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# DIFFERENCE EQUATIONS ON $\mathbb{R}_*^+$, OF THE FORM $u_{n+2} = \dfrac{f(u_{n+1})}{u_n+\lambda}$, $\lambda > 0$, WITH APPLICATIONS TO PERTURBATIONS OF DYNAMICAL SYSTEMS

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*Dedicated to the 70th birthday of Prof. Mustafa Kulenović*

ABSTRACT. There is classical difference equation on $\mathbb{R}_*^+$

$$u_{n+2}u_n = f(u_{n+1}), \tag{1}$$

that is in particular applied to several symmetric special QRT-applications. We study perturbations of this equation (1) by

$$u_{n+2}(u_n+\lambda) = f(u_{n+1}), \quad \lambda > 0, \tag{2}$$

showing general theorems of regarding permanence, convergence or divergence, and attraction of the fixed point.

## 1. INTRODUCTION

We know the behaviour of solutions of (1) when they are equations associated with a symmetric dynamics QRT-system: there is a fixed point (or ”equilibrium”) $(\ell_1, \ell_1)$ in (1), and if $M_0 := (u_0, u_1) \neq (\ell_1, \ell_1)$, the sequence $u_n$ diverges, and the orbit of the point $M_n = (u_n, u_{n+1})$ lies in the biquadratic curve with equation $Q_1(x,y) - Kxy = 0$ which passes through the point $M_0$, being periodic or dense. Moreover the quantity $G(x,y) = \frac{Q_1(x,y)}{xy}$ is invariant for the map: $(x,y) \mapsto (X,Y) := T_1(x,y) = \left(y, \frac{f(y)}{x}\right)$: $G(X,Y) = G(x,y)$.

A celebrated QRT-difference is Lyness' equation: $u_{n+2} = \frac{u_{n+1}+a}{u_n}$; the perturbed equation is $u_{n+2} = \frac{u_{n+1}+a}{u_n+\lambda}$, where $\lambda > 0$. This difference equation required a long period of time to be solved by [16]: if $a > 0$ and $u_0, u_1 > 0$, the sequence an converges to a limit, the fixed point. The method for proof of this theorem used algebra computer system (see also [13]).

In [2], the question is: for an analogous QRT-difference equation with a fixed point, it is the **problem (A)**: are the solutions of (2), when (1) was of type symmetric QRT, convergent to a limit of the fixed point for all $\lambda > 0$. This was the case in [16], and there are some cases in [2].

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When the difference (1) is not a QRT-difference (there is no invariant map or first integral), the problem for (2) is more mysterious. In this paper, we show some examples of equations (2) related to equations (1) developed in [9] and its bibliography, with even a non algebraic expression for $f$ (there are the cases for equations in sections 7, 9, 10, 11).

In every case we suppose that the function $f$ is of $C^1$ class. We suppose also that $f > 0$ on $]0,+\infty[$, and that $u_0$, $u_1 > 0$, for (1) and for (2), and then that we have $\forall n$ $u_n > 0$. Then, we always assume the following property, except in section 12

$$\textbf{for } (\mathbf{1}) \textbf{ and for } (\mathbf{2}), \textbf{the fixed point in } (\mathbf{0},+\infty) \textbf{ exists, and is unique}. \tag{1.1}$$

These fixed points are given by these equations

$$\ell^2 = f(\ell) \text{ for } (1), \text{ and } \ell(\ell+\lambda) = f(\ell) \text{ for } (2). \tag{1.2}$$

## 2. Attractivity of fixed point for equation (2)

Our first result regards all difference equations of type (2) when $f$ is a $C^1$ positive function in $]0,+\infty[$ satisfying hypothesis (1.2).

**Theorem 2.1.** *In addition to hypothesis (1.2), assume*

$$|f'(\ell)| < 2\ell+\lambda. \tag{2.1}$$

*Then the fixed point $L := (\ell,\ell)$ is attractive: there exists a neighbourhood $V$ of $L$ such that, if $M_{n_0} \in V$, then the sequence $M_n$ converges to $L$.*

*Proof.* The characteristic equation of differential of the map $T(x,y) := \left(\dfrac{f(x)}{y+\lambda}, x\right)$ at the fixed point is

$$P(t) := t^2 - t\frac{f'(\ell)}{\ell+\lambda} + \frac{\ell}{\ell+\lambda} = 0. \tag{2.2}$$

Recall the classical result (for Example [10]): the roots of equation $z^2 - pz - q = 0$ (where $p$, $q \in \mathbb{R}$) lie in the open disc $\{z \,|\, |z| < 1\}$ if and only if

$$|p| < 1 - q < 2.$$

It is clear that $P$ satisfies these inequalities. So the spectral radius of $\mathrm{diff}(T)(L)$ is less than 1. Since $f$ is a $C^1$ function, the last property holds for $\mathrm{diff}(T)(x,y)$ when $(x,y)$ approaches $L$. So $L$ is attractive. $\square$

## 3. Localisation of the initial point in order to obtain a permanent sequence, and a convergence condition.

The following result is a corollary of a Theorem 1 in [12], citing a result of [14].

**Theorem 3.1.** *We consider the difference equation (2). Let $I = [m,M] \subset \mathbb{R}_*^+$ be a non empty interval in which the following properties hold:*

*$f$ is increasing on $I$, and*

$$\frac{f(M)}{M} - m \leq \lambda \leq \frac{f(m)}{m} - M. \tag{3.1}$$

(a) *Then, if $M_{n_0} = (u_{n_0}, u_{n_0+1})$ belongs to $I^2$, then $\forall n \geq n_0$, $u_n \in I$, and the sequence is permanent (see* [10]*: $\exists \alpha, \beta$ such that $\forall n, 0 < \alpha \leq u_n \leq \beta$).*

(b) *Moreover, if $(u_{n_0}, u_{n_0+1})$ is also in $[m, M]$, then the sequence converges to the limit $\ell$.*

***Remark*** 3.1. Sometime, we also add the condition (3.1): we use the first condition of (3.1), and put the relation $x \mapsto g(x) := \frac{f(x)+x^2}{x}$ is decreasing on $I$ (which gives the second condition of (3.1)).

*Proof.* (a) Assume that $m \leq u_n$, $u_{n+1} \leq M$, and condition (3.1) holds. Then

$$\frac{f(m)}{M+\lambda} \leq u_{n+2} \leq \frac{f(M)}{m+\lambda},$$

and the inequalities $m \leq u_{n+2} \leq M$ can hold if

$$m \leq \frac{f(m)}{M+\lambda} \quad \text{and} \quad \frac{f(M)}{m+\lambda} \leq M,$$

(permitting an induction for $n \geq n_0$) or $\dfrac{f(M)}{M} - m \leq \lambda \leq \dfrac{f(m)}{m} - M$ (remark that (3.1) gives $\dfrac{f(M)}{M} + M \leq \dfrac{f(m)}{m} + m$).

Then by induction if $M_{n_0} \in I^2$, we have $\forall n \geqslant n_0$, $u_n \in I$. So, the interval $I$ is *invariant*. Hence the sequence is permanent.

(b) Let $F(x,y) = \frac{f(x)}{y+\lambda}$. Then $F$ is non decreasing in $x$ and non increasing in $y$. Suppose $F(m,M) = M$ and $F(M,m) = m$; then we have by (3.1)

$$M(M+\lambda) = f(m) \leq f(M) = m(m+\lambda),$$

and so $M = m$. From Theorem 1 of [12], this result implies that the sequence converges to the limit $\ell$. $\square$

**Theorem 3.2.** *Suppose that the hypothesis (1.1) holds and that the sequence $u_n$ is majorised by $M > 0$ for $n \geq n_0$.*

(a) *If $f$ is increasing in $]0, M]$ (so it has a limit $f(0_+)$ at 0) and*

$$\max_{x \in [m,M]} f'(x) < \lambda, \text{ where we put } m = \min\left(u_{n_0}, u_{n_0+1}, \frac{f(0_+)}{M+\lambda}\right), \tag{3.2}$$

*then the sequence $u_n$ converges to $\ell$.*

(b) *If $f$ is decreasing on $]0, M]$ and if*

$$\max_{x \in ]0,M]} f'(x) > -\lambda, \tag{3.3}$$

*then the sequence $u_n$ converges to $\ell$.*

*Proof.* (a) Since for $n \geq n_0$, we have $u_n \in [m,M]$, and so $S = \overline{\lim}(u_n)$ and $s = \underline{\lim}(u_n)$ exist. Clearly (putting $m = 0$ if $f(0_+) = 0$))

$$\frac{f(s)}{S+\lambda} \leq s \leq S \leq \frac{f(S)}{s+\lambda},$$

so $f(s) - \lambda s \leq Ss \leq f(S) - \lambda S$, and $\lambda(S-s) \leq f(S) - f(s)$. If $S > s$, $\lambda \leq \frac{f(S)-f(s)}{S-s} = f'(c)$ for some $c \in [m,M]$, in contradiction with condition (3.2). So $S = s$, and the sequence converges.
(b) The proof is similar to the proof of assertion (a). □

## 4. Applications to the difference equation $u_{n+2}(u_n + \lambda) = \dfrac{a + bu_{n+1} + u_{n+1}^2}{1 + u_{n+1}}$, with $a > 0$, $b > 0$ and $b \neq a + 1$

We are going to apply the previous theorems to some cases. It will be essential to ensure that the conditions of our theorem hold for some values of perturbation parameter $\lambda$ and some classes of function $f$.

We begin with the equation quoted in a title of this section. In the sequel we assume that

$$b \neq a + 1, \tag{4.1}$$

otherwise the function $f$ should simplify and the equation should become $u_{n+2} = \dfrac{u_{n+1} + a}{u_n + \lambda}$, whose behaviour is known (see [4], [16], [10], [8]).

By [2] this equation appears as the perturbation by $\lambda > 0$ of the equation where $\lambda = 0$, a QRT- difference equation associated with the family of symmetric bi-quadratic equations

$$xy(x+y) + x^2 + y^2 + b(x+y) + a - Kxy = 0.$$

***Remark*** 4.1. In [9], the authors quote the equation $u_{n+2}u_n = \frac{A+Bu_{n+1}+u_{n+1}^2}{1+Du_{n+1}}$. It is easy to see that the corresponding perturbed equation may be rewritten as the equation treated in the present section (let $v_n = Du_n$ and $\mu = \lambda D$).

This equation is of type (2), with $f(x) := \dfrac{a + bx + x^2}{x+1}$. Recall some results of [2] for the equation in the title of section 4.

**Lemma 4.1.** (i) *The sequence $u_n$ is permanent.*
(ii) *The associated equation possesses a unique attractive fixed point in* $]0, +\infty[$.
(iii) $1 \leq b \leq \lambda + 1$ *and* $b \geq \left(1 + \frac{1}{\lambda}\right)a$*, the sequence $u_n$ converges to the fixed point.*

Observe that by using Theorem 2.1 the attractive character of fixed point is easier to prove than in [2].

The curve $y = f(x)$ is a hyperbola, which can be written as $y = \frac{a-b+1}{t} + t + b - 2$, putting $x = -1 + t$. So the following facts hold:

(I) if $b > a+1$, $f$ is increasing in $[0,+\infty[$ ;
(II) if $a \le b < a+1$, $f$ is increasing in $[0,+\infty[$ ;
(III) if $a > b$, then if $x_0 = -1+\sqrt{a-b+1}$, $f$ is increasing in $[x_0,+\infty[$, decreasing in $[0,x_0]$.

## 4.1. **A partial result on convergence**

In the following paragraph, Theorem 3.2 is used to give a result on convergence with an initial condition, not given in [2].

**Proposition 4.1.** *Suppose that $b < a-1$ (it is the case* (III)*), and $\lambda \ge \frac{a}{x_0}$, where $x_0 = \sqrt{1+a-b}-1$. If $M_{n_0} \in ]0,x_0]^2$, then, the set of $(a,b)$ where the sequence converges to the positive fixed point, is not $\emptyset$.*

*Proof.* Since $b < a-1$, $x_0$ is well defined, and moreover $x_0 \ge \frac{a-b}{2}$. By induction we prove that $]0,x_0]$ is stable. Assume that $u_n$ and $u_{n+1}$ lie on this interval. Since $f$ is decreasing, one has $u_{n+2} \le \frac{f(0)}{\lambda} = \frac{a}{\lambda}$, which is majorised by $x_0$ if $\lambda \ge \frac{a}{x_0}$. So the induction step will hold if this inequality is true. We get $f'(x) = 1+\frac{b-a-1}{(x+1)^2}$. By our hypothesis, $f'$ is increasing and the condition of Theorem 3.2 can write as $f'(x_0) > -\lambda$, or $1-\frac{|b-a-1|}{(x_0+1)^2} > -\lambda$, that is $\lambda > \frac{1+a-b}{(\sqrt{a+1-b})^2} - 1 = 0$ which is true.

Now we see that the conditions on $(a,b)$, $b < a-1$ and $\lambda \ge \frac{a}{x_0}$ agree. The first one states that $(a,b)$ is under the line $b = a-1$, and the second gives by an easy computation $b \le -\dfrac{a^2}{\lambda^2} + \left(1-\dfrac{2}{\lambda}\right)a$. So, the point $(a,b)$ must lie under the parabola of Figure 1. These two conditions will give a non empty domain iff the positive root of $a_0 = \lambda(\lambda-2)$ of the corresponding polynomial is greater than 1. This implies $\lambda > 2$, and $a_0 \ge 1$ gives $\lambda \ge 1+\sqrt{2}$. Since the condition $\frac{a}{x_0} < 1+\sqrt{2}$ does not hold when $a \ge 1$, it remains that $\lambda \ge \frac{a}{x_0}$. □

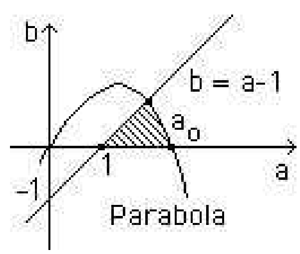


FIGURE 1. *Parabola under line*

## 4.2. **Another particular case of convergence**

We suppose now that $b > a+1$ (case (III)), which implies that $b > 1$ and $b > a$. So $f$ is increasing in $[0,+\infty[$, then in $[0,M]$ where $M$ is an upper bound of the sequence. One has $f'(x) = 1 + \dfrac{b-a+1}{(x+1)^2}$ so the maximum value of $f'$ in $[0,M]$ is $f'(0) = b-a > 1$. In order to apply the Lemma 3.2, one must have

$$\lambda > b-a.$$

As a result we have:

**Proposition 4.2.** *If $\lambda > 1$, a solution of difference equation*

$$u_{n+2}(u_n+\lambda) = \frac{a+bu_{n+1}+u_{n+1}^2}{1+u_{n+1}},\ a>0,\ b>0 \tag{4.2}$$

*converges to the positive fixed point when $(a,b)$ belongs to the non empty the infinite strip $a+1<b<a+\lambda$.*

It is of some interest to compare this convergence domain with the one quoted in [2], recalled here in Lemma 4.1: the assumption of the Lemma 4.1 was $\lambda \geq \dfrac{a}{b-a}$, but the $(a,b)$ domain is bounded (see Figure 2).

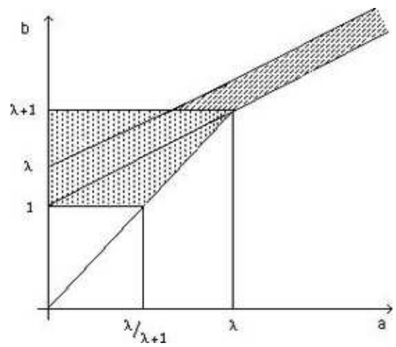


FIGURE 2. *Strip and other domain*

## 5. Application to difference equation perturbed the QRT-difference with a conics, a case of negative problem A

This perturbed difference equation is

$$u_{n+2}(u_n+\lambda) = a - u_{n+1} + u_{n+1}^2,\ a > \frac{1}{4},\ \lambda > 0. \tag{5.1}$$

This equation is the $\lambda > 0$ perturbed equation of $u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2$, $a > \frac{1}{4}$ (for positivity), which is of symmetric type QRT associated with the family of conic curves with equations

$$x^2+y^2-(x+y)+a-Kxy = 0,$$

$2-\frac{1}{a} \leq K < 2$ in the open set defined by $\dfrac{x^2+y^2-(x+y)+a}{xy} < 2$ and $x>0$, $y>0$ (see [3]).

Note by the Theorem 2.1 that the fixed point $\ell = \frac{a}{\lambda+1}$ is attractive.

### 5.1. **A first case of convergence**

We will see in the following subsection some cases of divergence for the sequence quoted in the title, but we begin by a case of convergence:

**Theorem 5.1.** *Assume that*

$$a > \frac{1}{2} \text{ and } \frac{3}{2}(\sqrt{2a}-1) \leq \lambda < 2a - \frac{1}{2} - \sqrt{\frac{a}{2}}, \tag{5.2}$$

*and set* $[m,M] = \left[\frac{1}{2}, \sqrt{\frac{a}{2}}\right]$*. Then, if* $(u_0,u_1) \in [m,M]^2$*, the solution* $u_n$ *of the difference equation (12) lies in* $[m,M]$ *and converges to the fixed point* $\ell = \frac{a}{\lambda+1}$*, which is an attractive point.*

*Proof.* The function $f$ is $f(x)=a-x+x^2$, and the assumption $|f'(\ell)| = \dfrac{|2a-\lambda-1|}{\lambda+1} < \dfrac{2a}{\lambda+1} + \lambda$ is easy to check. The function $x \mapsto f(x) = a - \frac{1}{4} + (x-1/2)^2$ is increasing in $\left[\frac{1}{2}, +\infty\right[$, and by (3.1) bis $x \mapsto g(x) = \frac{f(x)}{x} + x = \frac{a}{x} + 2x - 1$ is decreasing in $\left]0, \sqrt{\frac{a}{2}}\right]$. We must choose $[m,M] = \left[\dfrac{1}{2}, \sqrt{\dfrac{a}{2}}\right]$, which implies $a > \frac{1}{2}$.

The function $x \mapsto f'(x) = 2x - 1$ increases, and $f'\left(\sqrt{\dfrac{a}{2}}\right) = 2\sqrt{\dfrac{a}{2}} - 1$, so condition (3.2) is satisfied whenever $\lambda > 2\sqrt{\dfrac{a}{2}} - 1 = \sqrt{2a} - 1$.

The right hand side inequality (3.1) is $\lambda \leq \frac{f(m)}{m} - M$, that is $\lambda \leq 2a - \frac{1}{2} - \sqrt{\frac{a}{2}}$ which is compatible with previous inequality if $16a^2 - 10a + 1 > 0$. But it is true if $a > \frac{1}{2}$. The left hand side inequality $\frac{f(M)}{M} - m \leq \lambda$ gives $\dfrac{3}{2}(\sqrt{2a}-1) \leq \lambda$. So (5.2) is true. □

### 5.2. **A case of divergence: negative answer to the problem (A) of [2]**

Our aim is to prove the following.

**Theorem 5.2.** *We consider the difference equation*

$$u_{n+2} = \frac{a - u_{n+1} + u_{n+1}^2}{u_n + \lambda}, \ a > \frac{1}{4}, \ \lambda > 0, \ u_0 > 0, \ u_1 > 0.$$

*Assume that* $a > 1$ *and* $1 + \frac{1}{\sqrt{1+\frac{1}{\lambda}}} > \frac{a}{\lambda+1} > 1$ *. There exist two numbers* $\theta_0(a,\lambda) > 0$ *and* $\alpha_0(a,\lambda) > 1$ *such that the sequence* $u_n$ *diverges geometrically to* $+\infty$ *if* $(u_0,u_1)$ *belongs to the domain*

$$\Delta := \{u_1 > a\} \cap \{u_0 > 0\} \cap \{u_1 > \alpha_0 u_0 + \theta_0\}. \tag{5.3}$$

***Remark*** 5.1. The inequality in this theorem is also equivalent to $\dfrac{(a-1)^2}{2a-1} < \lambda < a-1$, e.g. $\lambda = 1,\ 2+\sqrt{2} > a > 2$.

*Proof.* We prove by induction the following hypothesis, searching some conditions about $\alpha > 0$, $\beta > 0$, $\theta > 0$, even if they impose some new conditions for $a$ and $\lambda$:

$$\begin{array}{ll} H_n(1) & u_{n+1} > a, \\ H_n^-(2) & \alpha u_n + \theta < u_{n+1}, \\ H_n^+(2) & u_{n+1} < \beta u_n. \end{array} \tag{5.4}$$

**First step:** $\{H_n(1)$ **and** $H_n^+(2)\} \Longrightarrow H_{n+1}^+(2)$

The inequality $u_{n+2} < \dfrac{a-u_{n+1}+u_{n+1}^2}{\frac{u_{n+1}}{\beta}+\lambda}$, leads to, by denoting $x := \frac{1}{u_{n+1}}$

$$u_{n+2} < \beta u_{n+1}\frac{1-x+ax^2}{1+\beta\lambda x}. \tag{5.5}$$

It suffices to show that $\frac{1-x+ax^2}{1+\beta\lambda x} - 1 < 0$, i.e $ax-(1+\lambda\beta) < 0$, or $x < \frac{1+\lambda\beta}{a}$, that is $u_{n+1} > \frac{a}{1+\lambda\beta}$. But this true from $H_n(1)$.

**Second step: if** $a > \lambda+1$**,** $\{H_n(1)$ **et** $H_n^-(2)\} \Longrightarrow H_{n+1}^-(2)$**,** $\alpha > \dfrac{a}{a-\lambda-1}$ **and if we choose** $\theta$ **such that** $\dfrac{1+\lambda\alpha}{1-\frac{1}{\alpha}} < \theta < a+\lambda\alpha$ **(it is possible by the previous inequality)**.

Label this inequalities

$$a > \lambda+1, \quad \alpha > \frac{a}{a-\lambda-1} \quad \text{and} \quad \frac{1+\lambda\alpha}{1-\frac{1}{\alpha}} < \theta < a+\lambda\alpha, \tag{5.6}$$

and denote

$$D_{\alpha,\theta} := \{(u_0,u_1)\,|\,u_0 > 0,\ u_1 > a,\ u_1 > \alpha u_0+\theta\}. \tag{5.7}$$

By majorising $u_n$ by $\frac{u_{n+1}-\theta}{\alpha}$ in the denominator of $u_{n+2}$, we get, as previously:

$$u_{n+2}-\theta > \frac{\alpha}{x}\frac{1-x+ax^2}{1+(\alpha\lambda-\theta)x} - \theta. \tag{5.8}$$

So we have $u_{n+2}-\theta > \alpha u_{n+1}\dfrac{1-x\big(1+\frac{\theta}{\alpha}\big)+x^2\big[a-\frac{\theta}{\alpha}(\lambda\alpha-\theta)\big]}{1-x(\theta-\lambda\alpha)}$, and it suffices to show that the ratio of the right hand side is greater than 1.

So we assume that $\theta > \lambda\alpha$, and that the following stronger relation is satisfied

$$\theta\big(1-\frac{1}{\alpha}\big) > 1+\lambda\alpha \quad \text{with} \quad a > \lambda+1. \tag{5.9}$$

Then we have

$$\frac{1-x\big(1+\frac{\theta}{\alpha}\big)+x^2\big[a-\frac{\theta}{\alpha}(\lambda\alpha-\theta)\big]}{1-x(\theta-\lambda\alpha)} - 1 = \frac{x\big(\theta-\lambda\alpha-1-\frac{\theta}{\alpha}\big)+x^2\big[a+\frac{\theta}{\alpha}(\theta-\lambda\alpha)\big)}{1-x(\theta-\lambda\alpha)}.$$

With the hypothesis of (5.9), the expression below is positive, if $x < \frac{1}{\theta-\lambda\alpha}$, which is true soon as the following hypothesis is true by $H_n(1)$

$$a > \theta - \lambda\alpha. \tag{5.10}$$

Then we see that implication $\{H_n(1) \text{ and } H_n^-(2)\} \Longrightarrow H_{n+1}^-(2)$ is true as soon as

$$\alpha > \frac{a}{a-\lambda-1} \quad \text{and} \quad \frac{1+\lambda\alpha}{1-\frac{1}{\alpha}} < \theta < a+\lambda\alpha. \tag{5.11}$$

**Third step: under the conditions of second step $H_n(1) \Longrightarrow H_{n+1}(1)$**

One has $u_{n+2} > u_{n+1}\,\alpha\frac{1-x+ax^2}{1-\mu x}$, with $\mu := \theta - \lambda\alpha$, then since $u_{n+1} > a$ we have to show $\alpha\frac{1-x+ax^2}{1-\mu x} > 1$. But $\frac{1-x+ax^2}{1-\mu x}$ is itself greater that 1, because $\mu = \theta - \lambda\alpha > 1$ and $x = \frac{1}{u_{n+1}} < \frac{1}{\mu}$, for $u_{n+1} > a > \mu$ for $H_n(1)$.

**Forth step: conditions for $(u_0, u_1)$ providing divergence**

We take $\alpha$ and $\theta$ as small as possible, that is

$$\alpha = \alpha_0 = \frac{a}{a-\lambda-1} > 1 \quad \text{and} \quad \theta = \theta_0 = \frac{1+\lambda\alpha_0}{1-\frac{1}{\alpha_0}} = \frac{a(a-1)}{a-\lambda-1}, \tag{5.12}$$

which yields following result

**Claim** If we have the inequality

$$a < (\lambda+1)\Big[1 + \frac{1}{\sqrt{1+\frac{1}{\lambda}}}\Big] \tag{5.13}$$

the function $\alpha \mapsto \frac{1+\lambda\alpha}{1-\frac{1}{\alpha}}$ increases on $\big[1+\sqrt{1+\frac{1}{\lambda}}, +\infty\big[$.

Note that $a < \theta_0$, and define

$$\Delta_{a,\lambda} := \big\{(u_0,u_1)\,|\,u_1 > \alpha_0 u_0 + \theta_0,\, u_0 > 0,\, u_1 > a\big\}. \tag{5.14}$$

If we pick a point in $M_0 = (u_0, u_1) \in \Delta_{a,\lambda}$, it belongs to a domain $D_{\alpha,\theta}$ of (5.10)

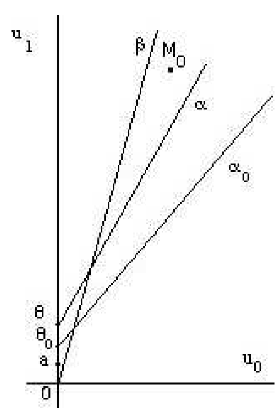


FIGURE 3. *Choose a point $M_0$*

satisfying (5.9), by the claim (see Figure 3). If we choose $\beta > \frac{u_1}{u_0}$, the initial conditions of the induction are satisfied. So the sequence $u_n$ goes to $+\infty$ (with a growth of geometric type).

This completes the proof of Theorem 5.2. □

***Remark*** 5.2. In fact, the fixed point of the difference equation in this section is always attractive (without conditions of $a > \frac{1}{4}$ and of $\lambda > 0$), as we have said at the top of the section.

**Corollary 5.1.** *The answer to* **problem (A) is negative**.

This clear, from the previous considerations.

### 5.3. **Another case of convergence**

**Theorem 5.3.** *Assume that* $\frac{a}{\lambda} < \frac{1}{2}$*. Then, if* $(u_0, u_1)$ *belongs to the open square* $\left]\frac{a-\frac{1}{4}}{\frac{1}{2}+\lambda}, \frac{1}{2}\right[^2$*, there exist m et M (depending of* $(u_0, u_1)$*) such that* $\forall n \geq 0$ *we have* $0 < m < u_n < M$*, and sequence* $u_n$ *converges to* $\ell$.

*Proof.* Let $M$ be some number satisfying $\dfrac{a}{\lambda} < M < \dfrac{1}{2}$, and assume $u_{n+1} < M < \frac{1}{2}$. Since function $x \mapsto a - \dfrac{1}{4} + \left(x - \dfrac{1}{2}\right)^2$ is decreasing on $[0, \frac{1}{2}]$, we have $u_{n+2} < \dfrac{a}{u_n + \lambda} < \dfrac{a}{\lambda} < M$. So inequality $\forall n\ u_n < M$ is proved by induction if it is true for $n = 0$ and $n = 1$.

Moreover, let $m = \dfrac{a - \frac{1}{4}}{M + \lambda}$. Then $m < M$ (because $M > \frac{a}{\lambda}$). It is clear that $u_{n+2} > m$ if $u_{n+1} < M$. So, if we assume that $u_0$ and $u_1$ strictly majorise $m$ with the conditions concerning $M$, the result holds.

Finally, assume that $(u_0, u_1)$ belongs in the open square $\left]\frac{a-\frac{1}{4}}{\frac{1}{2}+\lambda}, \frac{1}{2}\right[^2$. Then it exists some $M \in \left]\frac{a}{\lambda}, \frac{1}{2}\right[$ such that $\frac{a-\frac{1}{4}}{M+\lambda} < u_0,\ u_1 < M$, which implies the result.

But $x \mapsto f'(x) = 2x - 1$ is increasing. So this function is majorised in $\left[0, \frac{1}{2}\right]$ by $f'\left(\frac{1}{2}\right) = 0$, and condition (3.3) holds. This shows that the sequence $u_n$ converges.

## 6. Come back to the general equation (2): a sufficient condition for divergence

We return to the general equation (2), simply assuming that the function $f$ increases fast enough speedy. We have:

**Theorem 6.1.** *We consider the difference equation* $u_{n+2} = \dfrac{f(u_{n+1})}{u_n + \lambda}$*, where we assume only* $f(x) > 0$ *in* $]0, +\infty[$*. We suppose that:*

$$f(x) \geq Ax^m \text{ for } x \geq x_0 \geq 0, \text{ with :}$$
$$\{m > 2 \text{ and } A > 0\} \text{ or } \{m = 2 \text{ and } A > 1\}. \tag{6.1}$$

*Let $\alpha > 1$, Then, it exists a number $B > x_0$ such that if $u_1 \geq B$ and $u_1 \geq \alpha u_0$, the sequence $u_n$ verify $\forall n\, u_{n+1} \geq \alpha u_n$, and then $u_n$ diverges to $+\infty$.*

*Proof.* If $M_0 = (u_0, u_1)$ satisfies the previous hypothesis with $B > x_0$, where $B$ will be further defined, we take our induction hypothesis that $u_{n+1} \geq \alpha u_n$. Then the following inequality holds:

$$u_{n+2} \geq \frac{Au_{n+1}^m}{\frac{u_{n+1}}{\alpha} + \lambda}.$$

In order that the latter is greater than $\alpha u_{n+1}$, *id est* $Au_{n+1}^{m-1} \geq u_{n+1} + \lambda\alpha$, we need to have $u_{n+1} \geq B > x_0$, where this number $B$ depending only on $\alpha, \lambda, m, A, x_0$. We choose it to obtain $u_{n+1} \geq B$ in the induction hypothesis. Then, by induction, $u_{n+1} \geq \alpha u_n$, and consequently $u_n \to +\infty$. □

**Example 6.1.** *Suppose we have the difference equation $u_{n+2} = \dfrac{au_{n+1}^2 + bu_{n+1} + c}{u_n + \lambda}$, where $a > 1$ and $b^2 < 4ac$. Then the hypothesis of the theorem holds with some number A satisfying $a > A > 1$ and $x_0 \geq 0$ function of $a, b, c, A$. Then if $\alpha > 1$ is given for $u_1 \geq \alpha u_0$ and $u_1 \geq B$, where B depends on $A, a, b, c, \alpha, \lambda$, the sequence $u_n$ diverges to $+\infty$.*

**Problem 1.** Does there exist some domain $D$ in the first quadrant and a value of $\lambda > 0$ such that, if $M_0 := (u_0, u_1) \in D$, the solution $u_n$ stemming from $M_0$ of equation in part 4

$$u_{n+2}(u_n + \lambda) = \frac{a + bu_{n+1} + u_{n+1}^2}{1 + u_{n+1}}, \; a > 0, b > 0$$

diverges ?

**Another example.** Look at the difference equation:

$$u_{n+2} = \frac{au_{n+1}^3 + b}{u_n + \lambda}, \; a > 0, \; b > 0,$$

perturbed from some equation of [9]. Following Theorem 6.1, for all $\alpha > 1$, for $(u_0, u_1)$ in the infinite trapezoid $\{u_1 > B\} \cap \{u_1 \geq \alpha u_0\}$, the solution of the equation coming from this point tends to $+\infty$.

## 7. A NON ALGEBRAIC EXAMPLE

Now, we consider the difference equation

$$u_{n+2} = \frac{\ln(1 + u_{n+1})}{u_n + \lambda}, \; \lambda > 0, \; u_0 > 0, \; u_1 > 0. \tag{7.1}$$

The fixed point $(\ell, \ell)$ is the positive solution $\ell$ of equation

$$x(x + \lambda) = \ln(1 + x), \tag{7.2}$$

It exists and is unique only if $\lambda < 1$. To see this, we look at the concavity of the functions $x \mapsto x(x+\lambda)$ et $x \mapsto \ln(1+x)$ and their derivatives at 0. For $\lambda \geq 1$, the unique fixed point is 0.

If $f(x) := \ln(1+x)$, one see that $f$ increases on $[0,+\infty[$, and that $f'(x) = \dfrac{1}{1+x}$, so the attractivity condition for the fixed point is

$$\frac{1}{1+\ell} < 2\ell + \lambda,$$

that is $2\ell^2 + (\lambda+2)\ell + \lambda - 1 > 0$, or

$$\ell > \alpha(\lambda) := \frac{\sqrt{(\lambda+2)^2 + 8(1-\lambda)} - (\lambda+2)}{4} > 0. \tag{7.3}$$

This relation gives

$$\ln(1+\alpha(\lambda)) - \alpha(\lambda)(\alpha(\lambda)+\lambda) > 0.$$

Let $\phi(\lambda)$ be this function. It is clear that it is positive in $[0,1[$. But its derivative $\phi'$ is $\dfrac{\lambda+2-\sqrt{(\lambda-2)^2+8}}{4}$, which is negative in [0,1[ because $\lambda < 1$. As $\phi(1) = 0$, $\phi(\lambda)$ is positive on [0,1[. We conclude that the fixed point exists and is unique in $]0,+\infty[$ and is attractive.

Now, we search some upper bound for our sequence. Let $M_0(\lambda)$ the positive solution of the equation

$$\frac{\ln(1+x)}{x} = \lambda,$$

and let $M$such that $M \geq M_0(\lambda)$. We prove by induction that $u_n \leq M$ if $u_0,\ u_1 \leq M$. Assume that $u_n,\ u_{n+1} \leq M$. One has $u_{n+2} \leq \dfrac{\ln(1+M)}{\lambda}$ which is majorised by $M$ if $\dfrac{\ln(1+M)}{M} \leq \lambda$, true if $M \geq M_0(\lambda)$, for $x \mapsto \frac{\ln(1+x)}{x}$ is decreasing.

Now, assume that $\lambda \geq 1$. It is easy to see that if every interval $]0,M]$ is invariant, then the sequence is bounded. Taking into account Theorem 3.2, which is true if the fixed point is 0, and because $\max_{x\in[0,M]} f'(x) = 1$, the sequence converges to 0 if $\lambda > 1$. So we have:

**Proposition 7.1.** (a) *The fixed point in* $]0,+\infty[$ *of the difference equation*

$$u_{n+2} = \frac{\ln(1+u_{n+1})}{u_n+\lambda},\ 0<\lambda<1,\ u_0>0,\ u_1>0$$

*exists, is unique and attractive. If* $M_0(\lambda)$ *is the solution of* $\dfrac{\ln(1+x)}{x} = \lambda$, *then* $\forall M \geq M_0(\lambda)$ *the interval* $I := ]0,M]$ *is invariant: if* $(u_{n_0}, u_{n_0+1}) \in I^2$, *then* $\forall n \geq n_0\ u_n \in I$.

(b) *Si* $\lambda > 1$, *any solution of (7.2) converges to 0.*

**Problem 2.** If $\lambda < 1$, does there exist some $m > 0$ such that $\forall n\ u_n \geq m$ ? Does the sequence converge ? What happens if $\lambda = 1$ ? To prove divergence or convergence for some values (perhaps for all) of the initial point when $\lambda \leq 1$.

## 8. Study of difference equation $u_{n+2}(u_n+\lambda) = \frac{a}{1+u_{n+1}}$

We investigate some perturbations by $\lambda > 0$ of equation $u_{n+2}u_n = \frac{a}{1+u_{n+1}}$, that is

$$u_{n+2}(u_n+\lambda) = \frac{a}{1+u_{n+1}},\ \lambda > 0,\ a > 0,\ u_0, u_1 > 0. \tag{8.1}$$

We note first that the non perturbed equation (pour $\lambda = 0$) is a special symmetric QRT- equation associated with the family of biquadractic curves:

$$x^2y^2 + xy(x+y) + a(x+y) - Kxy = 0,$$

with invariant function $G(x,y) = xy+x+y+a\left(\frac{1}{x}+\frac{1}{y}\right)$. For this equation see [9], [1].

The first result is concerned with the permanent character of solutions.

**Proposition 8.1.** *For $n \geq 4$, one has*

$$m := \frac{a\lambda^2}{(\lambda+a)(\lambda^2+a)} \leq u_n \leq \frac{a}{\lambda} := M. \tag{8.2}$$

*Proof* For $n \geq 0$ we have $u_{n+2} \leq \frac{a}{\lambda}$ and the same majoration for $u_{n+3}$, from which $u_{n+4} \geq \frac{a}{(1+\frac{a}{\lambda})(\lambda+\frac{a}{\lambda})}$. □

***Remark*** 8.1. In fact, we can find some invariant intervals as large as possible which arbitrarily approach 0: $\forall \varepsilon, A$ with $0 < \varepsilon < A < +\infty$, there exist $m(\lambda)$ and $M(\lambda)$ verifying $0 < m(\lambda) < \varepsilon$ et $M(\lambda) > A$ such that the interval $[m(\lambda), M(\lambda)]$ is invariant: if $u_n$ and $u_{n+1}$ belong to this interval, so does is $u_{n+2}$ .

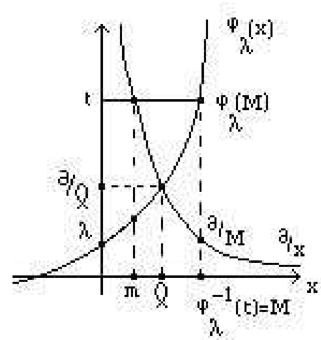


FIGURE 4. *Greater interval for permanent sequence*

If $\phi_\lambda(x) := (x+1)(x+\lambda)$, $x > 0$, $\lambda > 0$, one takes, for $t > 0$, $m(\lambda) = \frac{a}{t}$ and $M(\lambda) = \phi_\lambda^{-1}(t)$, choosing $t$ large enough, in particular greater than $\left(\frac{1+\lambda}{2}+\frac{a}{\lambda}\right)^2$.

Using Figure 4, the reader can easily verify this property.

The second result asserts in fact the convergence, (in other words, **the problem (A) has a positive solution for the case of the equation (8.1)**). .

**Proposition 8.2.** *For all $\lambda > 0$, the solutions of (8.2) converge to the number $\ell$, the unique positive solution of the equation $x(x+1)(x+\lambda) = a$, and the fixed point $L = (\ell, \ell)$ is attractive.*

*Proof.* We use Theorem 3.2(b): and must prove that, if $f(x) = \frac{a}{1+x}$, $\max_{x \in [m,M]} f'(x) > -\lambda$, or $\frac{a}{(1+M)^2} < \lambda$, that is to say $\dfrac{a}{(1+\frac{a}{\lambda})^2} < \lambda$, which always holds. At last, the Theorem 2.1 easily shows that the fixed point is attractive: the condition is $(\ell + 1)^2(2\ell+\lambda) - a > 0$, or $a\dfrac{\ell+1}{\ell} + \ell(\ell+1)^2 - a > 0$, which is always true. □

## 9. Another non algebraic example: $\mathrm{u_{n+2}} = \frac{\mathrm{u_{n+1}} \ln(1+\mathrm{u_{n+1}})}{\mathrm{u_n}+\lambda}$

This case is not be exactly included in this setting since the fixed point is $(0,0)$. Therefore, by extending the properties of subsections 1 and 2 to $[0,+\infty[$, a lot of results persist. As an example, one can see that *the fixed point is attractive*.

It is easy to prove the following result.

**Lemma 9.1.** *Given some solution difference equation in $\mathbb{R}_+^*$*

$$\mathrm{u_{n+2}} = \frac{\mathrm{u_{n+1}} \ln(1+\mathrm{u_{n+1}})}{\mathrm{u_n}+\lambda}, \ \lambda > 0, \tag{9.1}$$

*then there exists $n_0$ such that $u_{n_0+1} < u_{n_0}$, and then the solution tends to 0 (with a geometric or faster speed ) or the solution increases to $+\infty$.*

**Corollary 9.1.** *For $\lambda > 0$, every solution of (32) converges to 0.*

*Proof.* Let $r_n := \frac{u_{n+1}}{u_n}$. One has

$$\frac{r_{n+1}}{r_n} = \frac{u_n}{u_n+\lambda} \cdot \frac{\ln(1+u_{n+1})}{u_{n+1}} < \frac{\ln(1+u_{n+1})}{u_{n+1}}.$$

If $u_{n+1} \to +\infty$, this bound goes to 0, then for large enough $n$ $\dfrac{r_{n+1}}{r_n} < \dfrac{1}{2}$, and $r_n \to 0$. So for then for large enough $n$ the sequence $u_n$ decreases. But it cannot goes to infinity by the Lemma 9.1. Also by this lemma it tends to 0 . □

## 10. The difference equation $u_{n+2}(u_n+\lambda) = \sqrt{u_{n+1}}$

First, remark that if $u_{n+2} = \dfrac{\sqrt{u_{n+1}}}{u_n}$ (case where $\lambda = 0$), then $u_n \to 1$ if $n \to +\infty$ (let $v_n = \ln u_n$).

Clearly, the equation $\ell(\ell+\lambda) = \sqrt{\ell}$ or $\ell+\lambda = \frac{1}{\sqrt{\ell}}$, has a unique root $0 < \ell < 1$, as well as the null root 0. By Theorem 2.1 the fixed point $\ell$ is attractive. Moreover, defining $m(t) = \dfrac{1}{t^2}$ and $M(t) = t-\lambda$, we easily see by induction, that the interval $[m(t), M(t)]$ is invariant if $t \geq \lambda + \dfrac{1}{\lambda^2}$. In this case, $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'$ is majorised in the interval $[m, M]$ by $f'\big(m(t)\big) = \frac{t}{2} < \lambda$ if $2\lambda > t \geq \lambda + \dfrac{1}{\lambda^2}$ (which forces $\lambda > 1$).

**Proposition 10.1.** *If $\lambda > 1$, the solutions with $(u_0, u_1) \in [m(t), M(t)] := \Big[\dfrac{1}{t^2}, t-\lambda\Big]$, for $t$ such that $2\lambda > t \geq \lambda + \dfrac{1}{\lambda^2}$, from differences equation*

$$u_{n+2} = \frac{\sqrt{u_{n+1}}}{u_n+\lambda},\ \lambda > 0 \tag{10.1}$$

*converge to the fixed point $\ell \in ]0,1[$, solution of $x(x+\lambda) = \sqrt{x}$. This point is attractive.*

***Remark*** 10.1. If $t$ is large enough, the interval $[m(t), M(t)]$ clearly contains $u_0$ and $u_1$, and then all points $u_n$. But, it is is not sure that the condition of the Theorem 3.2 holds. So, we don't know if the sequence converges. Furthermore, the case $\lambda \leq 1$ is still open..

Finally we remark that $\ell \leq \frac{1}{\lambda^2}$, and that if $\lambda \leq 1$, then $\ell > \frac{1}{4}$, and if $\lambda > 1$, then $\ell > e^{-\lambda}$.

## 11. THE EQUATION $u_{n+2}(u_n+\lambda) = a + \sin u_{n+1}$ IN $\mathbb{R}_+^*$, WITH CONDITIONS $a > 1$ ET $\lambda > 0$

First, since $x \mapsto g(x) := x(x+\lambda) - a - \sin x$ is strictly convex in $[0, +\infty[$, it tends to $+\infty$ with $x$ and is $-a$ at $x = 0$, the fixed point $\ell$ exists and is unique. Moreover, one has $g'(\ell) = 2\ell + \lambda - \cos \ell > 0$. If $\lambda > 1$, one sees, by the mean of the tangent line to the curve $g$ at 0, that $\ell < \dfrac{a}{\lambda - 1}$. But, as $g''(x) = 2 + \sin x \geq 1$, we have, for $\lambda > 0$, $g(x) > -a + (\lambda-1)x + \frac{x^2}{2}$ for $x > 0$, which gives $\ell + \lambda < 1 + \sqrt{(\lambda-1)^2 + 2a}$. Then the following inequalities hold:

$$\cos(\ell) < 2\ell + \lambda \text{ and } \ell + \lambda < 1 + \sqrt{(\lambda-1)^2 + 2a}.$$

Finally we have:

**Theorem 11.1.** (a) *The difference equation*

$$u_{n+2} = \frac{a + \sin(u_{n+1})}{u_n + \lambda},\ a > 1,\ \lambda > 0,\ \text{in } \mathbb{R}_+^* \tag{11.1}$$

*has a unique fixed point $\ell$ satisfying* $\cos(\ell) < 2\ell + \lambda$ *and* $\ell + \lambda < 1 + \sqrt{(\lambda-1)^2 + 2a}$.

(b) *If* $2\ell+\lambda>1$*, the fixed point is attractive. Moreover, if* $\lambda>1$ *and* $\frac{a+1}{\lambda}\le\frac{\pi}{2}$*, then for* $u_0$ *and* $u_1$ *in interval* $]0,\frac{a+1}{\lambda}]$*, the sequence* $u_n$ *converges to* $\ell$*.*

*Proof.* It remains to prove (b). The fixed point is attractive if, when $f(x):=a+\sin x$, the inequality $|\cos\ell|<2\ell+\lambda$ holds, which proves the first assertion. Clearly, $u_{n+2}\le\frac{a+1}{\lambda}$. If this last number is majorised by $\frac{\pi}{2}$, the function $f$ increases in the interval $[0,\frac{a+1}{\lambda}]$, and $f'$ is majorised by 1. So if $\lambda>1$, we can use Theorem 3.2. □

## 12. Perturbation of the 'May's host parasitoid equation"

This equation is $u_{n+2}u_n=\frac{au_{n+1}^2}{1+u_{n+1}}$ (see [9] and [15]), where the perturbed version is

$$u_{n+2}(u_n+\lambda)=\frac{au_{n+1}^2}{1+u_{n+1}},\ \lambda>0,\ a>0,\ u_0>0,\ u_1>0. \tag{12.1}$$

By induction, it is easy to see that, if $u_0$ and $u_1$ are less than $\frac{\lambda}{a}$, then $\forall n\ u_n\le\frac{\lambda}{a}$ (so $\frac{\lambda}{a}$ is the greatest number $M$ such that $[0,M]$ is stable).The fixed points are solutions of $x(x+\lambda)(x+1)=ax^2$.

**Lemma 12.1.** *The possible fixed points are*

$$\begin{aligned}&\ell_0=0;\\&\text{if } a>(1+\sqrt{\lambda})^2,\text{ two numbers } \ell_1 \text{ and } \ell_2 \text{ with } 0<\ell_1<\sqrt{\lambda}<\ell_2;\\&\text{if } a=(1+\sqrt{\lambda})^2,\text{ the number } \ell_3=\sqrt{\lambda};\\&\text{if } a<(1+\sqrt{\lambda})^2,\text{ no fixed point except } 0.\end{aligned} \tag{12.2}$$

*Proof.* The proof is clear: we compare the slope $a$ of the straight line from 0 to the slope of tangent from 0 to the parabola with equation $y=(x+\lambda)(x+1)$, which is $(1+\sqrt{\lambda})^2$. This tangent touches the parabola at the point with abscissa $\sqrt{\lambda}$. □

First, we take the case where 0 is the unique fixed point.

**Proposition 12.1.** *Suppose that* $a<\min((1+\sqrt{\lambda})^2,\phi\lambda)$*, where* $\phi=\frac{1+\sqrt{5}}{2}$ *is the golden number. Then 0 is the unique fixed point, and every solution of (12.1) such that* $u_0$ *and* $u_1$ *are in* $]0,\frac{\lambda}{a}]$ *remains in* $]0,\frac{\lambda}{a}]$ *and converges to 0. If moreover* $a\le\lambda$*, the sequence converges to* 0 *a quadratic type, everywhere with the starting point* $(u_0,u_1)$.

*Proof.* First, if $a<1+\lambda+2\sqrt{\lambda}$, the unique fixed point is 0. Moreover, we have seen that the interval $]0,\frac{\lambda}{a}]$ is stable. So if $f(x)=\frac{ax^2}{x+1}$, the function $f$ is increasing, and $f'(x)=ax\frac{x+2}{(1+x)^2}$ is also increasing. Then $\max_{x\in]0,\frac{\lambda}{a}]}f'(x)=f'\left(\frac{\lambda}{a}\right)=\frac{a\lambda(\lambda+2a)}{(a+\lambda)^2}$. From Theorem 3.2 which we used, $u_n$ converges to 0 if this last number is strictly majorised by $\lambda$, *id est* if $a^2-a\lambda-\lambda^2<0$, or $a<\lambda\frac{1+\sqrt{5}}{2}$. Moreover, suppose that $a\le\lambda$. Then $\frac{u_{n+2}}{u_{n+1}}=\frac{u_{n+1}}{1+u_{n+1}}\frac{a}{u_n+\lambda}<\frac{a}{\lambda}\le 1$. Thus the sequence is decreasing, then $u_n\to 0$, and as one has $u_{n+2}<u_{n+1}^2$, the convergence is at least quadratic. □

What happens, with the hypothesis of the proposition, if $u_0$ or $u_1$ are greater than $\frac{\lambda}{a}$ ? And what happens if $a \geq (1+\sqrt{\lambda})^2$, where there are fixed points to 0 ? The situation depends strongly on the initial point $(u_0,u_1)$. For instance, if $a = (1+\sqrt{\lambda})^2$, where it is only a fixed point not zero $\ell_3 = \sqrt{\lambda}$, it is possible that the sequence tends to $\ell_3$ only than greater value; if it takes a smaller value, it seems to 0. And if there are two fixed points not zero $\ell_1 < \ell_2$, it seems that the sequence never tends to $\ell_1$, but it is stationary.

## 13. THE DIFFERENCE EQUATION $u_{n+2}(u_n+\lambda) = 1 + \dfrac{a}{u_{n+1}}$

First, find the fixed point.

**Proposition 13.1.** *The positive fixed point $\ell$ to difference equation*

$$u_{n+2}(u_n+\lambda) = 1 + \frac{a}{u_{n+1}},\ \lambda > 0,\ a > 0, \tag{13.1}$$

*exists and is unique, it is attractive, and we have*

$$\begin{aligned} &if\ \lambda = a,\ \ell = 1,\\ &if\ \lambda > a,\ 1 > \ell > \sqrt{\frac{a}{\lambda}},\\ &if\ \lambda < a,\ 1 < \ell < \sqrt{\frac{a}{\lambda}}. \end{aligned} \tag{13.2}$$

*Proof.* The equation of the fixed point is $x^2 = \dfrac{x+a}{x+\lambda}$, and a comparison of the parabola and of the hyperbola gives also the inequalities (13.2). Moreover, if $f(x) = 1 + \frac{a}{x}$, the condition of Theorem 2.1 is $\ell^2 > \frac{a}{2\ell+\lambda}$, which is right because $\ell^2 = \frac{\ell+a}{\ell+\lambda}$. □

Now it is possible to majorise a sequence $u_n$ solution of (13.1), and get the convergence of the solutions.

**Theorem 13.1.** *Every of solution of (13.1) is majorised by $M = \max(u_0,u_1,u_2,M_0)$ where*

$$M_0 = \frac{(\sqrt{a}+\sqrt{\lambda})^2 + a\lambda^2}{\lambda[(\sqrt{a}+\sqrt{\lambda})^2 - a]},$$

*and one solution is then permanent.*
*Every solution of (13.1) converges to $\ell$.*

*Proof.* One can easily see that

$$u_{n+3} = \frac{1}{u_{n+1}+\lambda} + a(u_n+\lambda)g(u_{n+1}),\ \text{where}\ g(x) = \frac{x}{(x+\lambda)(x+a)},$$

and that $g$ is maximum at $x = \sqrt{\lambda a}$, then $g(x) \leq \dfrac{1}{(\sqrt{\lambda}+\sqrt{a})^2}$.

Then, suppose $u_0, u_1$ and $u_2$ are majorised by a number $M$. Then we can majorise $u_{n+3}$ by $M$, and then use a recurrence if

$$\frac{1}{\lambda} + a(\lambda + M)\frac{1}{(\sqrt{\lambda}+\sqrt{a})^2} \leq M,$$

so $M\big[1 - \dfrac{a}{(\sqrt{\lambda}+\sqrt{a})^2}\big] \geq \dfrac{1}{\lambda} + \dfrac{a\lambda}{(\sqrt{\lambda}+\sqrt{a})^2}$. But one has obviously $(\sqrt{\lambda}+\sqrt{a})^2 > a$, so we have a recurrence that $\forall\, n\ u_n \leq M$, the condition $M \geq M_0$.

As $f(x) = 1 + \frac{a}{x}$ is decreasing and $f'$ is increasing, the maximum of $f'$ on $]0, M]$ is $-\frac{a}{M^2}$. By Theorem 3.2, if $\lambda > \frac{a}{M^2}$, the sequence is convergent. But we see that condition $\lambda > \frac{a}{M^2}$ is true ; it suffices to see condition that $\lambda > \frac{a}{M_0^2}$ is right as $\lambda$. Puting $\phi := \sqrt{\lambda} + \sqrt{a}$, this relation becomes

$$(a\lambda^2 + \phi^2)^2 > a\lambda(\phi^2 - a)^2,$$

or

$$a\lambda^2 + a + \lambda + 2\sqrt{a\lambda} > \sqrt{a\lambda}(\lambda + 2\sqrt{a\lambda}).$$

Puting $t = \sqrt{a}$, it is to prove that the polynomial in $t$

$$P(t) := t^2(\lambda-1)^2 + t\sqrt{\lambda}(2-\lambda) + \lambda$$

is positive on $]0, +\infty[$. This is clear for $\lambda = 1$ ; if not, $P$ is of second degree, with derivative at 0 equal to $\sqrt{\lambda}(2-\lambda)$, then the result is right if $2 \geq \lambda$ ; if not, the discriminant of $P$ is $\lambda^2(4-3\lambda) < 0$, which proves the final result. □

## 14. Other perturbed difference equation of one QRT equation with form $u_{n+2}u_n = f(u_{n+1})$

Here we study the difference equation

$$u_{n+2}(u_n + \lambda) = \frac{a + bu_{n+1} + u_{n+1}^2}{1 + u_{n+1}^2},\ \lambda, a, b > 0. \tag{14.1}$$

It is the perturbation by $\lambda > 0$ of one equation of [5], which is one QRT-equation associated to the family of elliptic quartics of the plane with equations

$$x^2y^2 + x^2 + y^2 + b(x+y) + a - Kxy = 0.$$

**Proposition 14.1.** (a) *Equation (14.1) has a unique fixed point $\ell$ in $]0, +\infty[$. If $\lambda \geq b + Ca$, then $\ell < \frac{1}{C}$. In particular, $\ell \to 0$ if $\lambda \to +\infty$. And $\ell$ is majorised by its limit when $\lambda \to 0$, it is the solution of $x^4 = a + bx$.*

(b) *If $\dfrac{1}{3} \leq a \leq 3$, the fixed point $\ell$ is attractive.*

*Proof.* (a) The number $\ell$ is the solution from equation $h(x) = 0$, where $h(x) := x^3 + \lambda x^2 + \lambda - \left(b + \frac{a}{x}\right)$. But $h$ increases on $-\infty$ to $+\infty$ when $x$ goes from 0 to $+\infty$, which gives existence and unicity of $\ell$. One has $h(1/C) = 1/C^3 + \lambda/C2 + [\lambda - (b + Ca)]$, then, if $\lambda \geq b + Ca$, $h(1/C) > 0$, and so $\ell < 1/C$. And it is easy which $h$ increases on $\lambda$.

(b) One has here $f(x) = \dfrac{a+bx+x^2}{x^2+1}$, and $f'(x) = \dfrac{-bx^2+2x(1-a)+b}{(x^2+1)^2}$, thus the condition for attractivity is $|b\ell^2+2\ell(a-1)-b| < (\ell^2+1)^2(2\ell+\lambda)$, and a sufficient condition for this is

$$b\ell^2+2\ell|a-1|+b < (\ell^2+1)^2(\ell+\lambda) = (\ell^2+1)\frac{a+b\ell+\ell^2}{\ell}.$$

But a sufficient condition from this inequality is $2\ell^2|a-1| \leq \ell^4+(a+1)\ell^2$, that is $2|a-1| \leq \ell^2+a+1$. And this last condition is clear if $\dfrac{1}{3} \leq a \leq 3$. □

**Proposition 14.2.** (a) *If $n \geq 2$, one has $0 < m < u_n < M$, where $M = \frac{1}{\lambda}\big(1+\sqrt{(a-1)^2+b^2}\big)$ and $m = \frac{\min(a,1)}{M+\lambda}$.*

(b) *If $\lambda \geq b$, the function $x \mapsto f(x) := \frac{a+bx+x^2}{1+x^2}$ is increasing on $[m,M]$.*

(c) *If $\lambda > b + \frac{1}{b}[-\min(a-1,0)]^2$, then $u_n$ converges to $\ell$.*

*Proof.* We first by remark that $f(x) = 1+h(x)$ where the function $x \mapsto h(x) := \frac{a-1+bx}{1+x^2}$ has its derivative on $[0,+\infty[$ which is canceled at the point $x_0 = \frac{1-a+\sqrt{(a-1)^2+b^2}}{b}$. The function has its maximum at this point, which is $\sqrt{(a-1)^2+b^2}$. So we have the majoration of $u_{n+2}$ by $M$.

Then one can we see that $f$ is increasing on $[0,x_0]$ and decreasing on $[x_0,+\infty[$, and as its maximum is at $x_0$. One see easily that, if $\lambda \geq b$, it is $M \leq x_0$. Hence the result of (b) from this proposition.

From (c), one see previously that if $\lambda \geq b$, then $f$ is increasing on $[m,M]$. By Theorem 3.2, it suffices to verify that $\max\limits_{[m,M]} f' < \lambda$. But $f' \geq 0$ on $[m,M]$, and $f'(x) \leq -bx^2+2x(1-a)+b := N(x)$. Studying variations of the fonction $N$, one see that if $a \geq 1$, then $\max N \leq N(0) = b$, thus one condition is $\lambda > b$. On the other hand, if $a \leq 1$, the maximum of $N$ is $N(\frac{1-a}{b}) = b + \frac{(1-a)^2}{b}$. Hence one has the sufficient condition of (c) from the proposition. □

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