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A STUDY OF A NON-LOCAL INITIAL VALUE PROBLEM FRACTIONALLY PERTURBED

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ABSTRACT. In this work, we study a class of fractionally nonlinearly perturbed first order differential equations, subject to a nonlocal initial condition on an unbounded interval, which is the novelty here. By means of the principle of contraction mapping to establish the existence result and by a Gronwall-like inequality, we obtain the asymptotic stability and the λ-stability of the zero solution. Finally, we give an illustrative example.

1. INTRODUCTION

Fractional order derivatives and integrals play an important role in describing many real-life phenomena with memory and hereditary properties from different fields of science and engineering such as chemistry, mechanics, thermoplasticity, viscoelasticity. For more details, see [13] and [12].

In recent years, many researchers have been interested in the subject of ordinary differential equations with nonlinearities involving fractional derivatives or integrals, for more details consult [1], [2], [14] and [16].

Motivated by the above cited papers, in this work we investigate the existence, uniqueness and stability of the solution for the following fractionally perturbed first-order differential equation

$$x'(t)+a(t)x(t)=f\left(t,x(t),I_{0^+}^{\beta}x(t)\right);\ \ t>0,\ \ \beta>0\,; \tag{1.1}$$

subject to the nonlocal initial condition

$$x(0)-\sum_{k=1}^{\infty}c_k x(t_k)=x_0;\ t_k\in\mathbb{R}^+, \tag{1.2}$$

where $I_{0^+}^{\beta}$ is the Riemann-Liouville integral operator; $a(t)$ is a given function; $\{t_k, k\geq 1\}$ is a positive increasing real sequence such that $\lim\limits_{k\to\infty} t_k=\infty$; $c_k\in\mathbb{R}$ with $c_k\neq 0; k=1,2...; x_0\in\mathbb{R}$. The nonlinear perturbation term $f(t,x,y)$ is continuous and bounded with respect to all of its variables.

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The stereotype of the integrodifferential first order equation is of the following form

$$x'(t)+a(t)x(t)=f\left(t,x(t),\int_0^t h(t,s)x(s)ds\right);\ t>0,$$

with singular kernel $h(t,s)$ in the case of the fractionally perturbed equation (1.1). In [1], the authors proved a local existence and uniqueness result concerning an initial value problem in $\mathbb{R}^N$ whose right-hand sides contained Riemann-Liouville fractional integrals of multiple orders.

In [4], Byszewski initiated the work on nonlocal initial value problems. He proposed a nonlocal condition of the following form

$$x(0)+g(t_1,t_2,...,t_p,x(.))=x_0, \tag{1.3}$$

where the symbol $x(.)$ meant that instead of "$\cdot$" we can only substitute elements of the sequence $(t_k)_{k=1,...,p}\subset[0,T]$. It is clear that the nonlocal condition gives more information on the sought solution which induces better results than the consideration of the usual initial condition given alone $x(0)=x_0$. For its wide range of applications, different types of equations with nonlocal initial conditions of the form (1.2) or (1.3), have been studied by many authors on finite interval, see for example [6] and [9]. In [6], Chen et al. considered a problem with the nonlocal initial condition (1.2) where $x_0=0$ on an infinite interval and we are motived by their approach.

One of the main properties of any solution of an ordinary differential equation is its stability which is studied by many methods for different classes of equations, see [1], [3], [5], [6], [8], [15], [17] and [18].

The rest of the paper is organized as follows. In Section 1, we will recall some basic definitions. In Section 2, we will use fixed point theory to show the existence of a unique solution for problem (1.1)-(1.2) and via some Gronwall type inequality we will etablish the asymptotic stability and the $\lambda-$stability of the zero solution. In the last Section, we will provide an example to illustrate the obtained results.

2. PRELIMINARIES

In this section, we will introduce some preliminaries that we will need in the rest of the work.

Definition 2.1. *The left-sided Riemann-Liouville fractional integral of order* $\alpha\in\mathbb{R}^+$ *for an integrable function* $f:\mathbb{R}^+\to\mathbb{R}$*, is defined by* ε

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\ for\ t>0,$$

where $\Gamma(\alpha)$ *is the Euler's gamma function.*

Remark 2.1. For $\alpha>0$, $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

Let us define the following spaces:

$C([0,+\infty),\mathbb{R})$, the space of continuous functions on $[0,+\infty)$ and

$BC([0,+\infty),\mathbb{R}) = \{x \in C([0,+\infty),\mathbb{R}) : \|x\|_\infty < \infty\}$ the space of continuous bounded functions on $[0,+\infty)$ where $\|x\|_\infty = \sup_{t\in[0,+\infty)} |x(t)|$.

By a solution of (1.1)-(1.2) we mean a continuous real-valued function $x(t)$ defined on the interval $[0,T)$, continuously differentiable on $(0,T)$ which satisfies equation (1.1) on $(0,T)$ for some positive constant $T > 0$ and condition (1.2) for $t = 0$. If the solution remains bounded, then $T = \infty$.

Let us give definitions of certain types of stability.

Definition 2.2. (i) *The zero solution of (1.1) is said to be stable if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $|x(0)| < \delta_\varepsilon$ implies that $|x(t)| < \varepsilon$ for $t \geq 0$.*

(ii) *The zero solution of (1.1) is said to be asymptotically stable if it is stable and $\lim_{t\to\infty} x(t) \to 0$.*

Definition 2.3. *Let the continuous positive function $\lambda(t)$ satisfy*

$$\lim_{t\to\infty} \lambda(t) = \infty;\ \lambda(t+s) \leq \lambda(t)\lambda(s) \text{ for } t,s \in \mathbb{R}^+. \tag{2.1}$$

Then, for any $x(0) \in \mathbb{R}$, the zero solution of (1.1) is said to be $\lambda-$stable if there exists a positive constant γ such that

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{\log\lambda(t)} \leq -\gamma.$$

Remark 2.2. (i) When $\lambda(t) = e^t$, the stability is called exponential.

(ii) When $\lambda(t) = 1+t$, the stability (1.1) is called polynomial .

(iii) When $\lambda(t) = \log(1+t)$, the stability is called logarithmical.

We will give a version of the Banach's fixed point theorem given in [10].

Theorem 2.1. *Let X be a Banach space and suppose $T : X \to X$ is contractive. Then T has a unique fixed point x^* and $T^n x \to x^*$ for each $x \in X$.*

The following Gronwall type inequality from Bykov and Salpagarov is given in [7] (Theorem 57 p35).

Theorem 2.2. *Let $u(t), v(t)$, $h(t,r)$ and $H(t,r,x)$ be nonnegative functions for $t \geq r \geq x \geq a$ and c_1, c_2 and c_3 be nonnegative constants not all zero. If*

$$u(t) \leq c_1 + c_2 \int_a^t \left[v(s)u(s) + \int_a^s h(s,r)u(r)\,dr \right] ds$$
$$+ c_3 \int_a^t \int_a^r \int_a^s H(r,s,x)u(x)\,dxdrds,$$

then, for $t \geq a$

$$u(t) \leq c_1 \exp\left[c_2 \int_a^t \left[v(s) + \int_a^s h(s,r)\,dr\right] ds + c_3 \int_a^t \int_a^r \int_a^s H(r,s,x)\,dxdrds\right].$$

3. Main results

In the first Theorem, we give the integral equation equivalent to problem *(1.1)-(1.2)*.

Theorem 3.1. *Let a and f be continuous functions and $x \in C[[0,T),\mathbb{R}]$. Then, $x(t)$ is the solution of initial value problem (1.1)-(1.2) if and only if it satisfies the following integral equation for $t \geq 0$*

$$\begin{aligned} x(t) = {} & \frac{x_0 e^{-\int_0^t a(r)dr}}{1 - \sum\limits_{k=1}^{\infty} undersetk = 1 c_k e^{-\int_0^{t_k} a(r)dr}} + \frac{e^{-\int_0^t a(r)dr}}{1 - \sum\limits_{k=1}^{\infty} c_k e^{-\int_0^{t_k} a(r)dr}} \\ & \times \sum_{k=1}^{\infty} undersetk = 1 c_k \int_0^{t_k} e^{-\int_s^{t_k} a(r)dr} f\left(s, x(s), I_{0^+}^{\beta} x(s)\right) ds \\ & + \int_0^t e^{-\int_s^t a(r)dr} f\left(s, x(s), I_{0^+}^{\beta} x(s)\right) ds. \end{aligned} \tag{3.1}$$

Proof. Multiplying both sides of equation *(1.1)* by $e^{\int_0^t a(s)ds}$ and then integrating from 0 to t , we get

$$x(t) = x(0) e^{-\int_0^t a(v)dv} + \int_0^t e^{-\int_s^t a(v)dv} \left(f\left(s, x(s), I_{0^+}^{\beta} x(s)\right)\right) ds.$$

Likewise,

$$x(t_k) = x(0) e^{-\int_0^{t_k} a(v)dv} + \int_0^{t_k} e^{-\int_s^{t_k} a(v)dv} f\left(s, x(s), I_{0^+}^{\beta} x(s)\right) ds,$$

then, the sum and (1.2) imply

$$\begin{aligned} x(0) = {} & \frac{x_0}{1 - \sum\limits_{k=1}^{\infty} c_k e^{-\int_0^{t_k} a(v)dv}} \\ & + \frac{1}{1 - \sum\limits_{k=1}^{\infty} c_k e^{-\int_0^{t_k} a(v)dv}} \sum_{k=1}^{\infty} c_k \int_0^{t_k} e^{-\int_s^{t_k} a(v)dv} f\left(s, x(s), I_{0^+}^{\beta} x(s)\right) ds, \end{aligned}$$

which leads to (3.1). On the other hand, if we derive (3.1) for $t > 0$, we obtain directly (1.1) and for $t = 0$ (3.1) implies (1.2). The proof is complete. □

Now, before we state the existence result based on Banach's fixed point theorem, we assume the following assumptions.

(A1) $a(t)$ is a positive continuous function on $\mathbb{R}^+$with $\lim\limits_{t\to\infty} \int_0^t a(v)\,dv = \infty$.

(A2) The series $\sum_{k=1}^{\infty} c_k$ converges to δ_0 which satisfies $\delta_0 < e^{\int_0^{t_1} a(v)dv}$.

(A3) There exist positive, Lebesgue integrable and continuous functions $L_1(t)$, $L_2(t)$ such that for each $t \in \mathbb{R}^+$; $(x,y), (x_1,y_1) \in \mathbb{R}\times\mathbb{R}$,

$$|f(t,x,y) - f(t,x_1,y_1)| \leq L_1(t)|x-x_1| + L_2(t)|y-y_1|$$

with $f(t,0,0) = 0$ and

$$L(t) = \int_0^t L_1(s)\, e^{\left(\int_0^s a(v)dv\right)} ds < \infty; \tag{3.2}$$

$$L_\beta(t) = \int_0^t \frac{L_2(s)\, s^\beta}{\Gamma(\beta+1)} e^{\left(\int_0^s a(v)dv\right)} ds < \infty.$$

Also, we denote

$$A = \inf_{t\geq 0}\left(\int_0^t a(v)\,dv\right);\ L = \sup_{t\geq 0} L(t);\ L_\beta = \sup_{t\geq 0} L_\beta(t) \tag{3.3}$$

$$K = L + L_\beta; \quad M = \left(\frac{e^{-A}\delta_0}{1-\delta_0 e^{\left(-\int_0^{t_1} a(v)dv\right)}} + 1\right) e^{-A} K.$$

Theorem 3.2. *Assume that (A1)-(A3) hold. If*

$$0 < M < 1, \tag{3.4}$$

then, problem (1.1)-(1.2) has a unique solution $x(t)$ in $BC([0,+\infty),\mathbb{R}) \cap C^1((0,+\infty),\mathbb{R})$.

Proof. To transform problem *(1.1)-(1.2)* into a fixed point problem, we define a nonempty closed subset of $BC([0,+\infty),\mathbb{R})$

$$S = \left\{ x \in BC([0,+\infty),\mathbb{R}) : x(0) = \sum_{k=1}^{\infty} c_k x(t_k) + x_0 \text{ with } \|x\|_\infty \leq R \right\}$$

for some positive constant R to be defined. $(S, \|.\|_\infty)$ is a Banach space. We define an operator P for any $x \in S$ by $Px(0) = \sum_{k=1}^{\infty} c_k x(t_k) + x_0$ and for $t > 0$ by

$$\begin{aligned} Px(t) = {} & \frac{x_0 e^{-\int_0^t a(v)dv}}{1-\sum_{k=1}^{\infty} c_k e^{-\int_0^{t_k} a(v)dv}} + \frac{e^{-\int_0^t a(v)dv}}{1-\sum_{k=1}^{\infty} c_k e^{-\int_0^{t_k} a(v)dv}} \\ & \times \sum_{k=1}^{\infty} c_k \int_0^{t_k} e^{-\int_s^{t_k} a(v)dv} f\left(s, x(s), I_{0^+}^\beta x(s)\right) ds \\ & + \int_0^t e^{-\int_s^t a(v)dv} f\left(s, x(s), I_{0^+}^\beta x(s)\right) ds. \end{aligned}$$

First, we will show that P maps S into itself. For each $x \in S$, it is clear that $Px(t) \in C([0,+\infty), \mathbb{R})$ and

$$\begin{aligned}|Px(t)| \leq & \frac{|x_0| e^{\left(-\int_0^t a(v)dv\right)}}{\left|1-\sum_{k=1}^{\infty} c_k e^{\left(-\int_0^{t_k} a(v)dv\right)}\right|} + \frac{e^{\left(-\int_0^t a(v)dv\right)}}{\left|1-\sum_{k=1}^{\infty} c_k e^{\left(-\int_0^{t_k} a(v)dv\right)}\right|} \sum_{k=1}^{\infty} |c_k| \\ & \times \int_0^{t_k} e^{\left(-\int_s^{t_k} a(v)dv\right)} \left(L_1(s)|x(s)| + L_2(s)\left|I_{0^+}^{\beta} x(s)\right|\right) ds \\ & + \int_0^t e^{\left(-\int_s^t a(v)dv\right)} \left(L_1(s)|x(s)| + L_2(s)\left|I_{0^+}^{\beta} x(s)\right|\right) ds.\end{aligned}$$

First, we have

$$\begin{aligned}\frac{1}{\left|1-\sum_{k=1}^{\infty} c_k e^{\left(-\int_0^{t_k} a(v)dv\right)}\right|} & < \frac{1}{1-\sum_{k=1}^{\infty} |c_k| e^{\left(-\int_0^{t_k} a(v)dv\right)}} \\ & < \frac{1}{1-\delta_0 e^{\left(-\int_0^{t_1} a(v)dv\right)}}.\end{aligned}$$

Also,

$$\begin{aligned}\left|I_{0^+}^{\beta} x(s)\right| & \leq \frac{1}{\Gamma(\beta)} \int_0^s \frac{|x(v)|}{(s-v)^{1-\beta}} dv \leq \frac{\sup_{0\leq v\leq s} |x(v)|}{\Gamma(\beta)} \int_0^s \frac{dv}{(s-v)^{1-\beta}} \\ & \leq \frac{\sup_{0\leq v\leq s} |x(v)|}{\Gamma(\beta)} \frac{s^{\beta}}{\beta}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|Px(t)| \leq & \frac{|x_0| e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1} a(v)dv\right)}} + \frac{e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1} a(v)dv\right)}} \sum_{k=1}^{\infty} |c_k| \\ & \times \int_0^{t_k} e^{\left(-\int_s^{t_k} a(v)dv\right)} \left(L_1(s)|x(s)| + \frac{L_2(s)}{\Gamma(\beta)} \frac{s^{\beta}}{\beta} \sup_{0\leq v\leq s} |x(v)|\right) ds \\ & + \int_0^t e^{\left(-\int_s^t a(v)dv\right)} \left(L_1(s)|x(s)| + \frac{L_2(s)}{\Gamma(\beta)} \frac{s^{\beta}}{\beta} \sup_{0\leq v\leq s} |x(v)|\right) ds.\end{aligned}$$

Then, we get

$$\begin{aligned}|Px(t)| \leq & \frac{|x_0| e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1} a(v)dv\right)}} + \frac{e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1} a(v)dv\right)}} \sum_{k=1}^{\infty} |c_k| \\ & \times \int_0^{t_k} \left(\sup_{0\leq s\leq t_k} |x(s)| L_1(s) + \sup_{0\leq v\leq s\leq t_k} |x(v)| \frac{L_2(s)}{\Gamma(\beta+1)} s^{\beta}\right) ds\end{aligned}$$

$$
\begin{aligned}
&+\sup_{0\le s\le t}|x(s)|\int_0^t\left(L_1(s)+\frac{L_2(s)}{\Gamma(\beta+1)}s^{\beta}\right)ds\\
\le\;& \frac{|x_0|\,e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}+\frac{e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}\\
&\times\sum_{k=1}^{\infty}|c_k|\sup_{0\le s\le t_k}|x(s)|\,e^{\left(-\int_0^{t_k}a(v)dv\right)}K(t_k)\\
&+\sup_{0\le s\le t}|x(s)|\,e^{\left(-\int_0^t a(v)dv\right)}K(t);
\end{aligned}
$$

where $K(t)=L(t)+L_{\beta}(t)$.

For each $x\in S$, we have $\|x\|_\infty\le R$, so we conclude that,

$$
\begin{aligned}
\|Px\|_\infty <\;& \frac{|x_0|\sup\limits_{t\ge 0}e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}+\frac{\sup\limits_{t\ge 0}e^{\left(-\int_0^t a(v)dv\right)}}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}\\
&\times\sum_{k=1}^{\infty}|c_k|\sup_{0\le s\le t_k}|x(s)|\,e^{\left(-\int_0^{t_k}a(v)dv\right)}K(t_k)+\|x\|_\infty\sup_{t\ge 0}\left(e^{\left(-\int_0^t a(v)dv\right)}K(t)\right)\\
<\;& \frac{|x_0|\,e^{-A}}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}+\left(\frac{e^{-A}\delta_0}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}+1\right)e^{-A}K\|x\|_\infty\\
<\;& MR+\frac{|x_0|\,e^{-A}}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}<R,
\end{aligned}
$$

for some R satisfying

$$
R>\frac{|x_0|\,e^{-A}}{\left[1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}\right](1-M)}.
$$

This means that $Px\in S$ for any $x\in S$. Let x and y be in S, then for $t\ge 0$, we have

$$
\begin{aligned}
&|Px(t)-Py(t)|\\
&\quad\le \frac{e^{\left(-\int_0^t a(v)dv\right)}}{1-\sum\limits_{k=1}^{\infty}c_k e^{\left(-\int_0^{t_k}a(v)dv\right)}}\sum_{k=1}^{\infty}c_k\int_0^{t_k}e^{\left(-\int_s^{t_k}a(v)dv\right)}\\
&\quad\quad\times\left|f\left(s,x(s),I_{0^+}^{\beta}x(s)\right)-f\left(s,y(s),I_{0^+}^{\beta}y(s)\right)\right|ds\\
&\quad\quad+\int_0^t e^{\left(-\int_s^t a(v)dv\right)}\left|f\left(s,x(s),I_{0^+}^{\beta}x(s)\right)-f\left(s,y(s),I_{0^+}^{\beta}y(s)\right)\right|ds.
\end{aligned}
$$

Hence, by using the Lipschitz condition and previous approximations, we obtain

$$
\begin{aligned}
\|Px-Py\|_\infty &\le\left(\frac{e^{-A}\delta_0}{1-\delta_0 e^{\left(-\int_0^{t_1}a(v)dv\right)}}+1\right)e^{-A}K\|x-y\|_\infty,\\
&\le M\|x-y\|_\infty.
\end{aligned}
$$

Then, from condition (3.4) we conclude that P is a contraction and in view of the contraction mapping principle, P has a unique fixed point in S which is a solution of (1.1) with initial nonlocal condition (1.2). Furthermore, it is clear from (1.1) that $x(t)$ is in $C^1((0,+\infty),\mathbb{R})$. This completes the proof. □

Theorem 3.3. *Under the conditions of Theorem 3.2, the zero solution of equation (1.1) with $x(0)=0$ is stable.*

Moreover if $\inf_{t\geq 0} a(t) = a_0 > 0$, then the zero solution is asymptotically stable.

Furthermore, if

$$\lim_{t\to\infty}\sup \frac{-a_0 t}{\log \lambda(t)} \leq -\gamma, \tag{3.5}$$

for some positive constant γ and some function $\lambda(t)$ satisfying (2.1), the zero solution of equation (1.1) is $\lambda-$stable.

Proof. First, we show that the zero solution of equation *(1.1) is stable*. Then, for all $t\geq 0$ and any given $\varepsilon>0$, there exists a positive real $\eta \leq \dfrac{1-e^K K_f}{e^K}\varepsilon$ such that if $|x(0)|<\eta$, then, we have

$$\begin{aligned} |x(t)| \leq\ & |x(0)|\, e^{\left(-\int_0^t a(v)dv\right)} + \int_0^t e^{\left(-\int_s^t a(v)dv\right)} \\ & \times \left[L_1(s)\,|x(s)| + \frac{L_2(s)}{\Gamma(\beta)}\int_0^s \frac{|x(v)|}{(s-v)^{1-\beta}}dv\right] ds. \end{aligned} \tag{3.6}$$

Hence,

$$|x(t)| \leq \eta e^{-A} + \|x\|_\infty e^{-A}\int_0^t e^{\left(\int_0^s a(v)dv\right)}\left[L_1(s) + \frac{L_2(s)\,s^\beta}{\Gamma(\beta+1)}\right] ds$$

which gives

$$|x(t)| < \frac{e^{-A}}{1-e^{-A}K}\eta < \varepsilon,$$

with $e^{-A}K < M < 1$. This implies that the zero solution is stable. On the other hand, taking $m(t) = |x(t)|\, e^{\left(\int_0^t a(v)dv\right)}$ we get from (3.6)

$$m(t) \leq m(0) + \int_0^t \left[L_1(s)\, e^{\left(\int_0^s a(v)dv\right)} m(s) + \int_0^s \frac{L_2(s)}{\Gamma(\beta)} \frac{e^{\left(\int_r^s a(v)dv\right)} m(r)}{(s-r)^{1-\beta}} dr\right] ds$$

and the application of like-Gronwall theorem 2.2 with $c_1 = m(0)$, $c_2 = 1$ and $c_3 = 0$ leads to

$$m(t) \leq m(0)\exp \int_0^t \left[L_1(s)\, e^{\left(\int_0^s a(v)dv\right)} + \int_0^s \frac{L_2(s)}{\Gamma(\beta)} \frac{e^{\left(\int_r^s a(v)dv\right)}}{(s-r)^{1-\beta}} dr\right] ds.$$

Then,

$$
\begin{aligned}
|x(t)| &< |x(0)| e^{-a_0 t} \exp \int_0^t e^{\left(\int_0^s a(v)dv\right)} \left[L_1(s) + \frac{L_2(s)}{\Gamma(\beta)} \int_0^s \frac{e^{\left(-\int_0^r a(v)dv\right)}}{(s-r)^{1-\beta}} dr \right] ds \\
&< |x(0)| e^{-a_0 t} \exp \int_0^t e^{\left(\int_0^s a(v)dv\right)} \left[L_1(s) + \frac{L_2(s)}{\Gamma(\beta)} \frac{s^\beta}{\beta} \sup_{0\le r\le s} e^{\left(-\int_0^r a(v)dv\right)} \right] ds \\
&< |x(0)| e^{-a_0 t} \exp \left(\int_0^t e^{\left(\int_0^s a(v)dv\right)} \left(L_1(s) + \frac{L_2(s)\, s^\beta}{\Gamma(\beta+1)} \right) ds \right),
\end{aligned}
$$

where $L(t)$ and $L_\beta(t)$ are defined and bounded by (3.2). This leads to

$$
|x(t)| < |x(0)| e^{-a_0 t} \exp\left(L + L_\beta\right) \tag{3.7}
$$

which implies that $|x(t)| \to 0$ when $t \to \infty$ and $x(t)$ is asymptotically stable.

For the $\lambda-$stability, we have by (3.7) for $t \ge 0$,

$$
\begin{aligned}
\frac{\log|x(t)|}{\log\lambda(t)} &< \frac{\log\left(|x(0)| e^{-a_0 t} \exp\left[L + L_\beta e^{-A}\right]\right)}{\log\lambda(t)} \\
&< \frac{\log|x(0)|}{\log\lambda(t)} - \frac{a_0 t}{\log\lambda(t)} + \frac{L + L_\beta e^{-A}}{\log\lambda(t)}.
\end{aligned}
$$

Hence,

$$
\limsup_{t\to\infty} \frac{\log|x(t)|}{\log\lambda(t)} \le \limsup_{t\to\infty} \frac{-a_0 t}{\log\lambda(t)} \le -\gamma;
$$

which is the desired result. The proof is complete. □

For the exponential stability $\limsup\limits_{t\to\infty} \dfrac{-a_0 t}{\log e^t} = -a_0 = -\gamma$, for the polynomial stability $\limsup\limits_{t\to\infty} \dfrac{-a_0 t}{\log(1+t)} = -\infty < -\gamma$ and for the logarithmical stability $\limsup\limits_{t\to\infty} \dfrac{-a_0 t}{\log(\log(1+t))} = -\infty < -\gamma.$

4. EXAMPLE

Finally, we will present an example to illustrate that the conditions of the above results are possible to satisfy. Consider the following fractionally perturbed first-order differential equation for $t > 0$,

$$
x'(t) + (t+2)x(t) = \frac{e^{-t^2 - Mt}}{100} \sin x(t) + \frac{t^{1/2} e^{-t^2/2 - Mt}}{100} \frac{\left|I_{0^+}^{3/2} x(t)\right|}{1 + \left|I_{0^+}^{3/2} x(t)\right|}; \tag{4.1}
$$

subject to the nonlocal initial condition

$$
x(0) - \sum_{k=1}^{\infty} \frac{1}{k^2} x(k) = \frac{1}{2}\,; t_k = k \in \mathbb{R}^+, \tag{4.2}
$$

where $a(t)=(t+2)$ is positive and continuous on $\mathbb{R}^+$ with $\int_0^t a(v)\,dv=\frac{t^2}{2}+2t\to\infty$ when $t\to\infty$ and then (A1) is satisfied. The second hypothesis (A2) is also satified. Indeed, we have

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}=1.6449=\delta_0<e^{\int_0^{t_1}a(v)dv}=\exp(\frac{5}{2})=12.182.$$

On the other hand, for $\beta=\frac{3}{2}$, $t>0$, we have

$$f\left(t,x(t),I_{0^+}^{\beta}x(t)\right)=\frac{e^{-t^2-2t}}{100}\sin x(t)+\frac{t^{1/2}e^{-t^2/2-2t}}{100}\frac{\left|I_{0^+}^{3/2}x(t)\right|}{1+\left|I_{0^+}^{3/2}x(t)\right|};$$

which satisfies (A3), for positive, Lebesgue integrable and continuous functions

$$L_1(t)=\frac{e^{-t^2-2t}}{100},\quad L_2(t)=\frac{t^{1/2}e^{-t^2/2-2t}}{100}$$

with $f(t,0,0)=0$. Some computations allow from (3.3)

$$\begin{aligned}L&=\sup_{t\geq 0}\int_0^t L_1(s)\,e^{\left(\int_0^s a(v)dv\right)}ds\\&=\sup_{t\geq 0}\int_0^t\frac{e^{-s^2-Ms}}{100}\left(e^{s^2/2+Ms}-1\right)ds\\&\leq\sup_{t\geq 0}\frac{1}{100}\int_0^t e^{-s^2/2}ds=\frac{5\sqrt{\pi}}{100}=8.8623\times 10^{-2};\end{aligned}$$

and

$$\begin{aligned}&A=0,\quad K=L+L_\beta=9.6146\times 10^{-2};\\&M=\left(\frac{\frac{\pi^2}{6}}{1-\frac{\pi^2}{6}\exp(-\frac{3}{2})}+1\right)K\\&\quad=3.5988\times 9.6146\times 10^{-2}=0.34601<1.\end{aligned}$$

Hence, in view of Theorem 3.2, problem (4.1)-(4.2) has a unique solution $x(t)$ in $BC([0,+\infty),\mathbb{R})\cap C^1((0,+\infty),\mathbb{R})$.

We have, $\inf\limits_{t\geq 0}a(t)=\inf\limits_{t\geq 0}(t+2)=2>0$, then, the zero solution of *(4.1)* is asymptotically stable and is in addition exponentially, polynomially and logarithmically stable since condition (3.5) is satisfied.

5. CONCLUSION

In this paper we studied a class of first order differential equations with nonlinear fractional perturbation and a nonlocal initial condition on an unbounded interval. We used the contraction mapping principle to obtain an existence result and by some like-Gronwall's inequality we etablish asymptotic stability and λ-stability of the zero solution, the last under suitable assumptions. The novelty in this work is that the perturbation of the linear differential equation of first order is nonlinear involving a fractional integral and this equation is subject to a multi-point non-local condition implying an infinite number of points on the positive real half-axis. Moreover, the result of stability generalizes strong stabilities such as the exponential one and weak ones such as the logarithmic one.

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