SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.18.02.10
Vol.18 (31), No.2 (2022), 297 – 312

DECOMPOSABLE EXTENSIONS BETWEEN RANK 1 MODULES IN GRASSMANNIAN CLUSTER CATEGORIES

DUSKO BOGDANIC AND IVAN-VANJA BOROJA

ABSTRACT. Rank 1 modules are the building blocks of the category $\mathrm{CM}(B_{k,n})$ of Cohen-Macaulay modules over a quotient $B_{k,n}$ of a preprojective algebra of affine type A. Jensen, King and Su showed in [8] that the category $\mathrm{CM}(B_{k,n})$ provides an additive categorification of the cluster algebra structure on the coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ of the Grassmannian variety of k-dimensional subspaces in $\mathbb{C}^n$. Rank 1 modules are indecomposable, they are known to be in bijection with k-subsets of $\{1,2,\ldots,n\}$, and their explicit construction has been given in [8]. In this paper, we give necessary and sufficient conditions for indecomposability of an arbitrary rank 2 module in $\mathrm{CM}(B_{k,n})$ whose filtration layers are tightly interlacing. We give an explicit construction of all rank 2 decomposable modules that appear as extensions between rank 1 modules corresponding to tightly interlacing k-subsets I and J.

1. INTRODUCTION

A categorification of the cluster algebra structure on the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ of the Grassmannian variety of k-dimensional subspaces in $\mathbb{C}^n$ has been given by Geiss, Leclerc, and Schroer [6, 7] in terms of a subcategory of the category of finite dimensional modules over the preprojective algebra of type A_{n-1}. Jensen, King, and Su [8] gave a new categorification of this cluster structure using the maximal Cohen-Macaulay modules over the completion of an algebra $B_{k,n}$ which is a quotient of the preprojective algebra of type A_{n-1}. Rank 1 modules are the building blocks of the category $\mathrm{CM}(B_{k,n})$ of Cohen-Macaulay modules over a quotient $B_{k,n}$ of a preprojective algebra of affine type A_{n-1}. Rank 1 modules are indecomposable, they are known to be in bijection with k-subsets of $[n]=\{1,2,\ldots,n\}$, and their explicit construction has been given in [8]. These are the building blocks of the category as any module in $\mathrm{CM}(B_{k,n})$ can be filtered by rank 1 modules (the filtration is noted in the profile of a module, [8, Corollary 6.7]). The number of rank 1 modules appearing in the filtration of a given module is called the rank of that module. In [4], we explicitly constructed all indecomposable rank 2 modules in tame cases.

2010 *Mathematics Subject Classification.* Primary 05E10; Secondary 16G50, 17B22.

Key words and phrases. Cohen-Macaulay modules, Grassmannian cluster categories, decomposable extensions.

In this paper, we give necessary and sufficient conditions for indecomposability of an arbitrary rank 2 module in $\mathrm{CM}(B_{k,n})$ whose filtration layers are tightly interlacing. Moreover, we construct explicitly all rank 2 decomposable Cohen-Macaulay $B_{k,n}$-modules that appear as middle terms in the short exact sequences where the end terms are rank 1 modules corresponding to tightly interlacing subsets. The central combinatorial notion throughout this paper is that of r-interlacing (Definition 2.3). If I and J are k-subsets of $\{1,\dots,n\}$, then I and J are said to be *r-interlacing* if there exist subsets $\{i_1,i_3,\dots,i_{2r-1}\}\subset I\setminus J$ and $\{i_2,i_4,\dots,i_{2r}\}\subset J\setminus I$ such that $i_1<i_2<i_3<\cdots<i_{2r}<i_1$ (cyclically) and if there are not exist larger subsets of I and of J with this property.

Denote by L_I the rank 1 module corresponding to the k-subset I. By [8, Proposition 5.6], $\mathrm{Ext}^1_B(L_I,L_J)\neq 0$ if and only if I and J are r-interlacing, where $r\geq 2$. In particular, rank 1 modules are rigid, i.e., $\mathrm{Ext}^1_B(L_I,L_I)\neq 0$ for every I. This means that if I and J are 1-interlacing, then the only module appearing as the middle term in short exact sequences with end terms L_I and L_J is the direct sum $L_I\oplus L_J$. For this reason, we will assume most of the time that I and J are r-interlacing with $r\geq 2$. Note that, since the Grassmannian cluster category $\mathrm{CM}(B_{k,n})$ is a 2-CY category, $\mathrm{Ext}^1_B(L_I,L_J)\cong\mathrm{Ext}^1_B(L_J,L_I)$, so we have the same arguments for the short exact sequences with L_I as the left term and L_J as the right term, and for the short exact sequences with L_J as the left term and L_I as the right term (cf. Theorem 3.7 in [1]).

The paper is organized as follows. In Section 2, we recall the definitions and key results about Grassmannian cluster categories. In Section 3, we study the filtration $I\mid J$, where $I=\{1,3,\dots,2r-1\}$ and $J=\{2,4,\dots,2r\}$, in the case $(r,2r)$. We explain how the general case of a module with tight r-interlacing filtration layers reduces to the case of the module with filtration $I\mid J$. For the filtration layers I and J of a module with profile $I\mid J$, we construct all decomposable rank 2 modules that are extensions of these rank 1 modules, i.e. we construct all decomposable modules that appear as middle terms in short exact sequences with I and J as end terms. In particular, we associate with every subset of peaks of the rim I a decomposable rank 2 module that is extension of L_J by L_I.

Our main results are Theorem 3.1 in which we give necessary and sufficient conditions for a rank 2 module with filtration $I\mid J$ to be indecomposable, and Theorem 3.2 in which we give an explicit construction of all rank 2 decomposable modules that appear as extensions between rank 1 modules corresponding to I and J.

2. Preliminaries

We follow closely the exposition from [1,2,4,8] in order to introduce notation and background results. Let Γ_n be the quiver of the boundary algebra, with vertices $1,2,\dots,n$ on a cycle and arrows $x_i: i-1\to i$, $y_i: i\to i-1$ (see Figure 1). We write $\mathrm{CM}(B_{k,n})$ for the category of maximal Cohen-Macaulay modules for the completed path algebra $B_{k,n}$ of Γ_n, with relations $xy-yx$ and x^k-y^{n-k} (at every vertex). The centre of $B_{k,n}$ is $Z := \mathbb{C}[|t|]$, where $t = \sum_i x_i y_i$.

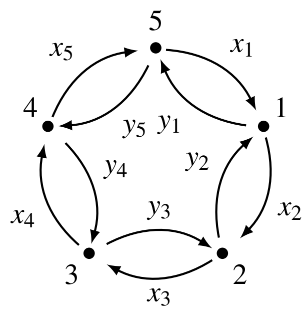


FIGURE 1. The quiver Γ_n for $n = 5$.

The algebra $B_{k,n}$ coincides with the quotient of the completed path algebra of the graph C (a circular graph with vertices $C_0 = \mathbb{Z}_n$ set clockwise around a circle, and with the set of edges, C_1, also labeled by $\mathbb{Z}_n$, with edge i joining vertices $i-1$ and i), i.e. the doubled quiver as above, by the closure of the ideal generated by the relations above (we view the completed path algebra of the graph C as a topological algebra via the m-adic topology, where m is the two-sided ideal generated by the arrows of the quiver, see [5, Section 1]). The algebra $B_{k,n}$, that we will often denote by B when there is no ambiguity, was introduced in [8, Section 3]. Observe that $B_{k,n}$ is isomorphic to $B_{n-k,n}$, so we will always assume that $k \leq \frac{n}{2}$.

The (maximal) Cohen-Macaulay B-modules are precisely those which are free as Z-modules. Such a module M is given by a representation $\{M_i : i \in C_0\}$ of the quiver with each M_i a free Z-module of the same rank (which is the rank of M).

Definition 2.1 ([8], Definition 3.5)**.** *For any $B_{k,n}$-module M and K the field of fractions of Z, the* **rank** *of M, denoted by* $\mathrm{rk}(M)$*, is* $\mathrm{rk}(M) = \mathrm{len}(M \otimes_Z K)$.

Note that $B \otimes_Z K \cong M_n(K)$, which is a simple algebra. It is easy to check that the rank is additive on short exact sequences, that $\mathrm{rk}(M) = 0$ for any finite-dimensional B-module (because these are torsion over Z) and that, for any Cohen-Macaulay B-module M and every idempotent e_j, $1 \leq j \leq n$, $\mathrm{rk}_Z(e_j M) = \mathrm{rk}(M)$, so that, in particular, $\mathrm{rk}_Z(M) = n\mathrm{rk}(M)$.

Definition 2.2 ([8], Definition 5.1)**.** *For any k-subset I of C_0, we define a rank* 1 *$B_{k,n}$-module*

$$L_I = (U_i,\ i \in C_0\,;\, x_i, y_i,\ i \in C_0)$$

as follows. For each vertex $i \in C_0$, the set $U_i = \mathbb{C}[[t]]$, e_i acts as the identity on U_i and $e_i U_j = 0$, for $i \neq j$. For each $i \in C_0$, set

$x_i \colon U_{i-1} \to U_i$ to be multiplication by 1 *if $i \in I$, and by t if $i \notin I$,*
$y_i \colon U_i \to U_{i-1}$ to be multiplication by t if $i \in I$, and by 1 *if $i \notin I$.*

The module L_I can be represented by a lattice diagram $\mathcal{L}_I$ in which $U_0, U_1, \dots, U_n$ are represented by columns of vertices (dots) from left to right (with U_0 and U_n to be identified), going down infinitely. The vertices in each column correspond to the natural monomial $\mathbb{C}$-basis of $\mathbb{C}[t]$. The column corresponding to U_{i+1} is displaced half a step vertically downwards (resp., upwards) in relation to U_i if $i+1 \in I$ (resp., $i+1 \notin I$), and the actions of x_i and y_i are shown as diagonal arrows. Note that the k-subset I can then be read off as the set of labels on the arrows pointing down to the right which are exposed to the top of the diagram. For example, the lattice diagram $\mathcal{L}_{\{1,4,5\}}$ in the case $k=3$, $n=8$, is shown in Figure 2.

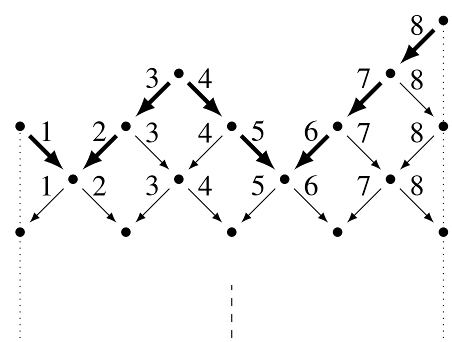


FIGURE 2. Lattice diagram of the module $L_{\{1,4,5\}}$

We see from Figure 2 that the module L_I is determined by its upper boundary, denoted by the thick lines, which we refer to as the *rim* of the module L_I (this is why we call the k-subset I the rim of L_I). Throughout this paper we will identify a rank 1 module L_I with its rim. Moreover, most of the time we will omit the arrows in the rim of L_I and represent it as an undirected graph.

We say that i is a *peak* of the rim I if $i \notin I$ and $i+1 \in I$. In the above example, the peaks of $I = \{1,4,5\}$ are 3 and 8. We say that i is a *valley* of the rim I if $i \in I$ and $i+1 \notin I$. In the above example, the valleys of $I = \{1,4,5\}$ are 1 and 5.

Proposition 2.1 ([8], Proposition 5.2)**.** *Every rank* 1 *Cohen-Macaulay $B_{k,n}$-module is isomorphic to L_I for some unique k-subset I of C_1.*

Every B-module has a canonical endomorphism given by multiplication by $t \in Z$. For L_I this corresponds to shifting $\mathcal{L}_I$ one step downwards. Since Z is central, $\mathrm{Hom}_B(M,N)$ is a Z-module for arbitrary B-modules M and N. If M,N are free Z-modules, then so is $\mathrm{Hom}_B(M,N)$. In particular, for any two rank 1 Cohen-Macaulay B-modules L_I and L_J, $\mathrm{Hom}_B(L_I,L_J)$ is a free module of rank 1 over $Z = \mathbb{C}[[t]]$, generated by the canonical map given by placing the lattice of L_I inside the lattice of L_J as far up as possible so that no part of the rim of L_I is strictly above the rim of L_J [8, Section 6].

Definition 2.3 (r-interlacing)**.** *Let I and J be two k-subsets of $\{1,\dots,n\}$. The sets I and J are said to be* r-interlacing *if there exist subsets $\{i_1,i_3,\dots,i_{2r-1}\} \subset I \setminus J$ and $\{i_2,i_4,\dots,i_{2r}\} \subset J \setminus I$ such that $i_1 < i_2 < i_3 < \cdots < i_{2r} < i_1$ (cyclically) and if there are not exist larger subsets of I and of J with this property. We say that I and J are* tightly r-interlacing *if they are r-interlacing and $|I \cap J| = k - r$.*

Definition 2.4. *A B-module is* rigid *if* $\mathrm{Ext}^1_B(M,M) = 0$.

If I and J are r-interlacing k-subsets, where $r < 2$, then $\mathrm{Ext}^1_B(L_I, L_J) = 0$, in particular, rank 1 modules are rigid (see [8, Proposition 5.6]).

Every indecomposable M of rank n in $\mathrm{CM}(B)$ has a filtration with factors L_{I_1}, $L_{I_2},\dots,L_{I_n}$ of rank 1. This filtration is noted in its *profile*, $\mathrm{pr}(M) = I_1 \mid I_2 \mid \dots \mid I_n$, [8, Corollary 6.7]. In the case of a rank 2 module M with filtration $L_I \mid L_J$ (i.e. with profile $I \mid J$), we picture this module by drawing the rim J below the rim I, in such a way that J is placed as far up as possible so that no part of the rim J is strictly above the rim I. We refer to this picture of M as its *lattice diagram.* Note that there is at least one point where the rims I and J meet (see Figure 3).

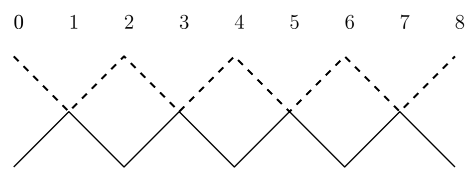
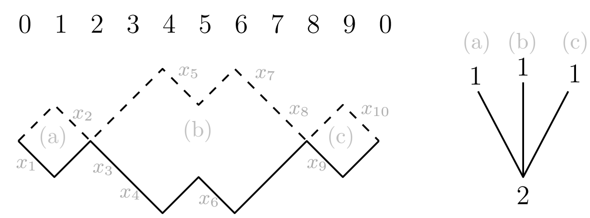


FIGURE 3. The lattice diagram of a module with filtration $L_{\{1,3,5,7\}} \mid L_{\{2,4,6,8\}}$.

The two rims in the lattice diagram of a rank 2 module M form a number of regions between the points where the two rims meet but differ in direction before and/or after meeting. We call these regions the *boxes* formed by the rims or by the profile. The term box is a combinatorial tool which is very useful in finding conditions for indecomposability. However, let us point out that the module M might be a direct sum in which case the lattice diagram is really a pair of lattice diagrams of rank 1 modules. We still view the corresponding diagram as forming boxes. If I and J are r-interlacing, then they form exactly r-boxes if and only if they are tightly r-interlacing. A lattice diagram with three boxes is shown in Figure 4. Moreover, the filtration layers of a module M give a poset structure. If M is a rank 2 module with r_1 boxes, with $r_1 \leq r$, the poset structure associated with M is $1^{r_1} \mid 2$, see Figure 4. The poset consists of a tree with one vertex of degree r_1 and r_1 leaves, it has dimension 1 at the leaves and dimension 2 at central vertex (we also refer to this as a *dimension lattice*). For background on the poset associated with an indecomposable module or its profile, we refer to [8, Section 6] and [3, Section 2].



$I = \{2,5,7,8,10\}$ $J = \{1,3,4,6,9\}$

FIGURE 4. The profile of a module with 4-interlacing layers forming three boxes with poset $1^3 \mid 2$. The dashed line shows the rim of L_I with arrows x_i, $i \in I$, indicated. The solid line below is the rim of L_J, with arrows x_i, $i \in J$, indicated.

A partial answer to the question of indecomposability of a rank 2 module in terms of its poset is given in the following proposition.

Proposition 2.2 ([2], Remark 3.2)**.** *Let $M \in \mathrm{CM}(B_{k,n})$ be an indecomposable module with profile $I \mid J$. Then I and J are r-interlacing and their poset is $1^{r_1} \mid 2$, where $r \geq r_1 \geq 3$.*

This result tells us that when dealing with rank 2 indecomposable modules, we can assume that the poset of such a module is of the form $1^{r_1} \mid 2$, for $r_1 \geq 3$.

Throughout the paper, our strategy to prove that a module is indecomposable is to show that its endomorphism ring does not have non-trivial idempotent elements. When we deal with a decomposable rank 2 module, in order to determine the summands of this module, we construct a non-trivial idempotent in its endomorphism ring, and then find corresponding eigenvectors at each vertex of the quiver and check the action of the morphisms x_i on these eigenvectors.

3. TIGHT r-INTERLACING

In this section we construct all rank 2 decomposable modules with the profile $I \mid J$ in the case when I and J are tightly r-interlacing k-subsets, i.e., when $|I \setminus J| = |J \setminus I| = r$ and non-common elements of I and J interlace.

We are interested in the modules M that are decomposable and appear as the middle term in a short exact sequence of the form:

$$0 \longrightarrow L_J \longrightarrow M \longrightarrow L_I \longrightarrow 0.$$

In [4], we defined a rank 2 module $\mathbb{M}(I,J)$ with filtration $L_I \mid L_J$ in a similar way as rank 1 modules are defined in $\mathrm{CM}(B_{k,n})$. We recall the construction here. Let $V_i := \mathbb{C}[|t|] \oplus \mathbb{C}[|t|]$, $i = 1, \ldots, n$. The module $\mathbb{M}(I,J)$ has V_i at each vertex $1, 2, \ldots, n$ of Γ_n. In order to have a module structure, for every i we need to define $x_i \colon V_{i-1} \to V_i$ and $y_i \colon V_i \to V_{i-1}$ in such a way that $x_i y_i = t \cdot \mathrm{id}$ and $x^k = y^{n-k}$.

Since L_J is a submodule of a rank 2 module $\mathbb{M}(I,J)$, and L_I is the quotient, if we extend the basis of L_J to the basis of the module $\mathbb{M}(I,J)$, then with respect to that basis all the matrices x_i, y_i must be upper triangular with diagonal entries from the set $\{1,t\}$. More precisely, the diagonal of x_i (resp. y_i) is $(1,t)$ (resp. $(t,1)$) if $i \in J \setminus I$, it is $(t,1)$ (resp. $(1,t)$) if $i \in I \setminus J$, (t,t) (resp. $(1,1)$) if $i \in I^c \cap J^c$, and $(1,1)$ (resp. (t,t)) if $i \in I \cap J$. The only entries in all these matrices that are left to be determined are the ones in the upper right corner.

Let us assume that we deal with the profile $\{1,3,\ldots,2r-1\} \mid \{2,4,\ldots,2r\}$ in the case $(r,2r)$. In the general case, all arguments are the same. Denote by b_i the upper right corner element of x_i. From $x_iy_i = t\cdot id$, we have that the upper right corner element of y_i is $-b_i$. From the relation $x^k = y^{n-k}$ it follows that $\sum_{i=1}^{2r} b_i = 0$. If $n=6$, $I=\{1,3,5\}$ and $J=\{2,4,6\}$, then our module $\mathbb{M}(I,J)$ is

$$V_0 \underset{\begin{pmatrix}1 & -b_1\\ 0 & t\end{pmatrix}}{\overset{\begin{pmatrix}t & b_1\\ 0 & 1\end{pmatrix}}{\rightleftarrows}} V_1 \underset{\begin{pmatrix}t & -b_2\\ 0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & b_2\\ 0 & t\end{pmatrix}}{\rightleftarrows}} V_2 \underset{\begin{pmatrix}1 & -b_3\\ 0 & t\end{pmatrix}}{\overset{\begin{pmatrix}t & b_3\\ 0 & 1\end{pmatrix}}{\rightleftarrows}} V_3 \underset{\begin{pmatrix}t & -b_4\\ 0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & b_4\\ 0 & t\end{pmatrix}}{\rightleftarrows}} V_4 \underset{\begin{pmatrix}1 & -b_5\\ 0 & t\end{pmatrix}}{\overset{\begin{pmatrix}t & b_5\\ 0 & 1\end{pmatrix}}{\rightleftarrows}} V_5 \underset{\begin{pmatrix}t & -b_6\\ 0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & b_6\\ 0 & t\end{pmatrix}}{\rightleftarrows}} V_0$$

FIGURE 5. A module with filtration $\{1,3,5\} \mid \{2,4,6\}$.

The question is how to determine the b_i's so that the module $\mathbb{M}(I,J)$ is decomposable. In [4], we dealt with the tame cases $(3,9)$ and $(4,8)$, and more generally, the 3-interlacing case, and we constructed all such modules and given criteria, in terms of divisibility by t of the sums $b_i + b_{i+1}$ (where i is odd), for the constructed module to be indecomposable. Moreover, in the case of a decomposable module, we determined the summands of such a module. In this paper we construct all decomposable modules in the general case of tight r-interlacing. We first consider the case $(r,2r)$ and show how the general case reduces to this case.

Assume first that $\mathbb{M}(I,J)$ is decomposable and that L_J is a direct summand of $\mathbb{M}(I,J)$. Then there exists a retraction $\mu = (\mu_i)_{i=1}^n$ such that $\mu_i \circ \theta_i = id$, where $(\theta_i)_{i=1}^n$ is the natural injection of L_J into $\mathbb{M}(I,J)$. Using the same basis as before, we can assume that $\mu_i = [1\ \alpha_i]$. From the commutativity relations we have $id \circ \mu_i = \mu_{i+1} \circ x_{i+1}$ for i odd, and $t\cdot id \circ \mu_i = \mu_{i+1} \circ x_{i+1}$ for i even. It follows that $\alpha_i = b_{i+1} + t\alpha_{i+1}$ for i odd, and $t\alpha_i = b_{i+1} + \alpha_{i+1}$ for i even. From this we have

$$t(\alpha_{2i} - \alpha_{2i+2}) = b_{2i+1} + b_{2i+2}, \text{ for } i = 0,\ldots,r-1.$$

Thus, if L_J is a direct summand of $\mathbb{M}(I,J)$, then $t|b_i + b_{i+1}$, for i odd, and we can easily find α_i, $i=1,\ldots,n$, satisfying previous equations. If only one of these divisibility conditions is not met, then L_J is not a direct summand of $\mathbb{M}(I,J)$. Note that if L_J is not a summand of $\mathbb{M}(I,J)$, it does not mean that M is indecomposable (cf. Theorem 3.12 in [2]). We will study the structure of the module $\mathbb{M}(I,J)$ in terms of the divisibility conditions the sums $b_i + b_{i+1}$ satisfy.

Let us now consider the general case, that is, let $\mathbb{M}(I,J)$ be the module as defined above, when I and J are tightly r-interlacing. Write $I \setminus J$ as $\{i_1,\dots,i_r\}$ and $J \setminus I$ as $\{j_1,\dots,j_r\}$ so that $1 \le i_1 < j_1 < i_2 < j_2 < \cdots < i_r < j_r \le n$. Define

$$x_{i_l} = \begin{pmatrix} t & b_{i_l} \\ 0 & 1 \end{pmatrix}, \quad x_{j_l} = \begin{pmatrix} 1 & b_{j_l} \\ 0 & t \end{pmatrix}, \quad y_{i_l} = \begin{pmatrix} 1 & -b_{i_l} \\ 0 & t \end{pmatrix}, \quad y_{j_l} = \begin{pmatrix} t & -b_{j_l} \\ 0 & 1 \end{pmatrix},$$

for $l = 1,2,\dots,r$ (see previous figure for $n = 6$). For $i \in I^c \cap J^c$, we set $x_i = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ and $y_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For $i \in I \cap J$, we set $x_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $y_i = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$. Also, we assume that $\sum_{l=1}^{n} b_l = 0$. Note that for $i \in (I^c \cap J^c) \cup (I \cap J)$ we define the matrices x_i and y_i to be diagonal, i.e. we assume that the upper right corner of x_i and y_i is 0 if $i \in (I^c \cap J^c) \cup (I \cap J)$. This is because if it were not 0, then by a suitable base change of the V_i, by changing the second basis element, we obtain a scalar matrix. By construction, $xy = yx$ and $x^k = y^{n-k}$ at all vertices, and $\mathbb{M}(I,J)$ is free over the centre of $B_{k,n}$. Hence, the following proposition holds.

Proposition 3.1. *The module* $\mathbb{M}(I,J)$ *as constructed above is in* $\mathrm{CM}(B_{k,n})$.

As in the case of the profile $\{1,3,\dots,2r-1\} \mid \{2,4,\dots,2r\}$, L_J is a direct summand of $\mathbb{M}(I,J)$ if and only if $t \mid b_{i_l} + b_{j_l}$, for all l. In order to determine the structure of the module $\mathbb{M}(I,J)$ when these divisibility conditions are not fulfilled (i.e., at least one of the sums $b_{i_l} + b_{j_l}$ is not divisible by t), we determine the structure of an endomorphism of this module. The following proposition is a generalization of Proposition 3.3 in [4].

For the rest of the paper, if $t^d v = w$, for a positive integer d, then $t^{-d} w$ denotes v.

Proposition 3.2. *For* $n \ge 6$, *let* I,J *be tightly* r*-interlacing,* $I \setminus J = \{i_1,\dots,i_r\}$, *and* $J \setminus I = \{j_1,\dots,j_r\}$, *where* $1 \le i_1 < j_1 < i_2 < j_2 < \cdots < i_r < j_r \le n$. *Let* $B_l := \sum_{g=1}^{l} (b_{i_g} + b_{j_g})$. *If* $\varphi = (\varphi_i)_{i=1}^n \in \mathrm{End}(\mathbb{M}(I,J))$, *then*

$$\varphi_{j_r} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\varphi_{i_l} = \begin{pmatrix} a + (B_{l-1} + b_{i_l}) t^{-1} c & tb + (d-a)(B_{l-1} + b_{i_l}) - (B_{l-1} + b_{i_l})^2 t^{-1} c \\ t^{-1} c & d - (B_{l-1} + b_{i_l}) t^{-1} c \end{pmatrix}, \tag{3.1}$$

$$\varphi_{j_l} = \begin{pmatrix} a + B_l t^{-1} c & b + t^{-1}((d-a)B_l - B_l^2 t^{-1} c) \\ c & d - B_l t^{-1} c \end{pmatrix},$$

$$\varphi_i = \varphi_{i-1}, \text{ for } i \in (I^c \cap J^c) \cup (I \cap J),$$

where $l = 1,2,\dots,r-1$, *with* $a,b,c,d \in \mathbb{C}[|t|]$, *and*

$$t \mid c, \quad t \mid (d-a)B_l - B_l^2 t^{-1} c. \tag{3.2}$$

Proof. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be an endomorphism of $\mathbb{M}(I,J)$, where each φ_i is an element of $M_2(\mathbb{C}[|t|])$ (matrices over the centre). We use commutativity relations $x_{i+1}\varphi_i = \varphi_{i+1}x_{i+1}$. That $x_{i_1}\varphi_{j_r} = \varphi_{i_1}x_{i_1}$ follows directly from $x_{i_1}x_{i_1-1}\cdots x_{j_r+1}\varphi_{j_r} = \varphi_{i_1}x_{i_1}x_{i_1-1}\cdots x_{j_r+1}$. Recall that $x_{i_1-1}, \dots, x_{j_r+1}$ are scalar matrices so they cancel out. If $\varphi_{j_r} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\varphi_{i_1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then $t \mid c$, $e = a + b_1t^{-1}c$, $f = tb + (d - a)b_1 - b_1^2t^{-1}c$, $g = t^{-1}c$, and $h = d - b_1t^{-1}c$. The rest is shown in the same way.

$$
\begin{array}{ccccccccc}
V_{j_r} & \underset{y_{jr+1}}{\overset{x_{jr+1}}{\rightleftarrows}} & V_{j_r+1} & \underset{y_{jr+2}}{\overset{x_{jr+2}}{\rightleftarrows}} & \cdots & \underset{y_{i_1-1}}{\overset{x_{i_1-1}}{\rightleftarrows}} & V_{i_1-1} & \underset{\begin{pmatrix} 1 & -b_1 \\ 0 & t \end{pmatrix}}{\overset{\begin{pmatrix} t & b_1 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} & V_{i_1} \\
\downarrow \varphi_{j_r} & & & & & & & & \downarrow \varphi_{i_1} \\
V_{j_r} & \underset{y_{jr+1}}{\overset{x_{jr+1}}{\rightleftarrows}} & V_{j_r+1} & \underset{y_{jr+2}}{\overset{x_{jr+2}}{\rightleftarrows}} & \cdots & \underset{y_{i_1-1}}{\overset{x_{i_1-1}}{\rightleftarrows}} & V_{i_1-1} & \underset{\begin{pmatrix} 1 & -b_1 \\ 0 & t \end{pmatrix}}{\overset{\begin{pmatrix} t & b_1 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} & V_{i_1}
\end{array}
$$

The only thing left to note is that if $i \in (I^c \cap J^c) \cup (I \cap J)$, then x_i is a scalar matrix (either identity or t times identity), so from $x_i\varphi_{i-1} = \varphi_i x_i$, we have $\varphi_{i-1} = \varphi_i$. □

By Remark 3.4 in [4], if φ is the morphism from the previous proposition, then it is sufficient to prove for a single index i that φ_i is idempotent in order to prove that φ is idempotent. Also, note that in our computations, for $i \in (I^c \cap J^c) \cup (I \cap J)$, x_i is a scalar matrix, it commutes with every other matrix and it cancels out in $x_i\varphi_{i-1} = \varphi_i x_i$, so it can be left out.

We now give necessary and sufficient conditions for the module $\mathbb{M}(I,J)$ to be indecomposable.

Theorem 3.1. *Let $\mathbb{M}(I,J)$ be as in the previous proposition. The module $\mathbb{M}(I,J)$ is indecomposable if and only if there exist indices i_g and i_l, where $g < l$, such that $t \mid b_{i_s} + b_{j_s}$, for $g < s < l$, $t \nmid b_{i_g} + b_{j_g}$, $t \nmid b_{i_l} + b_{j_l}$, and $t \nmid b_{i_g} + b_{j_g} + b_{i_l} + b_{j_l}$.*

Proof. As in the proof of the previous proposition, it is sufficient to consider the case $(k,n) = (r, 2r)$ of tight r-interlacing where $I = \{1,3,5,\dots,2r-1\}$ and $J = \{2,4,6,\dots,2r\}$. Let $i_{l_1}, i_{l_2}, \dots, i_{l_s}$ be all odd indices i (in cyclic ordering) such that $b_i + b_{i+1}$ is not divisible by t (note that $i_{l_g} + 1 = j_{l_g}$). We assume that there is at least one such index because if $t \mid b_i + b_{i+1}$ for all odd i, then $\mathbb{M}(I,J) \cong L_I \oplus L_J$.

Let $\varphi = (\varphi_i)_{i=0}^{n-1} \in \mathrm{End}(\mathbb{M}(I,J))$ be an idempotent homomorphism and assume that $\varphi_{j_r} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Again, denote by B_m the sum $\sum_{g=1}^{m}(b_{i_{l_g}} + b_{j_{l_g}})$. The divisibility conditions (3.2) from the previous proposition reduce to

$$t \mid c \text{ and } t \mid (d-a)B_i - B_i^2 t^{-1} c,\ i = 1,2,\ldots,s-1. \tag{3.3}$$

Without loss of generality we assume that $t \nmid B_2$. Relations (3.3) imply that

$$t \mid d - a - B_1 t^{-1} c \text{ and } t \mid d - a - B_2 t^{-1} c.$$

Thus, it must hold that $t \mid (b_{i_{l_2}} + b_{j_{l_2}})t^{-1}c$, and since $t \nmid b_{i_{l_2}} + b_{j_{l_2}}$, it must be that $t \mid t^{-1}c$, and subsequently that $t \mid d - a$.

From the fact that φ_{j_r} is idempotent and $t \mid c$ it follows that $t \mid a - a^2$ and $t \mid d - d^2$. Also, from $\varphi_{j_r}^2 = \varphi_{j_r}$ it follows that either $a = d$ or $a + d = 1$. If $a = d$, then $b = c = 0$ (otherwise $a = d = \frac{1}{2}$ and $\frac{1}{4} = bc$, which is not possible as c is divisible by t), and $a = d = 1$ or $a = d = 0$ giving us the trivial idempotents. If $a + d = 1$, then $t \mid a$ or $t \mid d$. Taking into account that $t \mid d - a$, we conclude that $t \mid a$ and $t \mid d$. This implies that $1 = a + d$ is divisible by t, which is not true. Thus, the only idempotent homomorphisms of $\mathbb{M}(I,J)$ are the trivial ones. Hence, $\mathbb{M}(I,J)$ is indecomposable.

Assume now that $t \mid b_{i_{l_g}} + b_{j_{l_g}} + b_{i_{l_{g+1}}} + b_{j_{l_{g+1}}}$ for every $g < s$. Then the divisibility conditions (3.3) for the endomorphism φ reduce to a single condition

$$t \mid d - a - (b_{i_{l_1}} + b_{j_{l_1}})t^{-1}c.$$

In order to find a non-trivial idempotent φ, we only need to find elements a, b, c, and d in such a way that $t \mid c$ and $t \mid d - a - (b_{i_{l_1}} + b_{j_{l_1}})t^{-1}c$. Recall that if $a = d$, then we only obtain the trivial idempotents because $t \mid c$. So it must be $a + d = 1$ if we want to find a non-trivial idempotent. If we choose $a = 1$, $d = 0$, then $t \mid 1 + (b_{i_{l_1}} + b_{j_{l_1}})t^{-1}c$. Thus, we can define $c = -t(b_{i_{l_1}} + b_{j_{l_1}})^{-1}$, and $b = 0$ since $a - a^2 = bc$ and $c \neq 0$, to get the idempotent $\varphi_{j_r} = \begin{pmatrix} 1 & 0 \\ -t(b_{i_{l_1}} + b_{j_{l_1}})^{-1} & 0 \end{pmatrix}$.

Since this is a non-trivial idempotent, the module $\mathbb{M}(I,J)$ is decomposable. $\square$

Remark 3.1. From the previous theorem, keeping the notation from the case $(k,n) = (r,2r)$, it follows that if $\mathbb{M}(I,J)$ is a decomposable module, then since $\sum_{i=1}^{n} b_i = 0$ there is an even number of odd i such that $t \nmid b_i + b_{i+1}$. If there was not were an odd number of odd i such that $t \nmid b_i + b_{i+1}$, then for two consecutive l_1 and l_2, it would hold that $t \nmid b_{i_{l_1}} + b_{i_{l_1}+1} + b_{i_{l_2}} + b_{i_{l_2}+1}$. Our aim is to determine all such decomposable modules, so for the rest of the paper we assume that there is an even number of odd indices i_{l_g} such that $t \nmid b_{i_{l_g}} + b_{i_{l_g}+1}$ and $t \mid b_{i_{l_g}} + b_{i_{l_g}+1} + b_{i_{l_{g+1}}} + b_{i_{l_{g+1}}+1}$ for every g.

Corollary 3.1. *If $n < 6$, there are no indecomposable rank 2 modules in* $\mathrm{CM}(B_{k,n})$.

The rest of the paper is dedicated to the determination of the summands of the module $\mathbb{M}(I,J)$ in the case when this module is decomposable. It is sufficient to study the case of the filtration $\{1,3,\ldots,2r-1\} \mid \{2,4,\ldots,2r\}$ when $k=r$ and $n=2r$. Then the general case of tight r-interlacing follows because the scalar matrices can be ignored since they do not affect any of the computations we conduct.

Denote $I=\{1,3,\ldots,2r-1\}$ and $J=\{2,4,\ldots,2r\}$. As before, assume that $x_i=\begin{pmatrix} t & b_i \\ 0 & 1 \end{pmatrix}$ for odd i and $x_i=\begin{pmatrix} 1 & b_i \\ 0 & t \end{pmatrix}$ for even i, and that $\sum_{i=1}^{2r} b_i=0$ so that we have a module structure, which we again denote by $\mathbb{M}(I,J)$.

The dimension lattice of a given module in $\mathrm{CM}(B_{k,n})$ is additive on short exact sequences. If $\mathbb{M}(I,J)$is the direct sum $L_X \oplus L_Y$, then from the short exact sequence

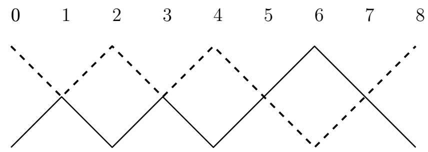
$$0 \longrightarrow L_J \longrightarrow L_X \oplus L_Y \longrightarrow L_I \longrightarrow 0$$

follows that the dimension lattices of L_X and L_Y add up to the sum of the dimension lattice of L_I and the dimension lattice of L_J. In terms of the rims, one way to combinatorially describe possible summands L_X and L_Y is by the fact that the rim of X has to be "taken out" from the lattice diagram of $L_X \oplus L_Y$ in such a way that the leftover part of the lattice diagram is the rim Y.

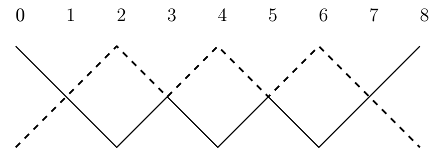
In terms of the peaks of the profile $I \mid J$, the rim X corresponds to a subset of the set of the peaks of I, and the rim Y corresponds to the complement of this set with respect to the set of peaks of I. To describe it in terms of the path we take in the lattice diagram of $I \mid J$ by travelling from left to right, we start from a peak of I and move to the right (we either go up or down in each step). If we are at a peak of I (resp. valley of J), then the next step has to be down (resp. up). If we are at a peak of J, which is also a valley of I, then we have a choice of going up or down. Eventually, to finish our trip, we have to return to the peak where we started off. The rim X is determined by the set of peaks of I we passed through during our trip (by abuse of notation we say that X passes through this set of peaks), and the rim Y is determined by the peaks of I we did not pass through.

The first four pictures in Figure 6 correspond to the case when X passes through a single peak of I (and Y passes through three peaks) when we travel from left to right through the lattice diagram of $I \mid J$. The next three pictures correspond to the case when X passes through two peaks of I (and Y passes through two peaks), and the last picture corresponds to the case when X passes through all peaks of I and Y passes through none. Obviously, there is symmetry in the argument so the case when X passes through one peak and Y through three peaks is the same as the case when X passes through three peaks and Y passes through one peak. In total, there are 2^{r-1} different cases.

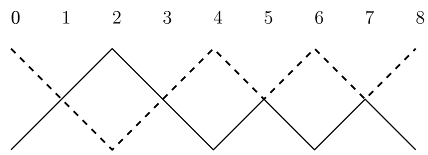
Example 3.1. *In the case $r=4$, there are eight possible choices for X and Y in such a way that the sum of the dimension lattices of X and Y is the sum of the dimension lattices of L_I and L_J. They are given in Figure 6.*



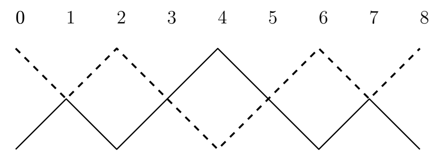
(A) $L_{\{1,3,5,6\}} \oplus L_{\{2,4,7,8\}}$



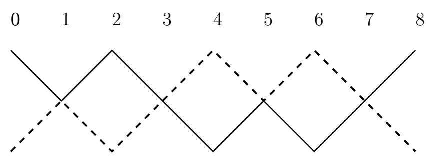
(B) $L_{\{1,2,4,6\}} \oplus L_{\{3,5,7,8\}}$



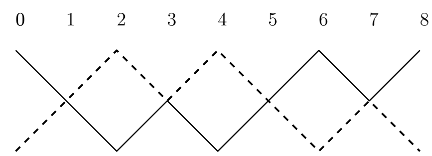
(C) $L_{\{1,2,5,7\}} \oplus L_{\{3,4,6,8\}}$



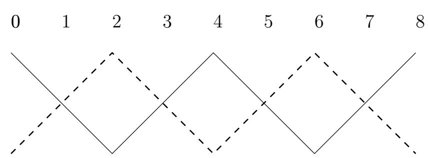
(D) $L_{\{1,3,4,7\}} \oplus L_{\{2,5,6,8\}}$



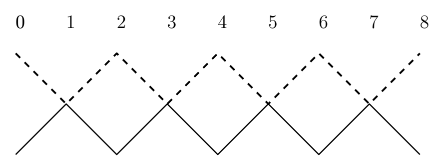
(E) $L_{\{1,3,4,6\}} \oplus L_{\{2,5,7,8\}}$



(F) $L_{\{1,2,4,7\}} \oplus L_{\{3,5,6,8\}}$



(G) $L_{\{1,2,5,6\}} \oplus L_{\{3,4,7,8\}}$



(H) $L_{\{1,3,5,7\}} \oplus L_{\{2,4,6,8\}}$

FIGURE 6. The pairs of profiles of decomposable extensions between $L_{\{1,3,5,7\}}$ and $L_{\{2,4,6,8\}}$.

Note that we only classify decomposable modules that are extensions of L_J by L_I, not all possible extensions (cf. Remark 3.9 in [4]).

For a given set X, i.e., for a given subset of the set of peaks of I, and the corresponding Y, we give the divisibility conditions for the sums $b_i + b_{i+1}$ so that the module $\mathbb{M}(I,J)$ is isomorphic to $L_X \oplus L_Y$. We denote by X' (resp. Y') the set of peaks of I that corresponds to X (resp. Y).

If X passes through every peak of I, then this is the case when $X = I$ and $Y = J$, i.e., the case of the direct sum $L_I \oplus L_J$. In terms of the divisibility conditions, this is the case when $t \mid b_i + b_{i+1}$, for every odd i.

Assume now that X' does not contain all peaks of I. This means that there is a peak, say $2j$, that belongs to X', such that the next peak $2j+2$ belongs to Y'.

Then $b_{2j+1}+b_{2j+2}$ is not divisible by t. If it were divisible by t, then $2j+2$ would belong to X' as we explain in the proof of the next theorem.

Recall that there has to be an even number of steps where we go from a valley to a peak or from a peak to a valley so that we can come back up to the starting point.

Theorem 3.2. *Let X' be a subset of the set of peaks of I, $\{0,2,4,\dots,2r-2\}$, Y' its complement, and X and Y corresponding k-subsets of $[n]$. Also, assume that $1 \le |X'| < r$. Starting from a peak in X', and moving to the right, define sums $b_i + b_{i+1}$ so that the following conditions hold:*

(1) *Assume that the current peak $2j$ belongs to X'. If the next peak $2j+2$ belongs to X', then $t \mid b_{2j+1}+b_{2j+2}$.*
(2) *Assume that the current peak $2j$ does not belong to X'. If the next peak $2j+2$ belongs to X', then $t \nmid b_{2j+1}+b_{2j+2}$.*
(3) *Assume that the current peak $2j$ belongs to X'. If the next peak $2j+2$ does not belong to X', then $t \nmid b_{2j+1}+b_{2j+2}$.*
(4) *Assume that the current peak $2j$ does not belong to X'. If the next peak $2j+2$ does not belong to X', then $t \mid b_{2j+1}+b_{2j+2}$.*

Additionally, we assume that $t \mid b_{i_1}+b_{i_1+1}+b_{i_2}+b_{i_2+1}$ for every two consecutive odd indices i_1 and i_2 such that $t \nmid b_{i_l}+b_{i_{l+1}}$, $l=1,2$. Then the module $\mathbb{M}(I,J)$ is isomorphic to the direct sum $L_X \oplus L_Y$.

Proof. By Theorem 3.1, the module is decomposable. We start our path at a peak from X'. Assume without loss of generality that this peak is 0 and that the next peak 2 does not belong to X', i.e., $t \nmid b_1+b_2$. Define an idempotent $\varphi_0 = \begin{pmatrix} 1 & 0 \\ -t(b_1+b_2)^{-1} & 0 \end{pmatrix}$. Its orthogonal complement is $\tilde{\varphi}_0 = \begin{pmatrix} 0 & 0 \\ t(b_1+b_2)^{-1} & 1 \end{pmatrix}$. From (3.1) we compute other idempotents φ_i. Let $B_l = \sum_{i=1}^{l} b_i$. For odd indices we get

$$\varphi_{2j+1} = \begin{pmatrix} 1-B_{2j+1}(b_1+b_2)^{-1} & -B_{2j+1}+B_{2j+1}^2(b_1+b_2)^{-1} \\ -(b_1+b_2)^{-1} & -B_{2j+1}(b_1+b_2)^{-1} \end{pmatrix},$$

$$\tilde{\varphi}_{2j+1} = \begin{pmatrix} B_{2j+1}(b_1+b_2)^{-1} & B_{2j+1}(1-B_{2j+1}(b_1+b_2)^{-1}) \\ (b_1+b_2)^{-1} & 1-B_{2j+1}(b_1+b_2)^{-1} \end{pmatrix},$$

and for even indices

$$\varphi_{2j} = \begin{pmatrix} 1-B_{2j}(b_1+b_2)^{-1} & -t^{-1}B_{2j}(1-B_{2j}(b_1+b_2)^{-1}) \\ -t(b_1+b_2)^{-1} & B_{2j}(b_1+b_2)^{-1} \end{pmatrix},$$

$$\tilde{\varphi}_{2j} = \begin{pmatrix} B_{2j}(b_1+b_2)^{-1} & t^{-1}B_{2j}(1-B_{2j}(b_1+b_2)^{-1}) \\ t(b_1+b_2)^{-1} & 1-B_{2j}(b_1+b_2)^{-1} \end{pmatrix}.$$

Let v_i (resp. w_i) be the eigenvector of φ_i (resp. $\tilde{\varphi}_i$) corresponding to the eigenvalue 1. The vectors w_i (resp. v_i) form a basis for L_X (resp. L_Y). We compute directly these eigenvectors. For an odd index we have

$$w_{2j+1} = \begin{pmatrix} B_{2j+1} \\ 1 \end{pmatrix}, \qquad v_{2j+1} = \begin{pmatrix} 1 - B_{2j+1}(b_1+b_2)^{-1} \\ -(b_1+b_2)^{-1} \end{pmatrix}.$$

Since $t \mid b_{i_1} + b_{i_1+1} + b_{i_2} + b_{i_2+1}$, for every two consecutive odd indices i_1 and i_2 such that $t \nmid b_{i_l} + b_{i_l+1}$, $l = 1,2$, when computing w_{2j} and v_{2j} we have to distinguish between the following cases. Let g be the number of indices in the set $[1, j]$ such that $t \nmid b_{2j-1} + b_{2j}$. If g is even (resp. odd), then $t \mid B_{2j}$ (resp. $t \nmid B_{2j}$). Therefore, if g is even, i.e., if B_{2j} is divisible by t, then

$$w_{2j} = \begin{pmatrix} t^{-1}B_{2j} \\ 1 \end{pmatrix}, \qquad v_{2j} = \begin{pmatrix} 1 - B_{2j}(b_1+b_2)^{-1} \\ -t(b_1+b_2)^{-1} \end{pmatrix}.$$

If g is odd, i.e., if B_{2j} is not divisible by t ($B_{2j} = b_1 + b_2 + tz$, for some z), then

$$w_{2j} = \begin{pmatrix} B_{2j} \\ t \end{pmatrix}, \qquad v_{2j} = \begin{pmatrix} t^{-1}(1 - B_{2j}(b_1+b_2)^{-1}) \\ -(b_1+b_2)^{-1} \end{pmatrix}.$$

Combinatorially, g is even (resp. odd) if and only if we are positioned at a peak (resp. valley) $2j$ after $(2j)$th step. This follows from the fact that $t \nmid b_{2j-1} + b_{2j}$ means that we are moving either from a peak to a valley, or from a valley to a peak. Since we started from a peak, if we are currently at a peak $2j$, then this means that we had an even number of the moves that correspond to the sums $b_{2i-1} + b_{2i}$ that are not divisible by t.

Consider the eigenvectors $v_0 = [1\ ,\ -t(b_1+b_2)^{-1}]^t$, $w_0 = [0\ ,\ 1]^t$ for φ_0 and its orthogonal complement. Then $x_1 w_0 = w_1$ and $x_2 w_1 = w_2$, so $1,2 \in X$. Also, $x_1 v_0 = t v_1$ and $x_2 v_1 = t v_2$, so $1,2 \notin Y$. We continue by moving to the right and consider the four cases from the statement of the theorem.

Case 1: Assume that the current peak $2j$ belongs to X'. If the next peak $2j+2$ belongs to X', then $t \mid b_{2j+1} + b_{2j+2}$. In this situation we are moving from a peak to another peak by going down and then up. Here, $t \mid B_{2j}$ and $t \mid B_{2j+2}$. Since $w_{2j} = \begin{pmatrix} t^{-1}B_{2j} \\ 1 \end{pmatrix}$, $w_{2j+1} = \begin{pmatrix} B_{2j+1} \\ 1 \end{pmatrix}$, and $w_{2j+2} = \begin{pmatrix} t^{-1}B_{2j+2} \\ 1 \end{pmatrix}$, we have $x_{2j+1} w_{2j} = w_{2j+1}$ and $x_{2j+2} w_{2j+1} = t w_{2j+2}$. Thus, $2j+1 \in X$ and $2j+2 \notin X$. Analogously, $x_{2j+1} v_{2j} = t v_{2j+1}$ and $x_{2j+2} v_{2j+1} = v_{2j+2}$. Hence, $2j+1 \notin Y$ and $2j+2 \in Y$.

Case 2: Assume that the current peak $2j$ is not in X'. If the next peak $2j+2$ belongs to X', then $t \nmid b_{2j+1} + b_{2j+2}$ and we are moving from a valley to a peak, i.e., $t \nmid B_{2j}$, $t \mid B_{2j+2}$, $w_{2j} = \begin{pmatrix} B_{2j} \\ t \end{pmatrix}$, $w_{2j+1} = \begin{pmatrix} B_{2j+1} \\ 1 \end{pmatrix}$, and $w_{2j+2} = \begin{pmatrix} t^{-1}B_{2j+2} \\ 1 \end{pmatrix}$. It follows that $x_{2j+1} w_{2j} = t w_{2j+1}$ and $x_{2j+2} w_{2j+1} = t w_{2j+2}$. Therefore, $2j+1 \notin X$ and $2j+2 \notin X$. Analogously, $x_{2j+1} v_{2j} = v_{2j+1}$ and $x_{2j+2} v_{2j+1} = v_{2j+2}$. Thus, $2j+1 \in Y$ and $2j+2 \in Y$.

Case 3: Assume that the current peak $2j$ belongs to X'. If the next peak $2j+2$ is not in X', then $t \nmid b_{2j+1} + b_{2j+2}$ and we move from a peak to a valley. Here, $t \mid B_{2j}$, $t \nmid$

B_{2j+2}, $w_{2j} = \begin{pmatrix} t^{-1}B_{2j} \\ 1 \end{pmatrix}$, $w_{2j+1} = \begin{pmatrix} B_{2j+1} \\ 1 \end{pmatrix}$, and $w_{2j+2} = \begin{pmatrix} B_{2j+2} \\ t \end{pmatrix}$. It follows that $x_{2j+1}w_{2j} = w_{2j+1}$ and $x_{2j+2}w_{2j+1} = w_{2j+2}$. Thus, $2j+1, 2j+2 \in X$. Analogously, $x_{2j+1}v_{2j} = tv_{2j+1}$ and $x_{2j+2}v_{2j+1} = tv_{2j+2}$. Therefore, $2j+1, 2j+2 \notin Y$.

Case 4: Assume that the current peak $2j$ is not in X'. If the next peak $2j+2$ is not in X', then $t \mid b_{2j+1} + b_{2j+2}$ and we move from a valley to a valley. Here, $t \nmid B_{2j}$ and $t \nmid B_{2j+2}$. In this case $w_{2j} = \begin{pmatrix} B_{2j} \\ t \end{pmatrix}$, $w_{2j+1} = \begin{pmatrix} B_{2j+1} \\ 1 \end{pmatrix}$, and $w_{2j+2} = \begin{pmatrix} B_{2j+2} \\ t \end{pmatrix}$. It follows that $x_{2j+1}w_{2j} = tw_{2j+1}$ and $x_{2j+2}w_{2j+1} = w_{2j+2}$. Hence, $2j+1 \notin X$ and $2j+2 \in X$. Analogously, $x_{2j+1}v_{2j} = v_{2j+1}$ and $x_{2j+2}v_{2j+1} = tv_{2j+2}$. Thus, $2j+1 \in Y$ and $2j+2 \notin Y$. □

Remark 3.2. Since X' was arbitrary in the previous theorem, and it corresponds to an arbitrary rim X and the module L_X that is, combinatorially, a possible summand of $\mathbb{M}(I,J)$, it follows that we proved that the modules constructed in the previous theorem are all possible decomposable modules with filtration $I \mid J$.

Example 3.2. *Consider Figure (F) from Example 3.1 (see Figure 6). Here, $X =$*

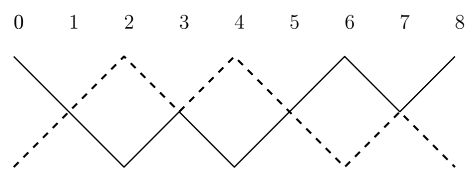


FIGURE 7. A pair of profiles for $L_{\{1,2,4,7\}} \oplus L_{\{3,5,6,8\}}$

$\{1,2,4,7\}$, $X' = \{0,6\}$, $Y = \{3,5,6,8\}$, and $Y' = \{2,4\}$. Define b_i, $i = 1,\dots,8$, as follows. We start at the peak 0, and we travel to the right by going through two points at each step. We first reach valley 2 by going down and down. Here, $t \nmid b_1 + b_2$ because of the third condition from the previous theorem. Then we reach valley 4 by going up and down. Here, $t \mid b_3 + b_4$ because of the fourth condition from the previous theorem. Next, we reach peak 6 by going up and up. By the second condition from the previous theorem, it must be $t \nmid b_5 + b_6$. Finally, we come back to the starting peak 0 by going down and then up. As stated in the first condition of the previous theorem, we have that $t \mid b_7 + b_8$. Therefore, if $t \nmid b_1 + b_2$, $t \mid b_3 + b_4$, $t \nmid b_5 + b_6$, and $t \mid b_7 + b_8$, then the module $\mathbb{M}(I,J)$ is isomorphic to $L_{\{1,2,4,7\}} \oplus L_{\{3,5,6,8\}}$. Note that we also have to make sure that $\sum_{i=1}^{8} b_i = 0$. For example, we can set $b_1 + b_2 = -(b_5 + b_6) = 1$ and $b_3 + b_4 = -(b_7 + b_8) = t$.

Remark 3.3. It is not too difficult to generalize Theorem 3.2, by taking analogous paths in the lattice diagram, to the general case when the layers of the profile $I \mid J$ are r-interlacing, $r \geq 3$, and the profile $I \mid J$ has r squared boxes, with poset $1^r \mid 2$ (we refer the reader to [4] for details on the notion of a box, the poset of a profile, and a branching point of a profile). The path is analogous to the path in the tight interlacing case, at each branching point (a point where the rims meet) we have an option to either go up or down. This path uniquely determines the summands L_X and L_Y. For instance, in [4, Example 4.14], in order to construct decomposable modules with the profile 2478 | 1356 we define $x_i = \begin{pmatrix} t & b_i \\ 0 & 1 \end{pmatrix}$, for $i = 2,4,7,8$, and $x_i = \begin{pmatrix} 1 & b_i \\ 0 & t \end{pmatrix}$, for $i = 1,3,5,6$, and assume that $\sum_{i=1}^{6} b_i + t(b_6 + b_7) = 0$. If $t \mid b_8 + b_1$, $t \mid b_2 + b_3$, and $t \mid b_4 + b_5$, then $\mathbb{M}(I,J) \cong L_I \oplus L_J$. If $t \nmid b_2 + b_3$, $t \mid b_4 + b_5$, $t \nmid b_8 + b_1$, then $\mathbb{M}(I,J) \cong L_{\{2,3,5,6\}} \oplus L_{\{1,4,7,8\}}$. If $t \mid b_2 + b_3$, $t \nmid b_4 + b_5$, $t \nmid b_8 + b_1$, then $\mathbb{M}(I,J) \cong L_{\{2,4,5,6\}} \oplus L_{\{1,3,7,8\}}$. If $t \nmid b_2 + b_3$, $t \nmid b_4 + b_5$, $t \mid b_8 + b_1$, then $\mathbb{M}(I,J) \cong L_{\{2,3,7,8\}} \oplus L_{\{1,4,5,6\}}$.

REFERENCES

[1] K. Baur and D. Bogdanic, *Extensions between Cohen–Macaulay modules of Grassmannian cluster categories*, J. Algebraic Combin., (2016), 1–36.

[2] K. Baur, D. Bogdanic, and A. G. Elsener, *Cluster categories from Grassmannians and root combinatorics*, Nagoya Math. J., (2019), 1–33.

[3] K. Baur, D. Bogdanic, A. G. Elsener, and J.-R. Li, *Rigid indecomposable modules in Grassmannian cluster categories*, arXiv:2011.09227, (2020)

[4] K. Baur, D. Bogdanic, and J.-R. Li, *Construction of rank* 2 *indecomposable modules in Grassmannian cluster categories*, Advanced Studies in Pure Mathematics, Volume 88 ”The McKay correspondence, Mutation and related topics”, Mathematical Society of Japan, (2022).

[5] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations. I. Mutations*, Selecta Math. (N.S.), vol. 14, no. 1, (2008), 59–119.

[6] C. Geiss, B. Leclerc, and J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math., vol. 165, no. 3, (2006), 589–632.

[7] C. Geiss, B. Leclerc, and J. Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble), vol. 58, no. 3, (2008), 825–876.

[8] B. T. Jensen, A. D. King, and X. Su, *A categorification of Grassmannian cluster algebras,* Proc. Lond. Math. Soc. (3), vol. 113, no. 2, (2016), 185–212.

(Received: December 30, 2020)
(Revised: January 28, 2022)

Dusko Bogdanic
University of Banja Luka
Faculty of Natural Sciences and Mathematics
Mladena Stojanovica 2, 78000 Banja Luka
Bosnia and Herzegovina
e-mail: *dusko.bogdanic@pmf.unibl.org*
and
Ivan-Vanja Boroja
University of Banja Luka
Faculty of Electrical Engineering
Patre 5, 78000 Banja Luka
Bosnia and Herzegovina
e-mail: *ivan-vanja.boroja@etf.unibl.org*