline{N}(r,\infty;\phi)+\overline{N}(r,0;\phi)+\overline{N}(r,\frac{-1}{V};\phi)+S(r,\phi)\\ &\leq \overline{N}(r,\infty;\phi)+T(r,\phi)-d_QT(r,\xi)+d_QN(r,0;\xi)+\overline{N}(r,\infty;\psi)\\ &\quad +S(r,\phi)+S(r,\psi),\end{aligned}$$

hence,

$$d_QT(r,\xi)\leq\overline{N}(r,\infty;\xi)+d_QN(r,0;\xi)+N(r,\infty;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}). \qquad (4.11)$$

Combining (4.9) and (4.11) and using Lemma 3.7, that is, $N(r,\infty;\mathcal{L})=\overline{N}(r,\infty;\mathcal{L})=S(r,\mathcal{L})$, we deduce that,

$$T(r,\xi)+T(r,\mathcal{L})\leq\frac{2}{d_Q}\overline{N}(r,\infty;\xi)+N(r,0;\xi)+N(r,0;\mathcal{L})+S(r,\xi)+S(r,\mathcal{L}),$$

which implies a contradiction.

Subcase 2.2. $V=-1$,
We obtain from (4.7) and (4.8) that, $\psi=\frac{U}{U+1-\phi}$ and $\phi=\frac{(U+1)\psi-U}{\phi}$. If $U+1\neq 0$, then, $\overline{N}(r,U+1;\phi)=\overline{N}(r,\infty;\psi)$ and $\overline{N}(r,\frac{U}{U+1};\psi)=\overline{N}(r,0;\phi)$. Now following the same argument as in Subcase 2.1. we arrive at a contradiction. Therefore $U+1=0$ and this implies that $\phi\psi=1$. Hence $Q(\xi)Q(\mathcal{L})=\rho^2$.

Subcase 2.3. $V=0$,
We obtain from (4.7) and (4.8) that, $\psi=\frac{\phi+U-1}{U}$ and $\phi=U\psi+1-U$. If $U-1\neq 0$, then, $\overline{N}(r,1-U;\phi)=\overline{N}(r,0;\psi)$ and $\overline{N}(r,\frac{U-1}{U};\psi)=\overline{N}(r,0;\phi)$. Now following the same argument as in Subcase 2.1. we arrive at a contradiction. Therefore $U-1=0$ and this implies that $\phi=\psi$. Hence $Q(\xi)=Q(\mathcal{L})$. This completes the proof of the theorem. □

5. Open Problems

We can pose the following problems from our results,
(i) Can theorem 2.1 be discussed under the concept of truncated sharing?
(ii) Can we replace the homogeneous differential polynomial from theorem 2.1 by a non-homogeneous differential polynomial?

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