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ULAM STABILITY FOR FIRST-ORDER NONLINEAR DYNAMIC EQUATIONS

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Dedicated to Professor Mustafa Kulenović on the occasion of his 70th birthday

ABSTRACT. The purpose of this paper is to investigate Ulam stability of first-order nonlinear dynamic equations on time scales. Based on the method of the Picard operator and using dynamic inequalities, we obtain four types of stability. In addition, as applications of our main result, we obtain new Ulam stability results for other nonlinear dynamic equations. An example is also provided to illustrate our main result.

1. INTRODUCTION

It is widely known that stability of solutions is one of the most important and interesting properties among various qualitative properties of solutions. In the existing literature, there are several stability theories, for both differential and difference equations (see e.g., [15, 20, 23, 24] for the discrete case and [25, 26] for PDEs) but the concept of Ulam stability has significant applications in various fields of mathematical analysis. This is mainly because Ulam stability essentially deals with the existence of an exact solution near to every approximate solution and is useful in the situation when it is difficult to find the exact solution. This kind of stability for functional equations was first discussed by Ulam [39] in his famous talk at the University of Wisconsin in 1940. He proposed to "provide an approximate solution for the exact solution in a simple form for a functional equation". One year later, D. H. Hyers [22] delivered an affirmative answer to this question. Thereafter, the results of Hyers were extended by many authors, but remarkable improvements were provided separately by T. Aoki [10], D. G. Bourgin [19], and Th. M. Rassias [31]. The problem of stability in Ulam sense for various kinds of differential, difference, integral equations etc. has been seriously studied by many researchers employing several techniques. In 2005, D. Popa [30] studied Ulam-type stability for difference equations. Some very recent studies on Ulam stability for difference equations can be found in [5–8, 11–13, 27]. I. A. Rus [33] presented four types of Ulam stability for differential equations

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$$x'(t) = f(t, x(t))$$

both in finite and infinite intervals. Also, in [34], he presented and discussed Ulam-type stability for the differential equation

$$x'(t) = p(t) + f(t, x(t))$$

in Banach spaces. Y. Shen and Y. Li [36], employing the method of variation of parameters, established the Ulam stability for linear differential equations of first-order, second-order, third-order, and nth order. In 2015, J. Huang et al. [21], adopting a fixed point method, investigated Hyers–Ulam as well as generalized Hyers–Ulam stability of nonlinear differential equations involving a Lipschitz condition on infinite intervals. Q. H. Alqifiary and S. M. Jung [3] proved Hyers–Ulam stability for second-order differential equations using Gronwall's inequality. Very recently, A. B. Makhlouf et al. [14] investigated Hyers–Ulam and Hyers–Ulam–Rassias stability for stochastic functional differential equations via the method of fixed point and stochastic analysis techniques.

S. András and A. R. Mészáros [9] studied Hyers–Ulam stability of some linear and nonlinear dynamic equations as well as integral equations on time scales. They employed both direct and operational methods, and based on the theory of Picard's operators, proposed a unified approach to Hyers–Ulam stability. Y. Shen [35], employing the method of integrating factor, investigated Ulam stability of the first-order linear dynamic equation

$$x^{\Delta}(t) = p(t)x(t) + f(t)$$

and its adjoint equation

$$x^{\Delta}(t) = -p(t)x^{\sigma}(t) + f(t)$$

on a finite interval. Also, D. R. Anderson and M. Ointsuka [4] established Hyers–Ulam stability of certain first-order linear homogeneous dynamic equations with constant coefficients. They extended the results given in [28, 29] to all time scales and also provided an application to a perturbed linear dynamic equation.

Most recently, in 2021, applying dynamic inequalities, M. A. Alghamdi et al. [1, 2] obtained several results on Hyers–Ulam and Hyers–Ulam–Rassias stability for the first-order dynamic equations

$$x^{\Delta}(t) = p(t)x(t) + f(t)$$

and

$$x^{\Delta}(t) = p(t)x(t) + f(t, x(t), h(x(t))) + g(t),$$

respectively.

In this paper, we investigate Ulam stability for the nonlinear dynamic equation (NDE) of the form

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t, x(t)), \quad t \in \mathbb{J}^{\kappa}, \tag{1.1}$$

where $\mathbb{J} := [a,b]_{\mathbb{T}}$, $a, b \in \mathbb{T}$ with $a < b$, $x : \mathbb{J} \to \mathbb{R}$ is the unknown function to be determined, $x^{\sigma} = x \circ \sigma$, x^{Δ} is the delta derivative of x, $p : \mathbb{T} \to \mathbb{R}$ is a positively regressive and rd-continuous function, $f : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous in its first variable and continuous in its second variable.

Based on the method of Picard operator and dynamic inequalities, we obtain results on stability of NDE (1.1). The results obtained in this paper are more general than the known results available in the literature and include the studies [2,4,21,33,36]. For the existence, uniqueness, and other properties of solutions of NDE (1.1), we refer to [18,37,38].

2. PRELIMINARIES

To understand the notation used in this paper, we include some preliminary material. The following material pertinent to time scales can be found in [16,17]. A nonempty closed subset of the real line $\mathbb{R}$ is called a time scale $\mathbb{T}$. We usually write $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T} < \infty$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.1. *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists $f^\Delta(t) \in \mathbb{R}$, a so-called delta derivative of f, with the following property: For any $\varepsilon > 0$ there is a neighbourhood N of t, such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon \big|\sigma(t) - s\big| \quad \textit{for all} \quad s \in N.$$

Definition 2.2. *A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at every right-dense point or maximal point in $\mathbb{T}$ and its left sided limits exist at left-dense points in $\mathbb{T}$. The symbol $\mathrm{C_{rd}}(\mathbb{T},\mathbb{R})$ will be used for the set of all such functions. If a function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous in its first variable and continuous in its second variable, then we write $f \in \mathrm{C_{rd}}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$.*

Remark 2.1. The family $\mathrm{C_{rd}}(\mathbb{J},\mathbb{R})$ of all rd-continuous functions from $\mathbb{J}$ into $\mathbb{R}$ forms a Banach space coupled with the norm $\|\cdot\|$ defined as $\|x\| := \sup_{t \in \mathbb{J}} |x(t)|$.

Definition 2.3. *We say that $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The symbol $\mathcal{R}(\mathbb{T},\mathbb{R})$ will be used for the set of all rd-continuous regressive functions. If $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, then p is said to be positively regressive, and $\mathcal{R}^+(\mathbb{T},\mathbb{R})$ denotes the set of all rd-continuous positively regressive functions.*

Definition 2.4. *For $p \in \mathcal{R}(\mathbb{T},\mathbb{R})$, the generalized exponential function $e_p(t,s)$ on the time scale $\mathbb{T}$ is defined as*

$$e_p(t,s) := \begin{cases} \exp\left(\displaystyle\int_s^t \frac{\mathrm{Log}\,|1 + \mu(r)p(r)|}{\mu(r)} \Delta r\right) & \textit{if } \mu(r) \neq 0, \\ \exp\left(\displaystyle\int_s^t p(r)\Delta r\right) & \textit{if } \mu(r) = 0. \end{cases}$$

For $p, q \in \mathcal{R}(\mathbb{T},\mathbb{R})$, we define

$$p \oplus q := p + q + \mu p q, \quad \ominus p := \frac{-p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Remark 2.2. We let

$$E_p := \sup_{s,t\in\mathbb{J}} |e_{\ominus p}(t,s)| > 0 \quad \text{and} \quad E_q := \sup_{s,t\in\mathbb{J}} |e_{\ominus \frac{q}{1-\mu q}}(t,s)| > 0.$$

In our investigation, we mainly use the following results and definition.

Theorem 2.1 (See [40, Theorem 2]). *Let $y, \mathcal{F} \in \mathrm{C_{rd}}(\mathbb{T},\mathbb{R}^+)$ with $\mathcal{F}$ a nondecreasing function and $\mathcal{G},\mathcal{H} \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$ with $\mathcal{G} \geq 0$, $\mathcal{H} \geq 0$. If*

$$y(t) \leq \mathcal{F}(t) + \int_a^t \mathcal{H}(s)\left[y(s) + \int_a^s \mathcal{G}(\tau)y(\tau)\Delta\tau\right]\Delta s \quad \text{for all} \quad t \in \mathbb{T}^\kappa,$$

then

$$y(t) \leq \mathcal{F}(t)e_{\mathcal{H}+\mathcal{G}}(t,a) \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

Definition 2.5 (See [32, Definition 2.1]). *Let (M,d) be a metric space. An operator $A : M \to M$ is said to be a Picard operator if there exists $u^* \in M$ with the following properties:*

(i) *$F_A = \{u^*\}$, where F_A is the fixed point set of A;*
(ii) *the sequence $\{A^n(u)\}_{n\in\mathbb{N}}$ converges to u^* for all $u \in M$.*

Lemma 2.1 (Abstract Gronwall lemma [32, Lemma 2.1]). *Let $(M,d,\leq)$ be an ordered metric space and $A : M \to M$ an increasing Picard operator ($F_A = u_A^*$). Then for $u \in M$, $u \leq A(s)$ implies $u \leq u_A^*$, while $u \geq A(s)$ implies $u \geq u_A^*$.*

Lemma 2.2 (See [18, Lemma 3.1]). *Let $a \in \mathbb{T}$, $f \in \mathrm{C_{rd}}(\mathbb{J}\times\mathbb{R},\mathbb{R})$, and $p \in (\mathbb{J},\mathbb{R})$. Then, x satisfies* (1.1) *if and only if*

$$x(t) = e_{\ominus p}(t,a)x(a) + \int_a^t e_{\ominus p}(t,s)f(s,x(s))\Delta s \quad \text{for all} \quad t \in \mathbb{J}. \tag{2.1}$$

Now, we introduce some basic definitions that will be used in this paper.

Definition 2.6. *We say that NDE* (1.1) *has Hyers–Ulam stability if there exists $K > 0$ with the following property: For any $\varepsilon > 0$, if $y \in \mathrm{C^1_{rd}}(\mathbb{J},\mathbb{R})$ is such that*

$$|y^\Delta(t) + p(t)y^\sigma(t) - f(t,y(t))| \leq \varepsilon \quad \text{for all} \quad t \in \mathbb{J}^\kappa, \tag{2.2}$$

then there exists $x \in \mathrm{C^1_{rd}}(\mathbb{J},\mathbb{R})$ satisfying (1.1) *such that*

$$|y(t) - x(t)| \leq K\varepsilon \quad \text{for all} \quad t \in \mathbb{J}. \tag{2.3}$$

Such $K > 0$ is known as HUS constant.

Definition 2.7. *We say that NDE* (1.1) *has generalized Hyers–Ulam stability if there exists $\theta_f \in \mathrm{C}(\mathbb{R}^+,\mathbb{R}^+)$, $\theta_f(0) = 0$ with the following property: For any $\varepsilon > 0$, if $y \in \mathrm{C^1_{rd}}(\mathbb{J},\mathbb{R})$ is such that*

$$|y^\Delta(t) + p(t)y^\sigma(t) - f(t,y(t))| \leq \varepsilon \quad \text{for all} \quad t \in \mathbb{J}^\kappa, \tag{2.4}$$

then there exists $x \in \mathrm{C^1_{rd}}(\mathbb{J},\mathbb{R})$ satisfying (1.1) *such that*

$$|y(t) - x(t)| \leq \theta_f(\varepsilon) \quad \text{for all} \quad t \in \mathbb{J}. \tag{2.5}$$

Definition 2.8. *Let $\mathcal{N}$ be a family of positive nondecreasing rd-continuous functions defined on $\mathbb{J}$. We say that NDE* (1.1) *has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ if there exists $K>0$ with the following property: For any $\phi\in\mathcal{N}$ and $\varepsilon>0$, if $y\in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ is such that*

$$|y^{\Delta}(t)+p(t)y^{\sigma}(t)-f(t,y(t))|\le\varepsilon\phi(t)\quad\text{for all}\quad t\in\mathbb{J}^{\kappa},\tag{2.6}$$

then there exists $x\in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ satisfying (1.1) *such that*

$$|y(t)-x(t)|\le K\varepsilon\phi(t)\quad\text{for all}\quad t\in\mathbb{J}.\tag{2.7}$$

Such $K>0$ is known as HURS constant.

Definition 2.9. *Let $\mathcal{N}$ be a family of positive nondecreasing rd-continuous functions defined on $\mathbb{J}$. We say that NDE* (1.1) *has generalized Hyers–Ulam–Rassias stability of type $\mathcal{N}$ if there exists $K>0$ with the following property: For any $\phi\in\mathcal{N}$, if $y\in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ is such that*

$$|y^{\Delta}(t)+p(t)y^{\sigma}(t)-f(t,y(t))|\le\phi(t)\quad\text{for all}\quad t\in\mathbb{J}^{\kappa},\tag{2.8}$$

then there exists $x\in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ satisfying (1.1) *such that*

$$|y(t)-x(t)|\le K\phi(t)\quad\text{for all}\quad t\in\mathbb{J}.\tag{2.9}$$

Such $K>0$ is known as GHURS constant.

Remark 2.3. A function $y\in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ satisfies (2.6) if there exists $\psi\in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ (which depends on y) with the following properties:

(i) $|\psi(t)|\le\varepsilon\phi(t)$ for all $t\in\mathbb{J}$,

(ii) $y^{\Delta}(t)+p(t)y^{\sigma}(t)=f(t,y(t))+\psi(t)$ for all $t\in\mathbb{J}^{\kappa}$.

Similar arguments hold for the inequalities (2.4) and (2.8).

3. Ulam stability

In this section we prove our main result of Ulam stability for NDE (1.1) and provide its applications.

Theorem 3.1. *Consider the NDE* (1.1)*. Assume that the following conditions are satisfied.*

(C_1) *Let $p\in\mathcal{R}^{+}(\mathbb{J},\mathbb{R})$ and $f\in C_{\mathrm{rd}}(\mathbb{J}\times\mathbb{R},\mathbb{R})$.*

(C_2) *There exists $L_f>0$ such that*

$$|f(t,u)-f(t,v)|\le L_f|u-v|\quad\text{for all}\quad t\in\mathbb{J}\quad\text{and}\quad u,v\in\mathbb{R}.\tag{3.1}$$

(C_3) *There exists $\eta>0$ such that for $\phi\in\mathcal{N}(\mathbb{J},\mathbb{R}^{+})$*

$$\int_a^t\phi(s)\Delta s\le\eta\phi(t)\quad\text{for all}\quad t\in\mathbb{J}.\tag{3.2}$$

If $E_pL_f(b-a)<1$, then the following assertions hold:

(i) *The NDE* (1.1) *has a unique solution* $x \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ *satisfying the initial condition* $x(a) = A$ *for any initial value* $A \in \mathbb{R}$.

(ii) *The NDE* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}$ *with HURS constant* $E_p(b-a)e_{E_pL_f}(b,a)$.

Proof. By Lemma 2.2, the NDE (1.1) with initial condition $x(a) = A$ is equivalent to the integral equation

$$x(t) = e_{\ominus p}(t,a)A + \int_a^t e_{\ominus p}(t,s)f(s,x(s))\Delta s \quad \text{for all} \quad t \in \mathbb{J}. \tag{3.3}$$

We first show (i). Fix $A \in \mathbb{R}$ and define $T : \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R}) \to \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ by

$$T(x)(t) := e_{\ominus p}(t,a)A + \int_a^t e_{\ominus p}(t,s)f(s,x(s))\Delta s. \tag{3.4}$$

We show that the operator T has a fixed point, and for this we use the contraction mapping principle. For any $x, y \in \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$, we can write

$$\begin{aligned}
&|T(x)(t) - T(y)(t)| \\
&\le |e_{\ominus p}(t,a)||A - A| + \int_a^t |e_{\ominus p}(t,s)||f(s,x(s)) - f(s,y(s))|\Delta s \\
&\overset{(\mathrm{C}_2)}{\le} E_p \int_a^t L_f|x(s) - y(s)|\Delta s \\
&\le E_pL_f(b-a)\|x - y\|.
\end{aligned}$$

Thus,

$$\|T(x) - T(y)\| \le E_pL_f(b-a)\|x - y\| \quad \text{for all} \quad x, y \in \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R}).$$

Since $E_pL_f(b-a) < 1$, the above inequality implies that the operator T is a contraction on $\mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$. So, T has a unique fixed point $x^* \in \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$, which is the unique solution of the NDE (1.1) satisfying $x^*(a) = A$.

Now we show (ii). Let $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ satisfy (2.6) and let $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ be the unique solution of (1.1) satisfying the initial condition $x(a) = y(a)$. Then (C_1) allows to write

$$x(t) = e_{\ominus p}(t,a)y(a) + \int_a^t e_{\ominus p}(t,s)f(s,x(s))\Delta s \quad \text{for all} \quad t \in \mathbb{J}.$$

Now, since $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ satisfies (2.6), by Remark 2.3, we can write

$$y^\Delta(t) + p(t)y^\sigma(t) = f(t,y(t)) + \psi(t) \quad \text{for all} \quad t \in \mathbb{J}^\kappa,$$

where $\|\psi(t)\| \le \varepsilon\phi(t)$ for all $t \in \mathbb{J}$. Thus,

$$\begin{aligned}
y(t) &= e_{\ominus p}(t,a)y(a) + \int_a^t e_{\ominus p}(t,s)(f(s,y(s)) + \psi(s))\Delta s \\
&= e_{\ominus p}(t,a)y(a) + \int_a^t e_{\ominus p}(t,s)f(s,y(s))\Delta s + \int_a^t e_{\ominus p}(t,s)\psi(s)\Delta s.
\end{aligned}$$

This gives

$$\begin{aligned}&\left|y(t)-e_{\ominus p}(t,a)y(a)-\int_a^t e_{\ominus p}(t,s)f(s,y(s))\Delta s\right|\\&\leq \int_a^t |e_{\ominus p}(t,s)||\psi(s)|\Delta s\\&\overset{(C_3)}{\leq} E_p(b-a)\eta\phi(t)\varepsilon \quad \text{for all} \quad t\in\mathbb{J}.\end{aligned} \tag{3.5}$$

Now, for $t\in\mathbb{J}$, we can write

$$\begin{aligned}|y(t)-x(t)| &= \left|y(t)-e_{\ominus p}(t,a)y(a)-\int_a^t e_{\ominus p}(t,s)f(s,y(s))\Delta s\right.\\&\quad\left.+\int_a^t e_{\ominus p}(t,s)f(s,y(s))\Delta s-\int_a^t e_{\ominus p}(t,s)f(s,x(s))\Delta s\right|\\&\leq \left|y(t)-e_{\ominus p}(t,a)y(a)-\int_a^t e_{\ominus p}(t,s)f(s,y(s))\Delta s\right|\\&\quad+\int_a^t |e_{\ominus p}(t,s)||f(s,y(s))-f(s,x(s))|\Delta s\\&\overset{(3.5)}{\leq} E_p(b-a)\eta\phi(t)\varepsilon+\int_a^t |e_{\ominus p}(t,s)||f(s,y(s))-f(s,x(s))|\Delta s\\&\overset{(C_2)}{\leq} E_p(b-a)\eta\phi(t)\varepsilon+L_f\int_a^t |e_{\ominus p}(t,s)(y(s)-x(s))|\Delta s.\end{aligned} \tag{3.6}$$

According to (3.6), we consider the operator $S:\mathrm{C_{rd}}(\mathbb{J},\mathbb{R})\to\mathrm{C_{rd}}(\mathbb{J},\mathbb{R})$ defined by

$$S(x)(t) := E_p(b-a)\eta\phi(t)\varepsilon+L_f\int_a^t e_{\ominus p}(t,s)x(s)\Delta s. \tag{3.7}$$

For $u,v\in\mathrm{C_{rd}}(\mathbb{J},\mathbb{R}^+)$, we can write

$$S(u)(t)-S(v)(t) = L_f\int_a^t e_{\ominus p}(t,s)(u(s)-v(s))\Delta s.$$

Then

$$|S(u)(t)-S(v)(t)| \leq L_f E_p(b-a)\|u-v\|.$$

Since $E_pL_f(b-a)<1$, we obtain that S is a contraction on $\mathrm{C_{rd}}(\mathbb{J},\mathbb{R})$, and using the Banach contraction principle, we see that S is a Picard operator and $F_S=\{u^*\}$. Then for $t\in\mathbb{J}$, we have

$$u^*(t) = E_p(b-a)\eta\phi(t)\varepsilon+L_f\int_a^t e_{\ominus p}(t,s)u^*(s)\Delta s.$$

We notice that u^* is increasing and

$$u^*(t) \leq E_p(b-a)\eta\phi(t)\varepsilon+\int_a^t E_pL_fu^*(s)\Delta s.$$

Employing the Gronwall inequality given in Theorem 2.1 to the above inequality with $y(t)=u^*(t)$, $\mathcal{F}(t)=E_p(b-a)\eta\phi(t)\varepsilon$, $\mathcal{H}(t)=E_pL_f$, and $G(t)=0$, we obtain

$$u^*(t) \leq E_p(b-a)\eta\phi(t)\varepsilon e_{E_pL_f}(t,a) \quad \text{for all} \quad t\in\mathbb{J}.$$

From (3.6), we have $u(t) \leq S(u)(t)$ for all $t \in \mathbb{J}$, where $u(t) = |y(t) - x(t)|$. Thus, S is an increasing Picard operator on $C_{rd}(\mathbb{J}, \mathbb{R})$. Now, in view of Lemma 2.1, we obtain $u(t) \leq u^*(t)$ for all $t \in \mathbb{J}$. This implies that

$$u(t) \leq E_p(b-a) e_{E_p L_f}(t,a) \varepsilon \eta \phi(t) \quad \text{for all} \quad t \in \mathbb{J}.$$

That is,

$$|y(t) - x(t)| \leq E_p(b-a) e_{E_p L_f}(t,a) \varepsilon \eta \phi(t) \quad \text{for all} \quad t \in \mathbb{J}.$$

Thus, the NDE (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with HURS constant $E_p(b-a)\eta e_{E_p L_f}(b,a)$. □

Corollary 3.1. *Assume* (C_1)–(C_3). *If* $E_p L_f(b-a) < 1$, *then* (1.1) *has generalized Hyers-Ulam-Rassias stability of type* $\mathcal{N}$ *with GHURS constant* $E_p(b-a)\eta e_{E_p L_f}(b,a)$.

Proof. In the proof of Theorem 3.1, if we take $\varepsilon = 1$, then we obtain

$$|y(t) - x(t)| \leq E_p(b-a) e_{E_p L_f}(b,a) \eta \phi(t) \quad \text{for all} \quad t \in \mathbb{J}.$$

This shows that NDE (1.1) has generalized Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with GHURS constant $E_p(b-a)\eta e_{E_p L_f}(b,a)$. □

Corollary 3.2. *Assume* (C_1)–(C_3). *If* $E_p L_f(b-a) < 1$, *then NDE* (1.1) *has Hyers–Ulam stability with HUS constant* $E_p(b-a)\eta e_{E_p L_f}(b,a)$.

Proof. In the proof of Theorem 3.1, if we take $\phi(t) \equiv 1$, then we obtain

$$|y(t) - x(t)| \leq E_p(b-a)\eta e_{E_p L_f}(b,a) \varepsilon \quad \text{for all} \quad t \in \mathbb{J}.$$

Thus NDE (1.1) has Hyers–Ulam stability with HUS constant $E_p(b-a)\eta e_{E_p L_f}(b,a)$. □

Corollary 3.3. *Assume* (C_1)–(C_3). *If* $E_p L_f(b-a) < 1$, *then NDE* (1.1) *has generalized Hyers–Ulam stability.*

Proof. Using $\theta_f(\varepsilon) = E_p(b-a)\eta e_{E_p L_f}(b,a)\varepsilon$, the result follows from Corollary 3.2. □

Now, as an application of Theorem 3.1, we shall discuss Hyers–Ulam–Rassias stability of the adjoint equation to (1.1), namely

$$x^{\Delta}(t) + q(t)x(t) = g(t, x(t)) \quad \text{for all} \quad t \in \mathbb{J}^{\kappa}, \tag{3.8}$$

where $q \in \mathcal{R}^+(\mathbb{J}, \mathbb{R})$ and $g \in C_{rd}(\mathbb{J} \times \mathbb{R}, \mathbb{R})$.

Theorem 3.2. *Consider the adjoint NDE* (3.8). *Assume that the following conditions are satisfied.*

- (C_4) *Let* $g \in C_{rd}(\mathbb{J} \times \mathbb{R}, \mathbb{R})$ *and* $q \in \mathcal{R}^+(\mathbb{J}, \mathbb{R})$ *be such that* $1 - \mu(t)q(t) > 0$ *for all* $t \in \mathbb{J}$.
- (C_5) *There exists* $L_g > 0$ *such that*

$$|g(t,u)-g(t,v)| \le L_g(1-\mu(t)q(t))|u-v| \tag{3.9}$$

for all $t \in \mathbb{J}$ *and* $u,v \in \mathbb{R}$.

(C_6) *There exists* $\eta > 0$ *such that for* $\phi \in \mathcal{N}(\mathbb{J},\mathbb{R}^+)$

$$\int_a^t \phi(s)\Delta s \le \eta\phi(t) \quad \text{for all} \quad t \in \mathbb{J}. \tag{3.10}$$

If $E_q L_g(1-\mu(t)q(t))(b-a) < 1$, *then the following assertions hold:*

(i) *The NDE* (3.8) *has a unique solution* $x \in C^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ *satisfying the initial condition* $x(a) = A$ *for any initial value* $A \in \mathbb{R}$.

(ii) *The NDE* (3.8) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}$ *with HURS constant* $E_q(b-a)\eta e_{E_q L_g}(b,a)$.

Proof. In view of the 'simple useful formula', putting $x^\sigma - \mu x^\Delta$ in place of x in (3.8), we obtain

$$x^\Delta(t) + q(t)(x^\sigma(t) - \mu(t)x^\Delta(t)) = g(t,x(t)).$$

Rearranging the terms, we can write this equation as

$$x^\Delta(t) + \left(\frac{q(t)}{1-\mu(t)q(t)}\right)x^\sigma(t) = \frac{g(t,x(t))}{1-\mu(t)q(t)}.$$

This equation is in the form of (1.1) with $p(t) := \frac{q(t)}{1-\mu(t)q(t)}$ and $f(t,x) := \frac{g(t,x)}{1-\mu(t)q(t)}$. In view of conditions (C_4) and (C_5), it is not difficult to show that (C_2) and (C_3) are verified. Hence, assertions (i) and (ii) follow from Theorem 3.1. □

Remark 3.1. Other Ulam stability results for NDE (3.8) can be derived by using Theorem 3.2.

Another application of Theorem 3.1 concerns Hyers–Ulam–Rassias stability of the NDE

$$x^\Delta(t) = F(t,x(t)) \quad \text{for all} \quad t \in \mathbb{J}^\kappa. \tag{3.11}$$

Theorem 3.3. *Consider the NDE* (3.11). *Assume that the following conditions are satisfied.*

(C_7) *Let* $F \in C_{\mathrm{rd}}(\mathbb{J}\times\mathbb{R},\mathbb{R})$ *and* $q \in \mathcal{R}^+(\mathbb{J},\mathbb{R})$ *be such that*

$$1-\mu(t)q(t) > |q(t)| \quad \text{for all} \quad t \in \mathbb{J}.$$

(C_8) *There exists* $L_F > \dfrac{|q(t)|}{1-\mu(t)q(t)}$ *such that*

$$|F(t,u)-F(t,v)| \le (L_F(1-\mu(t)q(t)) - |q(t)|)|u-v|$$

for all $t \in \mathbb{J}$ *and* $u,v \in \mathbb{R}$, *where* q *is as given in* (C_7).

(C_9) *There exists* $\eta > 0$ *such that for* $\phi \in \mathcal{N}(\mathbb{J},\mathbb{R}^+)$

$$\int_a^t \phi(s)\Delta s \le \eta\phi(t) \quad \text{for all} \quad t \in \mathbb{J}. \tag{3.12}$$

If $(E_q L_F(1-\mu(t)q(t)) - |q(t)|)(b-a) < 1$, *then the following assertions hold:*

(i) *The NDE* (3.11) *has a unique solution* $x \in C^1_{rd}(\mathbb{J},\mathbb{R})$ *satisfying the initial condition* $x(a) = A$ *for any initial value* $A \in \mathbb{R}$.
(ii) *The NDE* (3.11) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}$ *with HURS constant* $E_q(b-a)\eta e_{E_q L_F}(b,a)$.

Proof. Keeping in mind the 'simple useful formula' and rearranging the terms, we rewrite (3.11) as

$$x^\Delta(t) + \left(\frac{q(t)}{1-\mu(t)q(t)}\right) x^\sigma(t) = \frac{q(t)x(t)+F(t,x(t))}{1-\mu(t)q(t)} \quad \text{for all} \quad t \in \mathbb{J}^\kappa.$$

That is,

$$x^\Delta(t) + p(t)x^\sigma(t) = f(t,x(t)) \quad \text{for all} \quad t \in \mathbb{J}^\kappa,$$

where $p(t) := \frac{q(t)}{1-\mu(t)q(t)}$ and $f(t,x) := \frac{q(t)x+F(t,x)}{1-\mu(t)q(t)}$. Now, it remains to verify the condition (C_2) in Theorem 3.1. From (C_8), we have

$$|F(t,u)-F(t,v)| + |q(t)||u-v| \le L_F(1-\mu(t)q(t))|u-v|$$

for all $t \in \mathbb{J}$ and $u,v \in \mathbb{R}$. This gives

$$|F(t,u)-F(t,v)+q(t)(u-v)| \le L_F(1-\mu(t)q(t))|u-v|$$

for all $t \in \mathbb{J}$ and $u,v \in \mathbb{R}$. Rearranging the terms, we can write

$$\left| \frac{F(t,u)+q(t)u}{1-\mu(t)q(t)} - \frac{F(t,v)+q(t)v}{1-\mu(t)q(t)} \right| \le L_F|u-v|$$

for all $t \in \mathbb{J}$ and $u,v \in \mathbb{R}$. That is,

$$|f(t,u)-f(t,v)| \le L_F|u-v| \quad \text{for all} \quad t \in \mathbb{J} \quad \text{and} \quad u,v \in \mathbb{R}.$$

Hence, (C_2) is verified. Now, we are able to apply Theorem 3.1 and obtain that assertions (i) and (ii) follow from Theorem 3.1. □

Remark 3.2. Other Ulam stability results for NDE (3.11) can be derived by using Theorem 3.3.

4. Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $a = 2$, $b = 32$. Then $\mathbb{J} := [a,b] \cap \mathbb{T} = \{2,4,8,16,32\}$. Consider the NDE

$$x^\Delta(t) + tx^\sigma(t) = \frac{1}{e_t(t,2)} + \frac{(x^2(t)+5)^{1/2}}{8} \quad \text{for all} \quad t \in \mathbb{J}^\kappa \tag{4.1}$$

with the initial condition $x(2) = 2$ and the inequality

$$\left| y^\Delta(t) + ty^\sigma(t) - \frac{1}{e_t(t,2)} - \frac{(y^2(t)+5)^{1/2}}{8} \right| \le \varepsilon e_{\frac{1}{20}}(t,2) \quad \text{for all} \quad t \in \mathbb{J}^\kappa. \tag{4.2}$$

Here $f(t,x(t)) = \frac{1}{e_t(t,2)} + \frac{(x^2(t)+5)^{1/2}}{8}$ which satisfies (C_2) with $L_f = \frac{1}{8}$, and $p = t$, $\mu(t) = t$ for all $t \in \mathbb{T}$. Clearly, $1+\mu(t)p(t) = 1+t^2 > 0$. Thus $p \in \mathcal{R}^+(\mathbb{J},\mathbb{R})$. Also,

$$(\ominus p)(t) = \frac{-p(t)}{1+\mu(t)p(t)} = \frac{-t}{1+t^2}.$$

With these values, we obtain

$$\begin{aligned} e_{\ominus p}(t,s) &= \exp\left(\int_s^t \frac{1}{\mu(r)} \ln(1+\mu(r)\ominus p)\Delta r\right) \\ &= \exp\left(\int_s^t \frac{1}{r} \ln\left(1+r\left(\frac{-r}{1+r^2}\right)\right)\Delta r\right) \\ &= \exp\left(\int_s^t \frac{1}{r} \ln\left(1+\frac{-r^2}{1+r^2}\right)\Delta r\right) \\ &= \exp\left(\int_s^t \frac{1}{r} \ln\left(\frac{1}{1+r^2}\right)\Delta r\right) \\ &= \exp\left(\sum_{r=s}^{t/2} \ln\left(\frac{1}{1+r^2}\right)\right) \\ &= \prod_{r=s}^{t/2} \frac{1}{1+r^2} \quad \text{for} \quad s \le t \quad \text{and} \quad t \in \mathbb{J}. \end{aligned}$$

This leads to

$$E_p = \sup_{s,t\in\mathbb{J}} \left| \prod_{r=s}^{t/2} \frac{1}{1+r^2} \right| = \frac{1}{5}.$$

Further, we find that $E_p L_f(b-a) = \frac{1}{5}\frac{1}{8}(32-2) = \frac{3}{4} < 1$. Thus, all the conditions in Theorem 3.1 are satisfied. Therefore, (4.1) has a unique solution satisfying initial condition $x(2) = 2$. Now, let $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ be a solution of (4.2). Then, by Remark 2.3, there exists $\psi \in \mathrm{C}_{\mathrm{rd}}(\mathbb{J},\mathbb{R})$ such that $|\psi(t)| \le \varepsilon e_{\frac{1}{20}}(t,2)$ and

$$y^{\Delta}(t) + t y^{\sigma}(t) = \frac{1}{e_p(t,2)} + \frac{(y^2(t)+5)^{1/2}}{8} + \psi(t) \quad \text{for all} \quad t \in \mathbb{J}^{\kappa}. \tag{4.3}$$

By Lemma 2.2, we have

$$\begin{aligned} y(t) =& (y(2)+t-2)\prod_{r=2}^{t/2}\frac{1}{1+r^2} + \frac{1}{8}\sum_{s=2}^{t/2}\left(\prod_{r=s}^{t/2}\frac{1}{1+r^2}\right)(y^2(r)+5)^{1/2} \\ &+ \sum_{s=2}^{t/2}\left(\prod_{r=s}^{t/2}\frac{1}{1+r^2}\right)\psi(s) \quad \text{for all} \quad t \in \mathbb{J}. \end{aligned} \tag{4.4}$$

From Theorem 3.1, we find that the dynamic problem

$$\begin{aligned} x^{\Delta}(t) + t x^{\sigma}(t) &= \frac{1}{e_p(t,2)} + \frac{(x^2(t)+5)^{1/2}}{8};\ t \in \mathbb{J}^{\kappa}, \\ x(2) &= y(2) \end{aligned}$$

has a unique solution. According to Lemma 2.2, this unique solution is given by

$$
\begin{aligned}
x(t) &= y(2)e_{\ominus p}(t,2) + \int_2^t e_{\ominus p}(t,s)\left(\frac{1}{e_p(s,2)} + \frac{(x^2(s)+5)^{1/2}}{8}\right)\Delta s \\
&= y(2)e_{\ominus p}(t,2) + \int_2^t \left(e_{\ominus p}(t,2) + e_{\ominus p}(t,s)\frac{(x^2(s)+5)^{1/2}}{8}\right)\Delta s \\
&= y(2)e_{\ominus p}(t,2) + e_{\ominus p}(t,2)(t-2) + \frac{1}{8}\int_2^t e_{\ominus p}(t,s)(x^2(s)+5)^{1/2}\Delta s \\
&= (y(2)+t-2)e_{\ominus p}(t,2) + \frac{1}{8}\int_2^t e_{\ominus p}(t,s)(x^2(s)+5)^{1/2}\Delta s \\
&= (y(2)+t-2)\prod_{r=2}^{t/2}\frac{1}{1+r^2} + \frac{1}{8}\sum_{s=2}^{t/2}\left(\prod_{r=s}^{t/2}\frac{1}{1+r^2}\right)(x^2(r)+5)^{1/2}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
x(t) =&(y(2)+t-2)\prod_{r=2}^{t/2}\frac{1}{1+r^2} \\
&+\frac{1}{8}\sum_{s=2}^{t/2}\left(\prod_{r=s}^{t/2}\frac{1}{1+r^2}\right)(x^2(r)+5)^{1/2} \quad \text{for all} \quad t\in\mathbb{J}.
\end{aligned}
\tag{4.5}
$$

Now, from (4.4) and (4.5), we can write for $t\in\mathbb{J}$,

$$
\begin{aligned}
|y(t)-x(t)| &\le \sum_{s=2}^{t/2}\left(\prod_{r=s}^{t/2}\frac{1}{1+r^2}\right)|\psi(s)| \\
&\le \sum_{s=2}^{t/2}\left(\prod_{r=s}^{t/2}\frac{1}{1+r^2}\right)\varepsilon e_{\frac{1}{20}}(s,2) \\
&\le \varepsilon\frac{1}{5}\sum_{s=2}^{t/2} e_{\frac{1}{20}}(s,2) \\
&\le \varepsilon\frac{1}{5}e_{\frac{1}{20}}(32,2)30.
\end{aligned}
$$

Thus, $|y(t)-x(t)| \le 6\varepsilon e_{\frac{1}{20}}(32,2)$, $t\in\mathbb{J}$, which yields that (4.1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with HURS constant $6e_{\frac{1}{20}}(32,2)$.

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