SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.18.01.07
Vol.18 (31), No.1 (2022), 97 – 106

WEIGHTED HYERS-ULAM STABILITY FOR NONLINEAR NONAUTONOMOUS DIFFERENCE EQUATIONS

DAVOR DRAGIČEVIĆ

Dedicated to Professor Mustafa Kulenović on the occasion of his 70th birthday

ABSTRACT. Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear operators acting on an arbitrary Banach space X and admitting an exponential dichotomy. Furthermore, let $f_m\colon X\to X$, $m\in\mathbb{Z}$ be a sequence of Lipschitz maps. Provided that the Lipschitz constants of f_m are uniformly small, we show that a nonlinear difference equation

$$x_{m+1}=A_mx_m+f_m(x_m), \quad m\in\mathbb{Z},$$

exhibits various types of the *weighted* Hyers-Ulam stability property.

1. INTRODUCTION

Let $X=(X,\|\cdot\|)$ be an arbitrary Banach space, $(A_n)_{n\in\mathbb{Z}}$ a sequence of bounded linear operators on X and $(f_n)_{n\in\mathbb{Z}}$ a sequence of Lipschitz maps $f_n\colon X\to X$, $n\in\mathbb{Z}$. We consider the corresponding nonlinear nonautonomous difference equation given by

$$x_{n+1}=A_nx_n+f_n(x_n), \quad n\in\mathbb{Z}, \tag{1.1}$$

as well as the associated linear equation

$$x_{n+1}=A_nx_n, \quad n\in\mathbb{Z}. \tag{1.2}$$

We recall that (1.1) is said to be *Hyers-Ulam stable* if there exists $L>0$ with the property that for each $\delta>0$ and a sequence $(y_n)_{n\in\mathbb{Z}}\subset X$ such that

$$\sup_{n\in\mathbb{Z}}\|y_n-A_{n-1}y_{n-1}-f_{n-1}(y_{n-1})\|\le\delta, \tag{1.3}$$

there exists a solution $(x_n)_{n\in\mathbb{Z}}$ of (1.1) such that

$$\sup_{n\in\mathbb{Z}}\|x_n-y_n\|\le L\delta. \tag{1.4}$$

Hence, if (1.1) is Hyers-Ulam stable, in a vicinity of each approximate solution of (1.1) we can construct its exact solution.

We emphasize that it follows from [2, Theorem 3] that (1.1) is Hyers-Ulam stable provided that the following conditions hold:

2020 *Mathematics Subject Classification.* 34D20, 34D09.
Key words and phrases. shadowing; exponential dichotomy; nonautonomous dynamics.

- (1.2) admits an exponential dichotomy;
- the Lipschitz constants of f_n are uniformly small.

This result unified and extended several previously available results dealing with Hyers-Ulam stability of (1.2) in the case when $(A_n)_{n\in\mathbb{Z}}$ is either a constant or a periodic sequence (see [5–8, 22] and references therein). Moreover, the arguments and results developed in [2] inspired several other relevant contributions to the shadowing theory for nonautonomous difference and differential equations (see [1, 3, 4, 11–14]).

In addition, in [2] the authors dealt with a more general concept of the Hyers-Ulam stability, in which the size of an approximate solution as well as its deviation from an exact solution are not necessarily measured with respect to the l^∞-norm (as in (1.3) and (1.4)). Indeed, these quantities can be measured with respect to a wide collection of norms on sequence spaces.

In the present paper we study the *weighted* Hyers-Ulam stability of (1.1). This concept is motivated by the related notions of weighted shadowing property introduced in the context of smooth dynamical systems [10, 21]. The principal motivation for considering such notions are situations when the expression $\|y_n - A_{n-1}y_{n-1} - f_{n-1}(y_{n-1})\|$ in (1.3) cannot be well-controlled for *every* $n \in \mathbb{Z}$, or when it exhibits certain asymptotic behaviour when $|n| \to \infty$.

Our main result (see Theorem 3.1) implies that (1.1) has the weighted Hyers-Ulam stability property if the following conditions hold:

- (1.2) admits an exponential dichotomy;
- the Lipschitz constants of f_n are uniformly small;
- the sequence of weights exhibits the subexponential growth property (see Proposition 2.2).

We note that our arguments follow closely the approach developed in [2], by combining it with techniques from [10].

2. Preliminaries

2.1. **Banach sequence spaces**

Let $\mathcal{S}$ denote the set of all sequences $\mathbf{s} = (s_n)_{n\in\mathbb{Z}}$ of real numbers. We say that a linear subspace $B \subset \mathcal{S}$ is a *normed sequence space* if there exists a norm $\|\cdot\|_B \colon B \to \mathbb{R}_0^+$ on B such that if $\mathbf{s}' \in B$ and $|s_n| \le |s'_n|$ for $n \in I$, then $\mathbf{s} \in B$ and $\|\mathbf{s}\|_B \le \|\mathbf{s}'\|_B$. If in addition $(B, \|\cdot\|_B)$ is complete, we say that B is a *Banach sequence space*.

Let B be a Banach sequence space. We say that B is *admissible* if:

1. $\chi_{\{n\}} \in B$ and $\|\chi_{\{n\}}\|_B > 0$ for $n \in \mathbb{Z}$, where χ_A denotes the characteristic function of the set $A \subset \mathbb{Z}$;
2. for each $\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in B$ and $m \in \mathbb{Z}$, the sequence $\mathbf{s}^m = (s_n^m)_{n\in\mathbb{Z}}$ defined by $s_n^m = s_{n+m}$ belongs to B and $\|\mathbf{s}^m\|_B = \|\mathbf{s}\|_B$ for $\mathbf{s} \in B$ and $m \in \mathbb{Z}$.

Example 2.1. *The set* $l^\infty = \{\mathbf{s} \in \mathcal{S} : \sup_{n\in\mathbb{Z}}|s_n| < +\infty\}$ *is a Banach sequence space when equipped with the norm* $\|\mathbf{s}\| = \sup_{n\in\mathbb{Z}}|s_n|$.

Example 2.2. *For each* $p \in [1,\infty)$, *the set* $l^p = \{\mathbf{s} \in \mathcal{S} : \sum_{n\in\mathbb{Z}}|s_n|^p < +\infty\}$ *is a Banach sequence space when equipped with the norm* $\|\mathbf{s}\| = (\sum_{n\in\mathbb{Z}}|s_n|^p)^{1/p}$.

Example 2.3 (Orlicz sequence spaces)**.** *Let* $\phi\colon (0,+\infty) \to (0,+\infty]$ *be a nondecreasing nonconstant left-continuous function. We set* $\psi(t) = \int_0^t \phi(s)\,ds$ *for* $t \geq 0$. *Moreover, for each* $\mathbf{s} \in \mathcal{S}$, *let* $M_\phi(\mathbf{s}) = \sum_{n\in\mathbb{Z}} \psi(|s_n|)$. *Then*

$$B = \big\{\mathbf{s} \in \mathcal{S} : M_\phi(c\mathbf{s}) < +\infty \text{ for some } c > 0\big\}$$

is a Banach sequence space when equipped with the norm

$$\|\mathbf{s}\| = \inf\big\{c > 0 : M_\phi(\mathbf{s}/c) \leq 1\big\}.$$

Some important properties of admissible Banach sequence spaces are given in the following result (see [10, Proposition 1]).

Proposition 2.1. *Let B be an admissible Banach sequence space.*

1. *If* $\mathbf{s}^1 = (s_n^1)_{n\in\mathbb{Z}}$ *and* $\mathbf{s}^2 = (s_n^2)_{n\in\mathbb{Z}}$ *are sequences in* $\mathcal{S}$ *and* $s_n^1 = s_n^2$ *for all but finitely many* $n \in \mathbb{Z}$, *then* $\mathbf{s}^1 \in B$ *if and only if* $\mathbf{s}^2 \in B$.
2. *If* $\mathbf{s}^n \to \mathbf{s}$ *in B when* $n \to \infty$, *then* $s_m^n \to s_m$ *when* $n \to \infty$, *for* $m \in \mathbb{Z}$.
3. *For each* $\mathbf{s} \in B$ *and* $\lambda \in (0,1)$, *the sequences* $\mathbf{s}^1$ *and* $\mathbf{s}^2$ *defined by*

$$s_n^1 = \sum_{m\geq 0} e^{-\lambda m} s_{n-m} \quad \text{and} \quad s_n^2 = \sum_{m\geq 1} e^{-\lambda m} s_{n+m}$$

are in B, and

$$\|\mathbf{s}^1\|_B \leq \frac{1}{1-e^{-\lambda}}\|\mathbf{s}\|_B \quad \text{and} \quad \|\mathbf{s}^2\|_B \leq \frac{e^{-\lambda}}{1-e^{-\lambda}}\|\mathbf{s}\|_B. \tag{2.1}$$

Remark 2.1. For more information on admissible Banach sequence spaces (and their continuous time counterparts) and their role in the qualitative theory of nonautonomous systems, we refer to [16–20] and references therein.

2.2. **Weights**

We now introduce a class of *weights* introduced in [10].

More precisely, throughout this paper $\mathbf{w} = (w_k)_{k\geq 0}$ will be a sequence of real numbers such that there exists $t > 0$ so that

$$w_k \geq t \quad \text{for every } k \geq 0, \tag{2.2}$$

and with the property that for every $\lambda > 0$, there exist $\lambda', L > 0$ such that:

$$e^{-\lambda(m-n)}\frac{w_n}{w_m} \leq Le^{-\lambda'(m-n)} \quad \text{for every } m \geq n \geq 0, \tag{2.3}$$

and

$$e^{-\lambda(n-m)}\frac{w_n}{w_m} \leq Le^{-\lambda'(n-m)} \quad \text{for every } n \geq m \geq 0. \tag{2.4}$$

It turns out that (2.3) and (2.4) can be stated in a more transparent manner (see [10, Proposition 2]).

Proposition 2.2. *The following statements are equivalent:*
1. *for every* $\lambda > 0$*, there exist* $\lambda', L > 0$ *such that* (2.3) *and* (2.4) *hold;*
2. *for every* $\varepsilon > 0$ *there exists* $C > 0$ *such that*

$$\frac{w_n}{w_m} \le Ce^{\varepsilon|n-m|}, \quad m,n \ge 0. \tag{2.5}$$

The following result established in [10, Proposition 3] gives a large class of examples of sequences $\mathbf{w} = (w_k)_{k\ge 0}$ satisfying the above properties.

Proposition 2.3. *Assume that p is a polynomial with a positive leading coefficient such that* $p(k) > 0$ *for* $k \ge 0$*. Given* $w \ge 0$*, we define*

$$w_k = p(k)^w, \quad k \ge 0.$$

Then, the sequence $\mathbf{w} = (w_k)_{k\ge 0}$ *satisfies properties* (2.2)*,* (2.3) *and* (2.4).

2.3. **Sequence spaces induced by weights**

We now introduce a class of sequence spaces that will play a central role in our paper.

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space, B an admissible Banach sequence space and $\mathbf{w} = (w_k)_{k\ge 0}$ a sequence of weights satisfying (2.2), (2.3) and (2.4). Set

$$Y_{X,B,\mathbf{w}} = \left\{ (x_k)_{k\in\mathbb{Z}} \subset X : (w_{|k|}\|x_k\|)_{k\in\mathbb{Z}} \in B \right\}.$$

The following result can be established by arguing as in the proof of [10, Proposition 4].

Proposition 2.4. $Y_{X,B,\mathbf{w}}$ *is a Banach space with respect to the norm*

$$\|\mathbf{x}\|_{X,B,\mathbf{w}} = \|(w_{|k|}\|x_k\|)_{k\in\mathbb{Z}}\|_B.$$

2.4. **Exponential dichotomy**

Finally, we recall the notion of an *exponential dichotomy* (see [9, 15]). We continue to denote by $X = (X, \|\cdot\|)$ an arbitrary Banach space. By $\mathcal{B}(X)$ we will denote the space of all bounded linear operators on X, equipped with the operator norm that we will also denote by $\|\cdot\|$. Moreover, Id will denote the identity operator on X.

We say that a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$ admits an *exponential dichotomy* if there exist a sequence of projections P_n, $n \in \mathbb{Z}$ on X and constants $C, \lambda > 0$ such that:

- for $n \in \mathbb{Z}$,

$$P_{n+1}A_n = A_nP_n,$$

and $A_n|_{\operatorname{Ker} P_n} \colon \operatorname{Ker} P_n \to \operatorname{Ker} P_{n+1}$ is invertible;

- for $m \geq n$,
$$\|\mathcal{A}(m,n)P_n\| \leq Ce^{-\lambda(m-n)}, \tag{2.6}$$
where
$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & m > n; \\ \mathrm{Id} & m = n; \end{cases}$$
- for $m \leq n$,
$$\|\mathcal{A}(m,n)(\mathrm{Id}-P_n)\| \leq Ce^{-\lambda(n-m)}, \tag{2.7}$$
where
$$\mathcal{A}(m,n) := \left(\mathcal{A}(n,m)|_{KerP_m}\right)^{-1} : \ \mathrm{Ker}\,P_n \to \mathrm{Ker}\,P_m.$$

Example 2.4. *Let $A \in \mathcal{B}(X)$ be a hyperbolic linear operator, i.e. the spectrum of A is disjoint from the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Let $c > 0$ and assume that $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$ is a sequence such that*
$$\sup_{n\in\mathbb{Z}} \|A - A_n\| \leq c.$$
Provided that c is sufficiently small, it follows from [15, Theorem 7.6.7.] that the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy.

3. Main result

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space, B an admissible Banach sequence space and $\mathbf{w} = (w_k)_{k\geq 0}$ a sequence of weights satisfying properties (2.2), (2.3) and (2.4).

Moreover, let $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$ be an arbitrary sequence. We consider the associated linear difference equation given by
$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{3.1}$$
Given a sequence $(f_n)_{n\in\mathbb{Z}}$ of maps $f_n \colon X \to X$, we can consider the nonlinear difference equation given by
$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z}. \tag{3.2}$$
We now introduce the notion of a $(\delta, B, \mathbf{w})$-pseudotrajectory for (3.2).

Definition 3.1. *Let $\delta > 0$. We say that a sequence $(y_n)_{n\in\mathbb{Z}} \subset X$ is an $(\delta, B, \mathbf{w})$-*pseudotrajectory *for* (3.2) *if the sequence $(y_{n+1} - A_n y_n - f_n(y_n))_{n\in\mathbb{Z}}$ belongs to $Y_{X,B,\mathbf{w}}$ and*
$$\|(y_n - A_{n-1}y_{n-1} - f_{n-1}(y_{n-1}))_{n\in\mathbb{Z}}\|_{X,B,\mathbf{w}} \leq \delta. \tag{3.3}$$

Remark 3.1. In the particular case when $\mathbf{w} = (w_k)_{k\geq 0}$ is a constant sequence $w_k = 1$, the notion of an $(\delta, B, \mathbf{w})$-pseudotrajectory reduces to the notion of an (δ, B)-pseudotrajectory, which was introduced and studied in [2].

In the sequel, we will use the following simple result established in [10, Lemma 4.5].

Lemma 3.1. *For every $\lambda > 0$ there exists $\lambda', L > 0$ such that for every $n, m \in \mathbb{Z}$:*

1. $$e^{-\lambda(m-n)} \frac{w_{|n|}}{w_{|m|}} \le Le^{-\lambda'(m-n)}, \quad \textit{for } m \ge n;$$
2. $$e^{-\lambda(n-m)} \frac{w_{|n|}}{w_{|m|}} \le Le^{-\lambda'(n-m)}, \quad \textit{for } m \le n.$$

We are now in the position to formulate and establish the main result of our paper.

Theorem 3.1. *Assume that a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$ admits an exponential dichotomy. Furthemore, suppose that there exists $c > 0$ such that*

$$\|f_n(x) - f_n(y)\| \le c\|x - y\|, \quad \textit{for } x, y \in X \textit{ and } n \in \mathbb{Z}. \tag{3.4}$$

Then, provided that c is sufficiently small, there exists $L > 0$ with the following property: for each $\delta > 0$ and an $(\delta, B, \mathbf{w})$-pseudotrajectory $(y_n)_{n\in\mathbb{Z}} \subset X$ for (3.2), there exists a solution $(x_n)_{n\in\mathbb{Z}} \subset X$ of (3.2) such that

$$\|(x_n - y_n)_{n\in\mathbb{Z}}\|_{X,B,\mathbf{w}} \le L\delta. \tag{3.5}$$

Proof. Take $\delta > 0$ and a $(\delta, B, \mathbf{w})$-pseudotrajectory $(y_n)_{n\in\mathbb{Z}} \subset X$ for (3.2). Set

$$\mathcal{G}(m,n) = \begin{cases} \mathcal{A}(m,n)P_n & m \ge n; \\ -\mathcal{A}(m,n)(\mathrm{Id} - P_n) & m < n. \end{cases}$$

For a sequence $\mathbf{z} = (z_n)_{n\in\mathbb{Z}} \in Y_{X,B,\mathbf{w}}$, set

$$(\mathcal{T}\mathbf{z})_n = \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k), \quad n \in \mathbb{Z}.$$

By (2.6) and (2.7), we have that

$$\begin{aligned} \|(\mathcal{T}\mathbf{0})_n\| &\le \sum_{k=-\infty}^{n} \|\mathcal{A}(n,k)P_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k)\| \\ &\quad + \sum_{k=n+1}^{\infty} \|\mathcal{A}(n,k)P_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k)\| \\ &\le C\sum_{k=-\infty}^{n} e^{-\lambda(n-k)}\|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\| \\ &\quad + C\sum_{k=n+1}^{\infty} e^{-\lambda(k-n)}\|A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k\|, \end{aligned}$$

and thus

$$w_{|n|}\|(\mathcal{T}\mathbf{0})_n\| \le C\sum_{k=-\infty}^{n} e^{-\lambda(n-k)}\frac{w_{|n|}}{w_{|k|}}w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\| + C\sum_{k=n+1}^{\infty} e^{-\lambda(k-n)}\frac{w_{|n|}}{w_{|k|}}w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\|.$$

By invoking Lemma 3.1, we conclude that

$$\begin{aligned} w_{|n|}\|(\mathcal{T}\mathbf{0})_n\| &\le C'\sum_{k=-\infty}^{n} e^{-\lambda'(n-k)}w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\| \\ &\quad + C'\sum_{k=n+1}^{\infty} e^{-\lambda'(k-n)}w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\| \\ &= C'\sum_{j=0}^{\infty} e^{-\lambda' j}w_{|n-j|}\|A_{n-j-1}y_{n-j-1}+f_{n-j-1}(y_{n-j-1})-y_{n-j}\| \\ &\quad + C'\sum_{j=1}^{\infty} e^{-\lambda' j}w_{|n+j|}\|A_{n+j-1}y_{n+j-1}+f_{n+j-1}(y_{n+j-1})-y_{n+j}\|, \end{aligned} \tag{3.6}$$

for some constants $C',\lambda'>0$ that depend only on C,λ. We define sequences $(s_n)_{n\in\mathbb{Z}}$ and $(s_n^i)_{n\in\mathbb{Z}}$, $i=1,2$ of nonnegative numbers by

$$s_n = w_{|n|}\|A_{n-1}y_{n-1}+f_{n-1}(y_{n-1})-y_n\|,$$

$$s_n^1=\sum_{j=0}^{\infty} e^{-\lambda' j}s_{n-j} \quad \text{and} \quad s_n^2=\sum_{j=1}^{\infty} e^{-\lambda' j}s_{n+j}.$$

Then, (3.6) implies that

$$w_{|n|}\|(\mathcal{T}\mathbf{0})_n\| \le C'(s_n^1+s_n^2), \quad n\in\mathbb{Z}. \tag{3.7}$$

On the other hand, (3.3) together with Proposition 2.1 gives that

$$\|(s_n^1)_{n\in\mathbb{Z}}\|_B \le \frac{1}{1-e^{-\lambda'}}\delta \quad \text{and} \quad \|(s_n^2)_{n\in\mathbb{Z}}\|_B \le \frac{e^{-\lambda'}}{1-e^{-\lambda'}}\delta. \tag{3.8}$$

By combining (3.7) and (3.8), we have that

$$\|\mathcal{T}\mathbf{0}\|_{X,B,\mathbf{w}} \le M\delta, \tag{3.9}$$

where

$$M := C'\frac{1+e^{-\lambda'}}{1-e^{-\lambda'}} > 0.$$

Moreover, for $\mathbf{z}=(z_n)_{n\in\mathbb{Z}}\in Y_{X,B,\mathbf{w}}$ we have (using (3.4)) that

$$\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k\| \le \|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\|+c\|z_{k-1}\|,$$

for $k\in\mathbb{Z}$. Hence,

$$\begin{aligned}&w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k\|\\&\le w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\|+cw_{|k|}\|z_{k-1}\|\\&\le w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k\|+Dw_{|k-1|}\|z_{k-1}\|,\end{aligned}\tag{3.10}$$

for $k\in\mathbb{Z}$, where (see (2.5))

$$D:=\sup_{k\in\mathbb{Z}}\frac{w_{|k|}}{w_{|k-1|}}<+\infty.$$

In addition, using (2.6), (2.7) and Lemma 3.1, we have that

$$\begin{aligned}w_{|n|}\|(\mathcal{T}\mathbf{z})_n\|\le C'&\sum_{k=-\infty}^{n}e^{-\lambda'(n-k)}w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k\|\\&+C'\sum_{k=n+1}^{\infty}e^{-\lambda'(k-n)}w_{|k|}\|A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k\|,\end{aligned}\tag{3.11}$$

for $n\in\mathbb{Z}$. By combining (3.10) together with (3.11), and using the same arguments that enabled us to establish (3.9), we find that

$$\|\mathcal{T}\mathbf{z}\|_{X,B,\mathbf{w}}\le C'\frac{1+e^{-\lambda'}}{1-e^{-\lambda'}}\delta+C'D\frac{1+e^{-\lambda'}}{1-e^{-\lambda'}}\|\mathbf{z}\|_{X,B,\mathbf{w}}<+\infty.$$

In particular, $\mathcal{T}\colon Y_{X,B,\mathbf{w}}\to Y_{X,B,\mathbf{w}}$ is a well-defined map.

Now take $\mathbf{z}^i=(z_n^i)_{n\in\mathbb{Z}}\in Y_{X,B,\mathbf{w}}$, $i=1,2$. We have that

$$(\mathcal{T}\mathbf{z}^1)_n-(\mathcal{T}\mathbf{z}^2)_n=\sum_{k\in\mathbb{Z}}\mathcal{G}(n,k)(f_{k-1}(y_{k-1}+z_{k-1}^1)-f_{k-1}(y_{k-1}+z_{k-1}^2)),\ n\in\mathbb{Z}.$$

Furthermore, (3.4) implies that

$$\|f_{k-1}(y_{k-1}+z_{k-1}^1)-f_{k-1}(y_{k-1}+z_{k-1}^2)\|\le c\|z_{k-1}^1-z_{k-1}^2\|,\quad k\in\mathbb{Z}.$$

By proceeding as above, we conclude that

$$\|\mathcal{T}\mathbf{z}^1-\mathcal{T}\mathbf{z}^2\|_{X,B,\mathbf{w}}\le q\|\mathbf{z}^1-\mathbf{z}^2\|_{X,B,\mathbf{w}},\tag{3.12}$$

where

$$q:=cDC'\frac{1+e^{-\lambda'}}{1-e^{-\lambda'}}.$$

Provided that c is sufficiently small, we have that $q<1$ and thus $\mathcal{T}$ is a contraction on $Y_{X,B,\mathbf{w}}$. Let

$$L:=\frac{M}{1-q}>0,$$

and set

$$\mathcal{D}:=\{\mathbf{z}\in Y_{X,B,\mathbf{w}}:\|\mathbf{z}\|_{X,B,\mathbf{w}}\le L\delta\}.$$

Take $\mathbf{z}\in\mathcal{D}$. By (3.9) and (3.12), we have that

$$\|\mathcal{T}\mathbf{z}\|_{X,B,\mathbf{w}}\le\|\mathcal{T}\mathbf{z}-\mathcal{T}\mathbf{0}\|_{X,B,\mathbf{w}}+\|\mathcal{T}\mathbf{0}\|_{X,B,\mathbf{w}}\le qL\delta+M\delta=L\delta.$$

We conclude that $\mathcal{T}(\mathcal{D})\subset\mathcal{D}$, and thus $\mathcal{T}$ has a unique fixed point $\mathbf{z}=(z_n)_{n\in\mathbb{Z}}\in\mathcal{D}$. Hence,

$$\begin{aligned}
&z_{n+1}-A_nz_n \\
&=\sum_{k\le n+1}\mathcal{A}(n+1,k)P_k(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k) \\
&\quad-\sum_{k\ge n+2}\mathcal{A}(n+1,k)(\mathrm{Id}-P_k)(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k) \\
&\quad-\sum_{k\le n}\mathcal{A}(n+1,k)P_k(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k) \\
&\quad+\sum_{k\ge n+1}\mathcal{A}(n+1,k)(\mathrm{Id}-P_k)(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1})-y_k) \\
&=P_{n+1}(A_ny_n+f_n(y_n+z_n)-y_{n+1}) \\
&\quad+(\mathrm{Id}-P_{n+1})(A_ny_n+f_n(y_n+z_n)-y_{n+1}) \\
&=A_ny_n+f_n(y_n+z_n)-y_{n+1},
\end{aligned}$$

for $n\in\mathbb{Z}$. Consequently, setting $x_n=y_n+z_n$, $n\in\mathbb{Z}$, we conclude that $(x_n)_{n\in\mathbb{Z}}$ is a solution of (3.2). Moreover, since $\mathbf{z}\in\mathcal{D}$, we have that (3.5) holds. The proof of the theorem is completed. □

Remark 3.2. In the particular case when $\mathbf{w}=(w_k)_{k\ge 0}$ is the constant sequence $w_k=1$, Theorem 3.1 follows from [2, Theorem 3].

Corollary 3.1. *Assume that a sequence $(A_n)_{n\in\mathbb{Z}}\subset\mathcal{B}(X)$ admits an exponential dichotomy. Then, there exists $L>0$ with the following property: for each $\delta>0$ and an $(\delta,B,\mathbf{w})$-pseudotrajectory $(y_n)_{n\in\mathbb{Z}}\subset X$ for* (3.1)*, there exists a solution $(x_n)_{n\in\mathbb{Z}}\subset X$ of* (3.1) *such that* (3.5) *holds.*

Proof. The desired conclusion follows directly from Theorem 3.1 applied to the case when $f_n\equiv 0$, $n\in\mathbb{Z}$. □

Acknowledgments

The author is grateful to Lucas Backes for the productive collaboration on the topics related to the content of the present paper. He would also like to express his gratitude to referees for their useful comments.

References

[1] L. Backes and D. Dragičević, *Quasi-shadowing for partially hyperbolic dynamics on Banach spaces*, J. Math. Anal. Appl. **492** (2020),124445.

[2] L. Backes and D. Dragičević, *Shadowing for infinite dimensional dynamics and exponential trichotomies*, Proc. Roy. Soc. Edinburgh Sect. A **151** (2021), 863–884.

[3] L. Backes and D. Dragičević, *A general approach to nonautonomous shadowing for nonlinear dynamics*, Bull. Sci. Math. **170** (2021), 102996.

[4] L. Backes, D. Dragičević and L. Singh, *Shadowing for nonautonomous and nonlinear dynamics with impulses*, Monatsh. Math (2021). https://doi.org/10.1007/s00605-021-01629-2

[5] D. Barbu, C. Buse and A. Tabassum, *Hyers–Ulam stability and discrete dichotomy*, J. Math. Anal. Appl. **423** (2015), 1738–1752.

[6] N. Bernardes Jr., P.R. Cirilo, U. B. Darji, A. Messaoudi and E. R. Pujals, *Expansivity and shadowing in linear dynamics*, J. Math. Anal. Appl. **461** (2018), 796–816.
[7] C. Buse, D. O'Regan, O. Saierli and A. Tabassum, *Hyers–Ulam stability and discrete dichotomy for difference periodic systems*, Bull. Sci. Math. **140** (2016), 908–934.
[8] C. Buse, V. Lupulescu and D. O' Regan, *Hyers–Ulam stability for equations with differences and differential equations with time-dependent and periodic coefficients*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), 2175–2188.
[9] W. A. Coppel, *Dichotomies in stability theory*, Springer Verlag, Berlin, Heidelberg, New-York, 1978.
[10] D. Dragičević, *Admissibility, a general type of Lipschitz shadowing and structural stability*, Commun. Pure Appl. Anal. **14** (2015), 861–880.
[11] D. Dragičević, *Hyers–Ulam stability for a class of perturbed Hill's equations* Results Math **76**, 129 (2021). https://doi.org/10.1007/s00025-021-01442-1
[12] D. Dragičević, *On the Hyers–Ulam stability of certain nonautonomous and nonlinear difference equations* Aequat. Math. **95**, 829–840 (2021). https://doi.org/10.1007/s00010-020-00774-7
[13] D. Dragičević, *Hyers–Ulam stability for nonautonomous semilinear dynamics on bounded intervals*, Mediterr. J. Math. **18**, 71 (2021). https://doi.org/10.1007/s00009-021-01729-1
[14] D. Dragičević and M. Pituk, *Shadowing for nonautonomous difference equations with infinite delay*, Appl. Math. Lett. **120** (2021), 107284.
[15] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, **840**, Springer-Verlag, Berlin-New York, 1981.
[16] N. T. Huy, *Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line*, J. Funct. Anal. **235** (2006), 330–354.
[17] N. T. Huy, *Invariant manifolds of admissible classes for semi-linear evolution equations*, J. Differential Equations **246** (2009), 1820–1844.
[18] A. L. Sasu and B. Sasu, *Integral equations, dichotomy of evolution families on the half-line and applications*, Integral Equations Operator Theory **66** (2010), 113–140.
[19] A. L. Sasu and B. Sasu, *On the dichotomic behavior of discrete dynamical systems on the half-line*, Discrete Contin. Dyn. Syst. **33** (2013), 3057–3084.
[20] B. Sasu, *Input-output control systems and dichotomy of variational difference equations*, J. Difference Equ. Appl. **17** (2011) 889–913.
[21] D. Todorov, *Generalizations of analogs of theorems of Maizel and Pliss and their application in shadowing theory*, Discrete Contin. Dyn. Syst. **33** (2013), 4187 – 4205.
[22] A. Zada and B. Zada, *Hyers-Ulam stability and exponential dichotomy of discrete semigroup*, Appl. Math. E-Notes **19** (2019), 527–534.

(Received: January 23, 2022)
(Revised: March 12, 2022)

Davor Dragičević
University of Rijeka
Faculty of Mathematics
Radmile Matejčić 2, 51000 Rijeka, Croatia
e-mail: *ddragicevic@math.uniri.hr*