WEIGHTED HYERS-ULAM STABILITY FOR NONLINEAR NONAUTONOMOUS DIFFERENCE EQUATIONS

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Dedicated to Professor Mustafa Kulenović on the occasion of his 70th birthday

ABSTRACT. Let \((A_m)_{m \in \mathbb{Z}}\) be a sequence of bounded linear operators acting on an arbitrary Banach space \(X\) and admitting an exponential dichotomy. Furthermore, let \(f_m : X \to X, m \in \mathbb{Z}\) be a sequence of Lipschitz maps. Provided that the Lipschitz constants of \(f_m\) are uniformly small, we show that a nonlinear difference equation
\[
x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{Z},
\]
exhibits various types of the weighted Hyers-Ulam stability property.

1. INTRODUCTION

Let \(X = (X, \| \cdot \|)\) be an arbitrary Banach space, \((A_n)_{n \in \mathbb{Z}}\) a sequence of bounded linear operators on \(X\) and \((f_n)_{n \in \mathbb{Z}}\) a sequence of Lipschitz maps \(f_n : X \to X, n \in \mathbb{Z}\). We consider the corresponding nonlinear nonautonomous difference equation given by
\[
x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z},
\]
as well as the associated linear equation
\[
x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}.
\]
We recall that (1.1) is said to be Hyers-Ulam stable if there exists \(L > 0\) with the property that for each \(\delta > 0\) and a sequence \((y_n)_{n \in \mathbb{Z}} \subset X\) such that
\[
\sup_{n \in \mathbb{Z}} \|y_n - A_{n-1} y_{n-1} - f_{n-1}(y_{n-1})\| \leq \delta,
\]
there exists a solution \((x_n)_{n \in \mathbb{Z}}\) of (1.1) such that
\[
\sup_{n \in \mathbb{Z}} \|x_n - y_n\| \leq L\delta.
\]
Hence, if (1.1) is Hyers-Ulam stable, in a vicinity of each approximate solution of (1.1) we can construct its exact solution.

We emphasize that it follows from [2, Theorem 3] that (1.1) is Hyers-Ulam stable provided that the following conditions hold:

\begin{itemize}
  \item \textbf{2020 Mathematics Subject Classification.} 34D20, 34D09.
  \item \textbf{Key words and phrases.} shadowing; exponential dichotomy; nonautonomous dynamics.
\end{itemize}
(1.2) admits an exponential dichotomy;
the Lipschitz constants of $f_n$ are uniformly small.

This result unified and extended several previously available results dealing with
Hyers-Ulam stability of (1.2) in the case when $(A_n)_{n \in \mathbb{Z}}$ is either a constant or a
periodic sequence (see [5–8, 22] and references therein). Moreover, the arguments
and results developed in [2] inspired several other relevant contributions to the
shadowing theory for nonautonomous difference and differential equations (see [1,
3, 4, 11–14]).

In addition, in [2] the authors dealt with a more general concept of the Hyers-
Ulam stability, in which the size of an approximate solution as well as its deviation
from an exact solution are not necessarily measured with respect to the $l^\infty$-norm
(as in (1.3) and (1.4)). Indeed, these quantities can be measured with respect to a
wide collection of norms on sequence spaces.

In the present paper we study the weighted Hyers-Ulam stability of (1.1). This
concept is motivated by the related notions of weighted shadowing property intro-
duced in the context of smooth dynamical systems [10, 21]. The principal mo-
tivation for considering such notions are situations when the expression $\|y_n - A_{n-1}y_{n-1} - f_{n-1}(y_{n-1})\|$ in (1.3) cannot be well-controlled for every $n \in \mathbb{Z}$, or when it exhibits certain asymptotic behaviour when $|n| \to \infty$.

Our main result (see Theorem 3.1) implies that (1.1) has the weighted Hyers-
Ulam stability property if the following conditions hold:

• (1.2) admits an exponential dichotomy;
• the Lipschitz constants of $f_n$ are uniformly small;
• the sequence of weights exhibits the subexponential growth property (see
  Proposition 2.2).

We note that our arguments follow closely the approach developed in [2], by com-
bining it with techniques from [10].

2. PRELIMINARIES

2.1. Banach sequence spaces

Let $S$ denote the set of all sequences $s = (s_n)_{n \in \mathbb{Z}}$ of real numbers. We say that a
linear subspace $B \subset S$ is a normed sequence space if there exists a norm $\|\cdot\|_B : B \to \mathbb{R}^+_0$ on $B$ such that if $s' \in B$ and $|s_n| \leq |s'_n|$ for $n \in I$, then $s \in B$ and $\|s\|_B \leq \|s'\|_B$. If in addition $(B, \|\cdot\|_B)$ is complete, we say that $B$ is a Banach sequence space.

Let $B$ be a Banach sequence space. We say that $B$ is admissible if:

1. $\chi_{\{n\}} \in B$ and $\|\chi_{\{n\}}\|_B > 0$ for $n \in \mathbb{Z}$, where $\chi_A$ denotes the characteristic function
   of the set $A \subset \mathbb{Z}$;
2. for each $s = (s_n)_{n \in \mathbb{Z}} \in B$ and $m \in \mathbb{Z}$, the sequence $s^m = (s^m_n)_{n \in \mathbb{Z}}$ defined by
   $s^m_n = s_{n+m}$ belongs to $B$ and $\|s^m\|_B = \|s\|_B$ for $s \in B$ and $m \in \mathbb{Z}$.
Example 2.1. The set \( l^\infty = \{ s \in S : \sup_{n \in \mathbb{Z}} |s_n| < +\infty \} \) is a Banach sequence space when equipped with the norm \( ||s|| = \sup_{n \in \mathbb{Z}} |s_n| \).

Example 2.2. For each \( p \in [1, \infty) \), the set \( l^p = \{ s \in S : \sum_{n \in \mathbb{Z}} |s_n|^p < +\infty \} \) is a Banach sequence space when equipped with the norm \( ||s|| = (\sum_{n \in \mathbb{Z}} |s_n|^p)^{1/p} \).

Example 2.3 (Orlicz sequence spaces). Let \( \varphi : (0, +\infty) \to (0, +\infty) \) be a nondecreasing nonconstant left-continuous function. We set \( \psi = \varphi^{-1} \). Then
\[
\psi(\{s_n\}_n) = \sum_{n \in \mathbb{Z}} \varphi(|s_n|)
\]
is a Banach sequence space when equipped with the norm \( ||s|| = \inf \{ c > 0 : M_\psi(s/c) \leq 1 \} \).

Some important properties of admissible Banach sequence spaces are given in the following result (see [10, Proposition 1]).

Proposition 2.1. Let \( B \) be an admissible Banach sequence space.
1. If \( s^1 = (s^1_n)_{n \in \mathbb{Z}} \) and \( s^2 = (s^2_n)_{n \in \mathbb{Z}} \) are sequences in \( S \) and \( s^1_n = s^2_n \) for all but finitely many \( n \in \mathbb{Z} \), then \( s^1 \in B \) if and only if \( s^2 \in B \).
2. If \( s^1_n \to s \) in \( B \) when \( n \to \infty \), then \( s^1_n \to s_n \) when \( n \to \infty \), for \( m \in \mathbb{Z} \).
3. For each \( s \in B \) and \( \lambda \in (0, 1) \), the sequences \( s^1 \) and \( s^2 \) defined by
\[
s^1_n = \sum_{m \geq 0} e^{-\lambda m} s_{n-m} \quad \text{and} \quad s^2_n = \sum_{m \geq 1} e^{-\lambda m} s_{n+m}
\]
are in \( B \), and
\[
||s^1||_B \leq \frac{1}{1 - e^{-\lambda}} ||s||_B \quad \text{and} \quad ||s^2||_B \leq \frac{e^{-\lambda}}{1 - e^{-\lambda}} ||s||_B. \tag{2.1}
\]

Remark 2.1. For more information on admissible Banach sequence spaces (and their continuous time counterparts) and their role in the qualitative theory of nonautonomous systems, we refer to [16–20] and references therein.

2.2. Weights

We now introduce a class of weights introduced in [10].

More precisely, throughout this paper \( w = (w_k)_{k \geq 0} \) will be a sequence of real numbers such that there exists \( t > 0 \) so that
\[
w_k \geq t \quad \text{for every} \quad k \geq 0, \tag{2.2}
\]
and with the property that for every \( \lambda > 0 \), there exist \( \lambda', L > 0 \) such that:
\[
e^{-\lambda(m-n)} \frac{W_n}{W_m} \leq L e^{-\lambda'(m-n)} \quad \text{for every} \quad m \geq n \geq 0, \tag{2.3}
\]
and
\[
e^{-\lambda(n-m)} \frac{W_n}{W_m} \leq L e^{-\lambda'(n-m)} \quad \text{for every} \quad n \geq m \geq 0. \tag{2.4}
\]
It turns out that (2.3) and (2.4) can be stated in a more transparent manner (see [10, Proposition 2]).

**Proposition 2.2.** The following statements are equivalent:
1. for every \( \lambda > 0 \), there exist \( \lambda', L > 0 \) such that (2.3) and (2.4) hold;
2. for every \( \varepsilon > 0 \) there exists \( C > 0 \) such that
   \[
   \frac{w_n}{w_m} \leq Ce^{\varepsilon|n-m|}, \quad m, n \geq 0.
   \] (2.5)

The following result established in [10, Proposition 3] gives a large class of examples of sequences \( w = (w_k)_{k \geq 0} \) satisfying the above properties.

**Proposition 2.3.** Assume that \( p \) is a polynomial with a positive leading coefficient such that \( p(k) > 0 \) for \( k \geq 0 \). Given \( w \geq 0 \), we define
   \[ w_k = p(k)^w, \quad k \geq 0. \]
Then, the sequence \( w = (w_k)_{k \geq 0} \) satisfies properties (2.2), (2.3) and (2.4).

### 2.3. Sequence spaces induced by weights

We now introduce a class of sequence spaces that will play a central role in our paper.

Let \( X = (X, \| \cdot \|) \) be an arbitrary Banach space, \( B \) an admissible Banach sequence space and \( w = (w_k)_{k \geq 0} \) a sequence of weights satisfying (2.2), (2.3) and (2.4). Set
   \[ Y_{X,B,w} = \left\{ (x_k)_{k \in \mathbb{Z}} \subset X : (w_k \| x_k \|)_{k \in \mathbb{Z}} \in B \right\}. \]

The following result can be established by arguing as in the proof of [10, Proposition 4].

**Proposition 2.4.** \( Y_{X,B,w} \) is a Banach space with respect to the norm
   \[ \| X \|_{X,B,w} = \|(w_k \| x_k \|)_{k \in \mathbb{Z}}\|_B. \]

### 2.4. Exponential dichotomy

Finally, we recall the notion of an exponential dichotomy (see [9, 15]). We continue to denote by \( X = (X, \| \cdot \|) \) an arbitrary Banach space. By \( B(X) \) we will denote the space of all bounded linear operators on \( X \), equipped with the operator norm that we will also denote by \( \| \cdot \| \). Moreover, \( \text{Id} \) will denote the identity operator on \( X \).

We say that a sequence \( (A_n)_{n \in \mathbb{Z}} \subset B(X) \) admits an exponential dichotomy if there exist a sequence of projections \( P_n, n \in \mathbb{Z} \) on \( X \) and constants \( C, \lambda > 0 \) such that:

- for \( n \in \mathbb{Z} \),
  \[
  P_{n+1}A_n = A_n P_n,
  \]
  and \( A_n|_{\ker P_n} : \ker P_n \to \ker P_{n+1} \) is invertible;
• for $m \geq n$,
\[
\|A(m,n)P_n\| \leq Ce^{-\lambda(m-n)},
\]
where
\[
A(m,n) = \begin{cases} 
A_{m-1} \cdots A_n & m > n; \\
Id & m = n;
\end{cases}
\]
• for $m \leq n$,
\[
\|A(m,n)(Id - P_n)\| \leq Ce^{-\lambda(n-m)},
\]
where
\[
A(m,n) := \left( A(n,m)\mid_{\text{Ker}P_m} \right)^{-1}: \text{Ker}P_n \to \text{Ker}P_m.
\]

**Example 2.4.** Let $A \in \mathcal{B}(X)$ be a hyperbolic linear operator, i.e. the spectrum of $A$ is disjoint from the unit circle $S^1 \subset \mathbb{C}$. Let $c > 0$ and assume that $(A_n)_{n \in \mathbb{Z}} \subset \mathcal{B}(X)$ is a sequence such that
\[
\sup_{n \in \mathbb{Z}} \|A - A_n\| \leq c.
\]
Provided that $c$ is sufficiently small, it follows from [15, Theorem 7.6.7.] that the sequence $(A_n)_{n \in \mathbb{Z}}$ admits an exponential dichotomy.

## 3. Main Result

Let $X = (X, \| \cdot \|)$ be an arbitrary Banach space, $B$ an admissible Banach sequence space and $w = (w_k)_{k \geq 0}$ a sequence of weights satisfying properties (2.2), (2.3) and (2.4).

Moreover, let $(A_n)_{n \in \mathbb{Z}} \subset \mathcal{B}(X)$ be an arbitrary sequence. We consider the associated linear difference equation given by
\[
x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}.
\]
Given a sequence $(f_n)_{n \in \mathbb{Z}}$ of maps $f_n: X \to X$, we can consider the nonlinear difference equation given by
\[
x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z}.
\]

We now introduce the notion of a $(\delta, B, w)$-pseudotrajectory for (3.2).

**Definition 3.1.** Let $\delta > 0$. We say that a sequence $(y_n)_{n \in \mathbb{Z}} \subset X$ is a $(\delta, B, w)$-pseudotrajectory for (3.2) if the sequence $(y_{n+1} - A_n y_n - f_n(y_n))_{n \in \mathbb{Z}}$ belongs to $Y_{X,B,w}$ and
\[
\|(y_{n+1} - A_n y_n - f_n(y_n))_{n \in \mathbb{Z}}\|_{X,B,w} \leq \delta.
\]

**Remark 3.1.** In the particular case when $w = (w_k)_{k \geq 0}$ is a constant sequence $w_k = 1$, the notion of an $(\delta, B, w)$-pseudotrajectory reduces to the notion of an $(\delta, B)$-pseudotrajectory, which was introduced and studied in [2].
In the sequel, we will use the following simple result established in [10, Lemma 4.5].

**Lemma 3.1.** For every $\lambda > 0$ there exists $\lambda', L > 0$ such that for every $n, m \in \mathbb{Z}$:

1. $e^{-\lambda(m-n)} \frac{W[n]}{W[m]} \leq L e^{-\lambda'(m-n)}$, for $m \geq n$;
2. $e^{-\lambda(n-m)} \frac{W[n]}{W[m]} \leq L e^{-\lambda'(n-m)}$, for $m \leq n$.

We are now in the position to formulate and establish the main result of our paper.

**Theorem 3.1.** Assume that a sequence $(A_n)_{n \in \mathbb{Z}} \subset \mathcal{B}(X)$ admits an exponential dichotomy. Furthermore, suppose that there exists $c > 0$ such that

$$\|f_n(x) - f_n(y)\| \leq c \|x - y\|, \quad \text{for } x, y \in X \text{ and } n \in \mathbb{Z}. \quad (3.4)$$

Then, provided that $c$ is sufficiently small, there exists $L > 0$ with the following property: for each $\delta > 0$ and an $(\delta, B, w)$-pseudotrajectory $(y_n)_{n \in \mathbb{Z}} \subset X$ for $(3.2)$, there exists a solution $(x_n)_{n \in \mathbb{Z}} \subset X$ of $(3.2)$ such that

$$\|(x_n - y_n)_{n \in \mathbb{Z}}\|_{X, B, w} \leq L \delta. \quad (3.5)$$

**Proof.** Take $\delta > 0$ and a $(\delta, B, w)$-pseudotrajectory $(y_n)_{n \in \mathbb{Z}} \subset X$ for $(3.2)$. Set

$$G(m,n) = \begin{cases} A(m,n)P_n & m \geq n; \\ -A(m,n)(1d - P_n) & m < n. \end{cases}$$

For a sequence $z = (z_n)_{n \in \mathbb{Z}} \in Y_{X, B, w}$, set

$$(Tz)_n = \sum_{k \in \mathbb{Z}} G(n, k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k), \quad n \in \mathbb{Z}.$$ 

By (2.6) and (2.7), we have that

$$\|(Tz)_n\| \leq \sum_{k=-\infty}^{n} \|A(n,k)P_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k)\|$$

$$+ \sum_{k=n+1}^{\infty} \|A(n,k)P_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k)\|$$

$$\leq C \sum_{k=-\infty}^{n} e^{-\lambda(n-k)}\|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\|$$

$$+ C \sum_{k=n+1}^{\infty} e^{-\lambda(k-n)}\|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\|,$$
and thus
\[
\|w_{[n]}\|\|T^0\|_n \leq C \sum_{k=-\infty}^{n} e^{-\lambda(n-k)} \frac{w_{[n]}}{w_{[k]}} \|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\| \\
+ C \sum_{k=n+1}^{\infty} e^{-\lambda(k-n)} \frac{w_{[n]}}{w_{[k]}} \|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\|.
\]

By invoking Lemma 3.1, we conclude that
\[
w_{[n]}\|\|T^0\|_n \leq C' \sum_{k=-\infty}^{n} e^{-\lambda'(n-k)} w_{[k]} \|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\| \\
+ C' \sum_{k=n+1}^{\infty} e^{-\lambda'(k-n)} w_{[k]} \|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\| \\
= C' \sum_{j=0}^{\infty} e^{-\lambda j} w_{[n-j]} \|A_{n-j-1}y_{n-j-1} + f_{n-j-1}(y_{n-j-1}) - y_{n-j}\| \\
+ C' \sum_{j=1}^{\infty} e^{-\lambda j} w_{[n+j]} \|A_{n+j-1}y_{n+j-1} + f_{n+j-1}(y_{n+j-1}) - y_{n+j}\|,
\]
for some constants $C', \lambda' > 0$ that depend only on $C, \lambda$. We define sequences $(s_n)_{n \in \mathbb{Z}}$ and $(s'_n)_{n \in \mathbb{Z}}$, $i = 1, 2$ of nonnegative numbers by
\[
s_n = w_{[n]} \|A_{n-1}y_{n-1} + f_{n-1}(y_{n-1}) - y_n\|, \\
s'_n = \sum_{j=0}^{\infty} e^{-\lambda j} s_{n-j} \quad \text{and} \quad s''_n = \sum_{j=1}^{\infty} e^{-\lambda j} s_{n+j}.
\]
Then, (3.6) implies that
\[
w_{[n]}\|\|T^0\|_n \leq C'(s'_1 + s''_n), \quad n \in \mathbb{Z},
\]
(3.7)

On the other hand, (3.3) together with Proposition 2.1 gives that
\[
\|s'_1\|_{B} \leq \frac{1}{1 - e^{-\lambda'}} \delta \quad \text{and} \quad \|s''_n\|_{B} \leq \frac{e^{-\lambda'}}{1 - e^{-\lambda'}} \delta.
\]
(3.8)

By combining (3.7) and (3.8), we have that
\[
\|T^0\|_{X,B,w} \leq M\delta,
\]
(3.9)

where
\[
M := C'(1 + e^{-\lambda'}) \frac{1}{1 - e^{-\lambda'}} > 0.
\]

Moreover, for $z = (z_n)_{n \in \mathbb{Z}} \in Y_{X,B,w}$ we have (using (3.4)) that
\[
\|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k\| \leq \|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\| + c\|z_{k-1}\|,
\]
for $k \in \mathbb{Z}$. Hence,
that enabled us to establish (3.9), we find that

\[ w_{|k|} \| A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k \| \]
\[ \leq w_{|k|} \| A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k \| + cw_{|k|} \| z_{k-1} \| \]
\[ \leq w_{|k|} \| A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k \| + D w_{|k-1|} \| z_{k-1} \|, \]  

(3.10)

for \( k \in \mathbb{Z} \), where (see (2.5))

\[ D := \sup_{k \in \mathbb{Z}} \frac{w_{|k|}}{w_{|k-1|}} < +\infty. \]

In addition, using (2.6), (2.7) and Lemma 3.1, we have that

\[ w_{|n|} \|(Tz)_n\| \leq C' \sum_{k=-\infty}^{n} e^{-\lambda'(n-k)} w_{|k|} \| A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k \| \]
\[ + C' \sum_{k=n+1}^{\infty} e^{-\lambda'(k-n)} w_{|k|} \| A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k \|, \]  

(3.11)

for \( n \in \mathbb{Z} \). By combining (3.10) together with (3.11), and using the same arguments that enabled us to establish (3.9), we find that

\[ \| Tz \|_{X,B,w} \leq C' \frac{1 + e^{-\lambda'}}{1 - e^{-\lambda'}} \delta + C'D 1 + e^{-\lambda'} \| z \|_{X,B,w} < +\infty. \]

In particular, \( T : Y_{X,B,w} \to Y_{X,B,w} \) is a well-defined map.

Now take \( z' = (z'_n)_{n \in \mathbb{Z}} \in Y_{X,B,w}, i = 1,2 \). We have that

\[ (Tz^1)_n - (Tz^2)_n = \sum_{k \in \mathbb{Z}} G(n,k) (f_{k-1}(y_{k-1} + z'_1 - z_1 - z_{k-1}^2)) \]

for \( n \in \mathbb{Z} \). Furthermore, (3.4) implies that

\[ \| f_{k-1}(y_{k-1} + z'_1 - z_1 - z_{k-1}^2) - f_{k-1}(y_{k-1} + z_1 - z_{k-1}^2) \| \leq \epsilon \| z'_1 - z_1 - z_{k-1}^2 \|, \]

\( k \in \mathbb{Z} \).

By proceeding as above, we conclude that

\[ \| Tz^1 - Tz^2 \|_{X,B,w} \leq q \| z^1 - z^2 \|_{X,B,w}, \]  

(3.12)

where

\[ q := c DC' \frac{1 + e^{-\lambda'}}{1 - e^{-\lambda'}}. \]

Provided that \( c \) is sufficiently small, we have that \( q < 1 \) and thus \( T \) is a contraction on \( Y_{X,B,w} \). Let

\[ L := \frac{M}{1 - q} > 0, \]

and set

\[ D := \{ z \in Y_{X,B,w} : \| z \|_{X,B,w} \leq L \delta \}. \]

Take \( z \in D \). By (3.9) and (3.12), we have that

\[ \| Tz \|_{X,B,w} \leq \| Tz - T0 \|_{X,B,w} + \| T0 \|_{X,B,w} \leq qL \delta + M \delta = L \delta. \]

We conclude that \( T(D) \subseteq D \), and thus \( T \) has a unique fixed point \( z = (z_n)_{n \in \mathbb{Z}} \in D \). Hence,
\[ z_{n+1} - A_n z_n \]
\[ = \sum_{k \leq n+1} \mathcal{A}(n+1,k) P_k (A_{k-1} y_{k-1} + f_{k-1} (y_{k-1} + z_{k-1}) - y_k) \]
\[ - \sum_{k \geq n+2} \mathcal{A}(n+1,k) (\text{Id} - P_k) (A_{k-1} y_{k-1} + f_{k-1} (y_{k-1} + z_{k-1}) - y_k) \]
\[ + \sum_{k \geq n+1} \mathcal{A}(n+1,k) P_k (A_{k-1} y_{k-1} + f_{k-1} (y_{k-1} + z_{k-1}) - y_k) \]
\[ = P_{n+1}(A_n y_n + f_n (y_n + z_n) - y_{n+1}) \]
\[ + (\text{Id} - P_{n+1})(A_n y_n + f_n (y_n + z_n) - y_{n+1}) \]
\[ = A_n y_n + f_n (y_n + z_n) - y_{n+1}, \]
for \( n \in \mathbb{Z} \). Consequently, setting \( x_n = y_n + z_n, n \in \mathbb{Z} \), we conclude that \((x_n)_{n \in \mathbb{Z}}\) is a solution of (3.2). Moreover, since \( z \in \mathcal{D} \), we have that (3.5) holds. The proof of the theorem is completed. \( \square \)

**Remark 3.2.** In the particular case when \( w = (w_k)_{k \geq 0} \) is the constant sequence \( w_k = 1 \), Theorem 3.1 follows from [2, Theorem 3].

**Corollary 3.1.** Assume that a sequence \((A_n)_{n \in \mathbb{Z}} \subset \mathcal{B}(X)\) admits an exponential dichotomy. Then, there exists \( L > 0 \) with the following property: for each \( \delta > 0 \) and an \((\delta,B,w)\)-pseudotrajectory \((y_n)_{n \in \mathbb{Z}} \subset X\) for (3.1), there exists a solution \((x_n)_{n \in \mathbb{Z}} \subset X\) of (3.1) such that (3.5) holds.

**Proof.** The desired conclusion follows directly from Theorem 3.1 applied to the case when \( f_n \equiv 0, n \in \mathbb{Z} \). \( \square \)

**Acknowledgments**

The author is grateful to Lucas Backes for the productive collaboration on the topics related to the content of the present paper. He would also like to express his gratitude to referees for their useful comments.

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