SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.18.01.09
Vol.18 (31), No.1 (2022), 127 – 160

DYNAMICS OF A TWO-DIMENSIONAL COOPERATIVE SYSTEM OF POLYNOMIAL DIFFERENCE EQUATIONS WITH CUBIC TERMS

JASMIN BEKTEŠEVIĆ, VAHIDIN HADŽIABDIĆ, MIDHAT MEHULJIĆ, AND NAIDA MUJIĆ

Dedicated to the 70th birthday of Professor Mustafa R. S. Kulenović

ABSTRACT. In this paper we present a local dynamics and investigate the global behavior of the following system of difference equations

$$\begin{cases} x_{n+1} = ax_n^3 + by_n^3 \\ y_{n+1} = Ax_n^3 + By_n^3 \end{cases}, \quad n \in \mathbb{N}_0$$

with non-negative parameters and initial conditions x_0 and y_0 that are real numbers. We establish the relations for local stability of equilibriums and necessary and sufficient conditions for the existence of period-two solution(s). We then use this result to give global behavior results for special ranges of parameters and determine the basins of attraction of all equilibrium points.

1. INTRODUCTION

In this paper we study the local and global stability character, the periodic nature and the boundedness of solutions of the system of polynomial difference equations with cubic terms

$$\begin{cases} x_{n+1} = ax_n^3 + by_n^3 \\ y_{n+1} = Ax_n^3 + By_n^3 \end{cases}, \quad n \in \mathbb{N}_0 \tag{1.1}$$

where the parameters a, b, A, B are nonnegative numbers and initial conditions x_0 and y_0 are real numbers. In [2], the general second order difference equation is completely investigated and described the regions of initial conditions in the first quadrant for which all solutions tend to equilibrium points, period-two solutions, or the point at infinity, except for the case of infinitely many period-two solutions, are described. In [1], the case of infinitely many period-two solutions is completely investigated and the q corresponding difference equation is $x_{n+1} = ax_nx_{n-1} + ax_{n-1}^2 + bx_{n-1}$. In [3] we have extended our research to the general cubic polynomial difference equation where we give a class of examples of second order difference equations for which the Julia set can be found explicitly and is represented by a planar curve. Otherwise, the Julia set is the union of the

2010 *Mathematics Subject Classification.* 39A10, 39A23.
Key words and phrases. Difference equation, Equilibrium, Period-two solutions, Basin of Attraction, Global dynamics.

stable manifolds of some saddle equilibrium points or nonhyperbolic equilibrium points and/or period-two points. Asymptotic formulas for these manifoldsin both quadratic and cubic cases, were obtained in [4] and [5]. Furthermore , in [6] the behavior of all solutions of the difference equation of type

$$x_{n+1} = x_n^3 + x_{n-1}^3,$$

is described, where results are extended to hold in the whole real plane. All these results lead us to cosider the system (1.1). Our principal tool is the theory of monotone maps, and in particular cooperative maps, which guarantee the existence and uniqueness of the stable and unstable manifolds for the fixed points and periodic points. More precisely, we will use the results proved in [6] and [10] to describe the behavior of all solutions of the system (1.1).

Let $f_1(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ and $g_1(x) = b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0$ be two polynomials of degrees n and m, respectively. Their resultant (see [8, 9, 15]) $\operatorname{Res}(f_1, g_1)$ is the determinant of the $(m+n)\times(m+n)$ Sylvester matrix given by

$$\operatorname{Syl}(f_1, g_1) = \begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_m & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \end{pmatrix}$$

or

$$\operatorname{Res}(f_1, g_1) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m} (\alpha_i - \beta_j)$$

where $\alpha_i,\ i = 1,2,\ldots,n$ and $\beta_j,\ j = 1,2,\ldots,m$ are the zeros of the polynomials $f_1(x)$ and $g_1(x)$ respectively. In addition, for a polynomial $f_1(x)$ the most common definition of the discriminant is

$$\operatorname{Dis}(f_1) = a_n^{2n-2} \prod_{\substack{i,j \\ i<j}}^{n} (\alpha_i - \alpha_j)^2,$$

that is, if $a_n = 1$ then $\operatorname{Dis}(f_1)$ is the product of the squares of the differences of the polynomial roots α_i. By using the resultant, discriminant and Theorems 17 and 18 in [3] the global dynamics of polynomial cubic second order difference equation in the parametric regions where two distinct equilibrium points and a finite number of period-two solutions exist is described. If $g_1(x) = f_1'(x) = 0\cdot x^n + n\, a_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + ... + a_1$, then the $2n\times 2n$ Sylvester matrix $\operatorname{Syl}(f_1, f_1')$ is called the **discrimination** matrix.

Let D_k denote the determinant of the submatrix of $\mathrm{Syl}(f_1, f_1')$ formed by the first $2k$ rows and the first $2k$ columns for $k = 1, 2, \dots, n$. The $n-$tuple $\{D_1(f_1), D_2(f_1), \dots, D_n(f_1)\}$ is the **discriminant sequence** of polynomial $f_1(x)$. The list $\{\mathrm{sign}(D_1(f_1)), \mathrm{sign}(D_2(f_1)), \dots, \mathrm{sign}(D_n(f_1))\}$ is the **sign list** of the discriminant sequence $\{D_1(f_1), D_2(f_1), \dots, D_n(f_1)\}$.

A two-dimensional system of difference equations is of the form

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n), \ n = 0, 1, \dots$$

where $f, g : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^2$. A map $T(x,y) = (f(x,y), g(x,y))$ is called coopratіve if f and g are continuous functions defined on some subset of $\mathbb{R}^2$ with non-empty interior such that f and g are non-decreasing in all of its arguments. The well-known deMottoni-Schiaffino theorem (see [12,14]) claims that in this case for each $(x,y) \in \mathcal{D}$, the sequence $\{T^n(x,y)\}$ (resp. $\{T^{2n}(x,y)\}$) is eventually coordinate-wise monotonic. Consequently, every bounded sequence $\{T^n(x,y)\}$ (resp. $\{T^{2n}(x,y)\}$) converges to a fixed point of T or to a point on the boundary of $\mathcal{D}$.

The next two results can be proved by using the techniques of proof of Theorems 3, 5, 6, 7, 8, 9 and 10 in [6] applied to cooperative maps.

Theorem 1.1. *Let T be cooperative map on a rectangular region $\mathcal{R} \subseteq \mathbb{R}^2$ and assume that there is no minimal period-two solution of map T. Assume that $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ are two consecutive equilibrium points in North-East ordering that satisfy $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ and that E_1 is a local attractor and E_2 is a saddle point or a non-hyperbolic point with second characteristic root in interval $(-1, 1)$. Then the basin of attraction $\mathcal{B}(E_1)$ of E_1 is the region below the global stable manifold $\mathcal{W}^s(E_2)$ the graph of a strictly decreasing continuous function of the first coordinate on an interval. More precisely $\mathcal{B}(E_1) = \{(x,y) : \exists y_u : y < y_u, (x, y_u) \in \mathcal{W}^s(E_2)\}$.*

The basin of attraction $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ is exactly the global stable manifold of E_2. Any endpoints of the global stable manifold $\mathcal{W}^s(E_2)$ are exactly either fixed points or minimal period-two points. The curve $\mathcal{W}^u(E_2)$ is an unstable set, passing through the point E_2, and it is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of the global unstable manifold $\mathcal{W}^u(E_2)$ are fixed points of T.

Theorem 1.2. *Let T be a cooperative map on a rectangular region $\mathcal{R} \subseteq \mathbb{R}^2$ and assume that there is no minimal period-two solution of map T. Assume that $E_1(x_1, y_1)$, $E_2(x_2, y_2)$ and $E_3(x_2, y_2)$ are three consecutive equilibrium points in North-East ordering that satisfy $(x_1, y_1) \preceq_{ne} (x_2, y_2) \preceq_{ne} (x_3, y_3)$ and that E_2 is a local attractor and E_1, E_3 are a saddle points. Then the basin of attraction $\mathcal{B}(E_2)$ of E_2 is the region between the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$, where $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ are the graphs of a strictly decreasing continuous functions of the* *first coordinate on an interval. More precisely* $\mathcal{B}(E_1) = \{(x,y) : \exists y_u, y_l : y_l < y < y_u,\ (x,y_l) \in \mathcal{W}^s(E_1), (x,y_u) \in \mathcal{W}^s(E_3)\}$. *The basins of attraction* $\mathcal{B}(E_1) = \mathcal{W}^s(E_1)$ *and* $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ *are exactly the global stable manifolds of* E_1 *and* E_3.

The requirement that E_1 and E_3 are saddle points can be replaced by the requirement that at least one of them is non-hyperbolic point with corresponding conditions. Also, see [6], Theorem 1.2 can be extended to the case when map T has a finite number of equilibrium points.

The next theorem follows from Theorem 1.1.1 in [13]

Theorem 1.3. *Let* T *be the function defined by* $T(x,y) = (f(x,y), g(x,y))$ *where* $f,g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. *Let* $J_T(E)$ *be the Jacobian matrix evaluated at an eqilibrium point* E *of the function* T. *Set* $\mathcal{S} = tr(J_T(E))$ *and* $\mathcal{D} = \det(J_T(E))$ *the trace and determinant of* $J_T(E)$ *respectively. The following statements hold:*

(i) *if* $|\mathcal{S}| < 1+\mathcal{D}$ *and* $\mathcal{D} < 1$ *then* E *is locally asymptotically stable (sink) (or* $(\mathcal{S}-\mathcal{D}-1)(\mathcal{S}+\mathcal{D}+1) < 0$*),*
(ii) *if* $|\mathcal{S}| > |1+\mathcal{D}|$ *then* E *is a saddle point (or* $(\mathcal{S}-\mathcal{D}-1)(\mathcal{S}+\mathcal{D}+1) > 0$*),*
(iii) *if* $|\mathcal{S}| < |1+\mathcal{D}|$ *and* $|\mathcal{D}| > 1$ *then* E *is repeller,*
(iv) *if* $|\mathcal{S}| = |1+\mathcal{D}|$ *(or* $(\mathcal{S}-\mathcal{D}-1)(\mathcal{S}+\mathcal{D}+1) = 0$*) or* $\mathcal{D} = 1$ *and* $|\mathcal{S}| \leq 2$ *then* E *is a nonhyperbolic point.*

The following theorem is from [16]. Let D_k be determinant of the submatrix of $\mathrm{Discr}(p)$ dicrimination matrix of polynomial $p(x)$ formed by the first $2k$ rows and the first $2k$ columns $k = 1,2,\dots,n$.

Theorem 1.4. *Let* $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ *be a polynomial with real coefficients. If the number of sign changes of the revised sign list of*

$$\{D_1(p), D_2(p), \dots, D_n(p)\}$$

is v*, the the number of pairs of distinct conjugate imaginary roots of* $p(x)$ *equals* v*. Furthermore, if the number of non-vanishing (non-zero) members of the revised sign list is* l*, then the number of the distinct real roots of* $p(x)$ *is* $l-2v$*.*

In addition to Theorem 1.4 let

$$\{s_1, s_2, \dots, s_n\} = \{\mathrm{sign}(D_1(p)), \mathrm{sign}(D_2(p)), \dots, \mathrm{sign}(D_n(p))\}$$

be the sign list of the discriminant sequence $\{D_1(p), D_2(p), \dots, D_n(p)\}$ of the polynomial $p(x)$. If $\left\{s_i, s_{i+1}, s_{i+2}, \dots, s_{i+j-1}, s_{i+j}\right\}$ is a part of the given sign list such that $s_i \neq 0$, $s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0$ and $s_{i+j} \neq 0$ then we construct the revised sign list where the term s_{i+r} will be replaced with $(-1)^{\left\lfloor \frac{r+1}{2} \right\rfloor} s_i$, $r = 1,2,\dots,j-1$. So, the section $\{s_i, 0, 0, \dots, 0, s_{i+j}\}$ will be replaced by $\{s_i, -, s_i, -s_i, s_i, s_i, -s_i, -s_i, \dots, s_{i+j}\}$.

The following two well known theorems are very useful in determining the number of positive zeros of polynomial.

Theorem 1.5. *Let $P(x) = a_0x^{b_0} + a_1x^{b_1} + \ldots + a_nx^{b_n}$ where a_i, $i = \overline{0,n}$ are nonzero real numbers and $0 \le b_0 < b_1 < \ldots < b_n$ are integers. Then $P(x) = 0$ has an even number of positive zeros, counting multiplicities, if and only if $a_0a_n > 0$.*

Theorem 1.6. *Let $P(x) = a_0x^{b_0} + a_1x^{b_1} + \ldots + a_nx^{b_n}$ where a_i, $i = \overline{0,n}$ are nonzero real numbers and $0 \le b_0 < b_1 < \ldots < b_n$ are integers.The number of positive zeros of $P(x) = 0$, counting multiplicities, is either equal to $v(P)$ or less than that by an even number, where $v(P)$ denotes the number of sign changes in the sequence $a_0, a_1, \ldots, a_n$.*

In this paper, figures 7-12 are obtained by using Mathematica 9.0, with the boundaries of the basins of attraction obtained by using the software package Dynamica (see [11]).

2. Equilibrium points

The map T associated to system (1.1) is given by

$$T(x,y) = (f(x,y), g(x,y)) = \left(ax^3 + by^3, Ax^3 + By^3\right). \tag{2.1}$$

The equilibrium points of the system (1.1) are the solutions of the system

$$\begin{aligned} a\overline{x}^3 + b\overline{y}^3 &= \overline{x} \\ A\overline{x}^3 + B\overline{y}^3 &= \overline{y}. \end{aligned} \tag{2.2}$$

If (u,v) is solution of the system (2.2), where $u \ge 0$ and $v \ge 0$, then $(-u,-v)$ is solution of the system (2.2). Indeed,

$$\begin{aligned} a(-u)^3 + b(-v)^3 &= -\left(au^3 + bv^3\right) = -u \\ A(-u)^3 + B(-v)^3 &= -\left(Au^3 + Bv^3\right) = -v. \end{aligned}$$

Similarly, if $(-u,v)$ is solution of the system (2.2), where $u \ge 0$ and $v \ge 0$, then $(u,-v)$ is solution of the system (2.2)

$$\begin{aligned} a(-u)^3 + bv^3 = -au^3 + bv^3 = -u &\Leftrightarrow au^3 - bv^3 = au^3 + b(-v)^3 = u \\ A(-u)^3 + Bv^3 = -Au^3 + Bv^3 = v &\Leftrightarrow Au^3 - Bv^3 = Au^3 + B(-v)^3 = -v. \end{aligned}$$

One can conclude that we have the symmetry of the first and third quadrant and the second and fourth quadrant. Now, it is clear that it is enough to observe case where $\overline{x} \in \mathbb{R}$ and $\overline{y} \ge 0$. Clearly, the case when $a = 0$ $(b = 0)$ reduces to the case when $A = 0$ $(B = 0)$ by replacing x_n and y_n, so it will be avoided.

2.1. case $b = 0$, $a > 0$, $A > 0$, $B > 0$

The system (2.2) is equivalent to

$$\begin{aligned} a\overline{x}^3 &= \overline{x} \\ A\overline{x}^3 + B\overline{y}^3 &= \overline{y}. \end{aligned}$$

Clearly, $a\bar{x}^3 = \bar{x}$ if and only if $\bar{x}(\sqrt{a}\bar{x}-1)(\sqrt{a}\bar{x}+1) = 0$ which implies $\bar{x}_1 = 0$, $\bar{x}_2 = \frac{\sqrt{a}}{a}$, $\bar{x}_3 = -\frac{\sqrt{a}}{a}$.

(i) If $\bar{x}_1 = 0$, then $B\bar{y}^3 = \bar{y}$ if and only if $\bar{y}\left(\sqrt{B}\bar{x}-1\right)\left(\sqrt{B}\bar{x}+1\right) = 0$ and It follows immediately that the solutions $\left\{(0,0),\left(0,\frac{\sqrt{B}}{B}\right),\left(0,-\frac{\sqrt{B}}{B}\right)\right\}$ are equilibrium points of system (1.1).

(ii) If $\bar{x}_2 = \frac{\sqrt{a}}{a}$, then
$$B\bar{y}^3 - \bar{y} + \frac{A\sqrt{a}}{a^2} = 0. \tag{2.3}$$
Set $h(y) = By^3 - y + \frac{A\sqrt{a}}{a^2}$. Since $h(0) = \frac{A\sqrt{a}}{a^2} > 0$ and $h(-\infty) = -\infty$, then Eq.(2.3) always has negative solution $\bar{y}$. Furthermore, $h'(y) = 3By^2 - 1$. Since $B > 0$ that implies $h'(y) = 0$ if and only if $y_{1,2} = \pm\frac{1}{\sqrt{3B}}$ where we set $y_1 < 0 < y_2$. Hence, $h(y_1)h(y_2) = \frac{A^2}{a^3} - \frac{4}{27B}$, $h(0) = \frac{A\sqrt{a}}{a^2} > 0$ and we get the following:

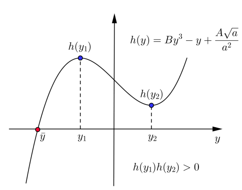

FIGURE 1

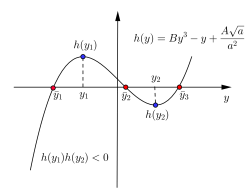

FIGURE 2

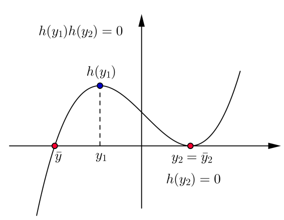

FIGURE 3

- if $h(y_1)h(y_2) > 0$ if and only if $27A^2B > 4a^3$, then $\bar{y}$ is unique negative solution of Eq.(2.3) which implies $\left(\frac{\sqrt{a}}{a},\bar{y}\right)$ is equilibrium point of the system (1.1) (see Figure 1).
- if $h(y_1)h(y_2) < 0$ if and only if $27A^2B < 4a^3$, that implies there are three different real solutions of Eq.(2.3) $\bar{y}_1 = \bar{y}$, $\bar{y}_2 \in (0,y_2)$, $\bar{y}_3 \in (y_2,+\infty)$ and three equilibrium points of the system (1.1) $\left\{\left(\frac{\sqrt{a}}{a},\bar{y}_1\right),\left(\frac{\sqrt{a}}{a},\bar{y}_2\right),\left(\frac{\sqrt{a}}{a},\bar{y}_3\right)\right\}$ (see Figure 2).

- if $h(y_1)h(y_2) = 0$ if and only if $27A^2B = 4a^3$, that implies there are two different real solutions of Eq.(2.3). If $h(y_1) = 0$, then $\bar{y} = y_1$ and that implies $h(0) < 0$ which is impossible. Thus $h(y_2) = 0$, so $\bar{y}$, y_2 are solutions of Eq.(2.3) and points $\left\{\left(\frac{\sqrt{a}}{a}, \bar{y}\right), \left(\frac{\sqrt{a}}{a}, y_2\right)\right\}$ are equilibrium points of the system (1.1) (see Figure 3).

(iii) If $\bar{x}_2 = -\frac{\sqrt{a}}{a}$, then

$$B\bar{y}^3 - \bar{y} - \frac{A\sqrt{a}}{a^2} = 0. \tag{2.4}$$

Set $g(y) = By^3 - y - \frac{A\sqrt{a}}{a^2}$. Since $g(0) = -\frac{A\sqrt{a}}{a^2} < 0$ and $g(+\infty) = +\infty$, then Eq.(2.4) always has positive solution $\bar{y}_+$. Furthermore, $g'(y) = 3By^2 - 1$. Since $B > 0$ that implies $g'(y) = 0$ if and only if
$y_{1,2} = \pm\frac{1}{\sqrt{3B}}$ where we set $y_1 < 0 < y_2$. Hence, $g(y_1)g(y_2) = \frac{A^2}{a^3} - \frac{4}{27B}$ and $g(0) = -\frac{A\sqrt{a}}{a^2} < 0$. By using the fact that we have the symmetry of the first and third quadrant and the second and fourth quadrant that immediately leads to the following statements:

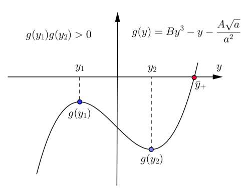

FIGURE 4

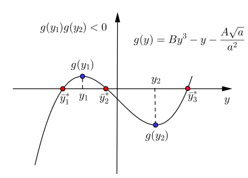

FIGURE 5

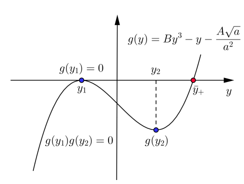

FIGURE 6

- if $g(y_1)g(y_2) > 0$ if and only if $27A^2B > 4a^3$, then $\bar{y}_+ = -\bar{y}$ ($\bar{y}$ is negative solution of Eq.(2.3)) is unique positive solution of Eq.(2.4) which implies $\left(-\frac{\sqrt{a}}{a}, \bar{y}_+\right)$ is equilibrium point of the system (1.1) (see Figure 4).
- if $g(y_1)g(y_2) < 0$ if and only if $27A^2B < 4a^3$ that implies there are three different real solutions of Eq.(2.4) $\bar{y}_1^* \in (-\infty, y_1)$, $\bar{y}_2^* \in (y_1, 0)$, $\bar{y}_3^* = \bar{y}_+ \in (y_2, +\infty)$ and three equilibrium points of the system (1.1) $\left\{\left(-\frac{\sqrt{a}}{a}, \bar{y}_1^*\right), \left(-\frac{\sqrt{a}}{a}, \bar{y}_2^*\right), \left(-\frac{\sqrt{a}}{a}, \bar{y}_3^*\right)\right\}$, where $\bar{y}_1^* = -\bar{y}_3$, $\bar{y}_2^* = -\bar{y}_2$ and $\bar{y}_2, \bar{y}_3$ are different real solutions of Eq.(2.3) (see Figure 5).

- if $g(y_1)\,g(y_2)=0$ if and only if $27A^2B=4a^3$ that implies there are two different real solutions of Eq.(2.4). If $g(y_2)=0$, then $\overline{y}_+=y_2$ and that implies $g(0)>0$ which is impossible. Thus it must be $g(y_1)=0$ so y_1, $\overline{y}_+$ are solutions of Eq.(2.4) and points $\left\{\left(-\frac{\sqrt{a}}{a},y_1\right),\left(-\frac{\sqrt{a}}{a},\overline{y}_+\right)\right\}$ are equilibrium points of the system (1.1) (see Figure 6).

All this leads to the following theorem:

Theorem 2.1. *Assume that $b=0$, $a>0$, $A>0$, $B>0$. Then*

(i) *the points* $\left\{E_0(0,0),E_1\left(0,\frac{\sqrt{B}}{B}\right),E_2\left(0,-\frac{\sqrt{B}}{B}\right)\right\}$ *are equilibrium points of the system* (1.1),

(ii) *if* $27A^2B>4a^3$ *then* $E_3\left(\frac{\sqrt{a}}{a},\overline{y}\right)$ *and* $E_4\left(-\frac{\sqrt{a}}{a},-\overline{y}\right)$ *are equilibrium points of the system* (1.1), *where $\overline{y}$ is unique negative solution of Eq.*(2.3),

(iii) *If* $27A^2B<4a^3$, *then* $E_5\left(\frac{\sqrt{a}}{a},\overline{y}_1\right)$, $E_6\left(\frac{\sqrt{a}}{a},\overline{y}_2\right)$, $E_7\left(\frac{\sqrt{a}}{a},\overline{y}_3\right)$, $E_8\left(-\frac{\sqrt{a}}{a},-\overline{y}_3\right)$, $E_9\left(-\frac{\sqrt{a}}{a},-\overline{y}_2\right)$, $E_{10}\left(-\frac{\sqrt{a}}{a},-\overline{y}_1\right)$ *are equilibrium points of the system* (1.1), *where* $\overline{y}_1$, $\overline{y}_2$ *and* $\overline{y}_3$ *are three different real solutions of Eq.*(2.3),

(iv) *if* $27A^2B=4a^3$, *then* $E_{11}\left(\frac{\sqrt{a}}{a},\overline{y}\right)$, $E_{12}\left(\frac{\sqrt{a}}{a},\frac{1}{\sqrt{3B}}\right)$, $E_{13}\left(-\frac{\sqrt{a}}{a},-\overline{y}\right)$, $E_{14}\left(-\frac{\sqrt{a}}{a},-\frac{1}{\sqrt{3B}}\right)$ *are equilibrium points of the system* (1.1), *where $\overline{y}$ is the negative solution of Eq.*(2.3).

2.2. **case** $a=0$**,** $b>0$**,** $A>0$**,** $B>0$

The system (2.2) is equivalent to

$$b\overline{y}^3=\overline{x}$$
$$A\overline{x}^3+B\overline{y}^3=\overline{y}.$$

Hence, $Ab^3\overline{y}^9+B\overline{y}^3=\overline{y}$ if and only if $\overline{y}\left(Ab^3\overline{y}^8+B\overline{y}^2-1\right)=0$ which implies $\overline{y}_1=0$ or

$$Ab^3\overline{y}^8+B\overline{y}^2-1=0. \tag{2.5}$$

Set $\overline{y}^2=t>0$. Then we get the following equation $P(t)=Ab^3t^4+Bt-1=0$. Since $P(0)=-1$ and $P(+\infty)=+\infty$ one can see that the polynimial $P(t)$ has at least the one positive zero. Furthermore $Ab^3\cdot(-1)<0$, then by applying Theorem 1.5 the last equation has an odd number of positive zeros, counting multiplicities. The number of sign changes in the sequence -1, B, Ab^3 is $v(P)=1$ and Theorem 1.6 implies the number of positive zeros of $P(t)=0$, counting multiplicities, is either equal to $v(P)$ or less than that by an even number. All this leads that $P(t)$ has exactly the one positive zero. Hence, the equation $Ab^3\overline{y}^8+B\overline{y}^2-1=0$ has two symmetric solutions. Let $\overline{y}_-$ and $\overline{y}_+$ denote that solutions. Now,

$$1=Ab^3\overline{y}_+^8+B\overline{y}_+^2\geq 2\sqrt{ABb^3\overline{y}_+^{10}}=2b\sqrt{ABb}\,\overline{y}_+^5$$

which implies

$$\overline{y}_{+} \in \left(0, \frac{1}{\sqrt[5]{2b\sqrt{ABb}}}\right] \text{ and } \overline{y}_{-} \in \left[\frac{-1}{\sqrt[5]{2b\sqrt{ABb}}}, 0\right).$$

It follows immediately that the solutions $\left\{(0,0), \left(b\overline{y}_{-}^{3}, \overline{y}_{-}\right), \left(b\overline{y}_{+}^{3}, \overline{y}_{+}\right)\right\}$ are equilibrium points of the system (1.1).

Theorem 2.2. *Assume that $a = 0$, $b > 0$, $A > 0$, $B > 0$. Then the points $E_0\,(0,0)$, $E_1\left(b\overline{y}_{-}^{3}, \overline{y}_{-}\right)$, $E_2\left(b\overline{y}_{+}^{3}, \overline{y}_{+}\right)$ are equilibrium points of the system (1.1), where $\overline{y}_{-}$ and $\overline{y}_{+}$ denote symmetric solutions of Eq.(2.5).*

2.3. **case** $a > 0$**,** $b > 0$**,** $A > 0$**,** $B > 0$ **and** $aB = bA$

From $aB = bA$ we get $\frac{a}{A} = \frac{b}{B} = k > 0$. Now, the equilibrium points of the system (1.1) are solutions of

$$Ak\overline{x}^{3} + Bk\overline{y}^{3} = \overline{x}$$
$$A\overline{x}^{3} + B\overline{y}^{3} = \overline{y},$$

which yields $\overline{x} = k\overline{y}$ and $\overline{y}((Ak^3 + B)\overline{y}^2 - 1) = 0$ if and only if $\overline{y}_1 = 0$, $\overline{y}_2 = -\frac{A}{\sqrt{a^3+A^2B}}$, $\overline{y}_3 = \frac{A}{\sqrt{a^3+A^2B}}$. Hence, solutions

$$\left\{(0,0), \left(-\frac{a}{\sqrt{a^3+A^2B}}, -\frac{A}{\sqrt{a^3+A^2B}}\right), \left(\frac{a}{\sqrt{a^3+A^2B}}, \frac{A}{\sqrt{a^3+A^2B}}\right)\right\}$$

are equilibrium points of the system (1.1).

Theorem 2.3. *Assume that $a > 0$, $b > 0$, $A > 0$, $B > 0$ and $aB = bA$. Then the points $E_0\,(0,0)$, $E_1\left(-\frac{a}{\sqrt{a^3+A^2B}}, -\frac{A}{\sqrt{a^3+A^2B}}\right)$, $E_2\left(\frac{a}{\sqrt{a^3+A^2B}}, \frac{A}{\sqrt{a^3+A^2B}}\right)$ are equilibrium points of the system (1.1).*

2.4. **case** $a > 0$**,** $b > 0$**,** $A > 0$**,** $B > 0$ **and** $aB \neq bA$

In this case we have

$$aB\overline{x}^{3} + bB\overline{y}^{3} = B\overline{x}$$
$$bA\overline{x}^{3} + bB\overline{y}^{3} = b\overline{y},$$

so by subtracting the last two lines and after some calculation we obtain

$$\overline{y} = \frac{B}{b}\overline{x} + \frac{bA - aB}{b}\overline{x}^{3}. \tag{2.6}$$

This implies the following equation

$$\overline{x}\left(a\overline{x}^{2} + b\overline{x}^{2}\left(\frac{B}{b} + \frac{bA - aB}{b}\overline{x}^{2}\right)^{3} - 1\right) = 0. \tag{2.7}$$

Obviously $\bar{x}_1 = 0$ and point $(0,0)$ is equilibrium point of the system (1.1). Set $\bar{x}^2 = t > 0$ and $p(t) = at + bt\left(\frac{B}{b} - \frac{aB-bA}{b}t\right)^3 - 1$. Since $p(0) = -1$ and $p\left(\frac{1}{a}\right) = \frac{bA^3}{a^4} > 0$ thus $p(t)$ has at least one positive zero at $\left(0, \frac{1}{a}\right)$. One can show that the following holds:

$$p(t) = -1 + \frac{1}{b^2}\left(ab^2 + B^3\right)t + \frac{3B^2}{b^2}(bA - aB)t^2 + \\ + \frac{3B}{b^2}(bA - aB)^2 t^3 + \frac{1}{b^2}(bA - aB)^3 t^4. \tag{2.8}$$

and

$$p'(t) = \frac{1}{b^2}\left(ab^2 + B^3\right) + \frac{6B^2}{b^2}(bA - aB)t + \frac{9B}{b^2}(bA - aB)^2 t^2 + \\ + \frac{4}{b^2}(bA - aB)^3 t^3.$$

We will consider two different cases:

(i) If $bA - aB > 0$, then by applying Theorems 1.5 and 1.6 to the polynomial $p(t)$ given by (2.8) we obtain that $p(t)$ has exactly the one positive zero at $\left(0, \frac{1}{a}\right)$. More precisely, equation (2.7) has two symmetric zeros $\bar{x}_- \in \left(-\frac{1}{\sqrt{a}}, 0\right)$ and $\bar{x}_+ \in \left(0, \frac{1}{\sqrt{a}}\right)$ which implies solutions $\left\{(0,0), (\bar{x}_-, \bar{y}_-), (\bar{x}_+, \bar{y}_+)\right\}$ are equilibrium points of the system (1.1) where $\bar{y}_-$ and $\bar{y}_+$ are given by (2.6) with corresponding $\bar{x}$.

(ii) If $bA - aB < 0$, then by applying Theorems 1.5 and 1.6 on polynomial $p(t)$ given by (2.8) we obtain that $p(t)$ has either two or four positive zeros. Let $\mathrm{Syl}(p, p')$ be the Sylvester matrix of $p(t)$ and $p'(t)$

$$\begin{pmatrix}
\frac{(bA-aB)^3}{b^2} & \frac{3B(bA-aB)^2}{b^2} & \frac{3B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & -1 & 0 & 0 & 0 \\
0 & \frac{4(bA-aB)^3}{b^2} & \frac{9B(bA-aB)^2}{b^2} & \frac{6B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & 0 & 0 & 0 \\
0 & \frac{(bA-aB)^3}{b^2} & \frac{3B(bA-aB)^2}{b^2} & \frac{3B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & -1 & 0 & 0 \\
0 & 0 & \frac{4(bA-aB)^3}{b^2} & \frac{9B(bA-aB)^2}{b^2} & \frac{6B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & 0 & 0 \\
0 & 0 & \frac{(bA-aB)^3}{b^2} & \frac{3B(bA-aB)^2}{b^2} & \frac{3B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & -1 & 0 \\
0 & 0 & 0 & \frac{4(bA-aB)^3}{b^2} & \frac{9B(bA-aB)^2}{b^2} & \frac{6B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & 0 \\
0 & 0 & 0 & \frac{(bA-aB)^3}{b^2} & \frac{3B(bA-aB)^2}{b^2} & \frac{3B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2} & -1 \\
0 & 0 & 0 & 0 & \frac{4(bA-aB)^3}{b^2} & \frac{9B(bA-aB)^2}{b^2} & \frac{6B^2(bA-aB)}{b^2} & \frac{ab^2+B^3}{b^2}
\end{pmatrix}$$

Let D_k be determinant of the submatrix of $\mathrm{Syl}(p, p')$ formed by the first $2k$ rows and the first $2k$ columns $k = 1,2,3,4$. Hence,

$$D_1 = \frac{4}{b^4}(bA - aB)^6 > 0,$$

$$D_2 = \frac{3B^2}{b^8}(bA - aB)^{10} > 0,$$
$$D_3 = \frac{6}{b^{10}}(bA - aB)^{12}\left(aB^3 + 2bAB^2 - 6a^2b^2\right),$$
$$D_4 = \frac{1}{b^{10}}(bA - aB)^{12}\left(-27a^4b^2 + 192ab^2A^2B + 6a^2bAB^2 + 4a^3B^3 - 256b^3A^3 - 27A^2B^4\right)$$

and

$$\{\operatorname{sign}(D_1), \operatorname{sign}(D_2), \operatorname{sign}(D_3), \operatorname{sign}(D_4)\} = \{1, 1, \operatorname{sign}(D_3), \operatorname{sign}(D_4)\}. \tag{2.9}$$

Now, if $\operatorname{sign}(D_3) = -1$, then it follows that $\operatorname{sign}(D_4) \in \{0, -1\}$, otherwise if $\operatorname{sign}(D_4) = 1$ Theorem 1.4 yields that the polynomial $p(t)$ has no real roots which is impossible. Hence, if $aB^3 + 2bAB^2 - 6a^2b^2 < 0$ then by applying Theorem 1.4 the polynomial $p(t)$ has exactly two real roots. If $\operatorname{sign}(D_3) = 0$ and $\operatorname{sign}(D_4) \neq 0$ then the revised sign list has the form $\{1, 1, -1, \operatorname{sign}(D_4)\}$ which yields that $\operatorname{sign}(D_4) = -1$, otherwise the polynomial $p(t)$ has no real roots which is impossible. Now, for $\operatorname{sign}(D_4) = -1$ by applying Theorem 1.4 the polynomial $p(t)$ has exactly two real roots. Finally, if $\operatorname{sign}(D_3) = 1$ then the revised sign list has the form $\{1, 1, 1, \operatorname{sign}(D_4)\}$. Therefore, if $\operatorname{sign}(D_4) \in \{0, 1\}$ then by applying Theorem 1.4 the polynomial $p(t)$ has exactly four real roots, otherwise the polynomial $p(t)$ has exactly two real roots. All this we present at the following table where we set $p_1 = bA - aB < 0$, $p_2 = aB^3 + 2bAB^2 - 6a^2b^2$ and $p_3 = -27a^4b^2 + 192ab^2A^2B + 6a^2bAB^2 + 4a^3B^3 - 256b^3A^3 - 27A^2B^4$:

p_1, p_2, p_3	number of real roots of Eq.(2.8)	symmetric real zeros of Eq.(2.7)
$p_1 < 0$, $p_2 < 0$	2	4
$p_1 < 0$, $p_2 = 0$, $p_3 \neq 0$	2	4
$p_1 < 0$, $p_2 > 0$, $p_3 < 0$	2	4
$p_1 < 0$, $p_2 > 0$, $p_3 \geq 0$	4	8

TABLE 1

Therefore, Eq.(2.7) has four or eight symmetric zeros according to Table 1 which yields that the system (1.1) has four or eight equilibrium points.

Theorem 2.4. *Assume that $a > 0$, $b > 0$, $A > 0$, $B > 0$ and $aB \neq bA$. Then the point $E_0(0,0)$ is equilibrium point of the system (1.1) and*

(i) *if $bA - aB > 0$, then $E_1(\overline{x}_-, \overline{y}_-)$, $E_2(\overline{x}_+, \overline{y}_+)$ are equilibrium points of the system (1.1) where $\overline{x}_-$ and $\overline{x}_+$ are two symmetric zeros of Eq.(2.7) with corresponding $\overline{y}$ given by (2.6);*

(ii) *if $bA - aB < 0$, then the number of symmetric real zeros of Eq.(2.7) is given by table (1).*

3. Local Stability of Equilibrium Solutions

The Jacobian matrix of the map T given by (2.1) evaluated in an equilibrium point $(\bar{x},\bar{y})$ is

$$J_T(\bar{x},\bar{y}) = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x},\bar{y}) & \frac{\partial f}{\partial y}(\bar{x},\bar{y}) \\ \frac{\partial g}{\partial x}(\bar{x},\bar{y}) & \frac{\partial g}{\partial y}(\bar{x},\bar{y}) \end{pmatrix} = \begin{pmatrix} 3a\bar{x}^2 & 3b\bar{y}^2 \\ 3A\bar{x}^2 & 3B\bar{y}^2 \end{pmatrix}. \tag{3.1}$$

The determinant and trace of (3.1) are

$$\begin{aligned} \det(J_T(\bar{x},\bar{y})) = 9\bar{x}^2\bar{y}^2(aB-bA) \quad \text{and} \\ \operatorname{tr}(J_T(\bar{x},\bar{y})) = 3\left(a\bar{x}^2+B\bar{y}^2\right). \end{aligned} \tag{3.2}$$

The eigenvalues of (3.1) are

$$\lambda = \frac{3}{2}\left(a\bar{x}^2+B\bar{y}^2+\sqrt{\left(a\bar{x}^2+B\bar{y}^2\right)^2+4\bar{x}^2\bar{y}^2(bA-aB)}\right)$$

$$\mu = \frac{3}{2}\left(a\bar{x}^2+B\bar{y}^2-\sqrt{\left(a\bar{x}^2+B\bar{y}^2\right)^2+4\bar{x}^2\bar{y}^2(bA-aB)}\right)$$

with corresponding eigenvectors

$$E_\lambda = \left(\frac{a\bar{x}^2-B\bar{y}^2+\sqrt{\left(a\bar{x}^2+B\bar{y}^2\right)^2+4\bar{x}^2\bar{y}^2(bA-aB)}}{2A\bar{x}^2},1\right)$$

$$E_\mu = \left(\frac{a\bar{x}^2-B\bar{y}^2-\sqrt{\left(a\bar{x}^2+B\bar{y}^2\right)^2+4\bar{x}^2\bar{y}^2(bA-aB)}}{2A\bar{x}^2},1\right).$$

Theorem 3.1. *$E_0(0,0)$ is locally asymptotically stable.*

Proof. Since $\det(J_T(E_0)) = 0$ and $\operatorname{tr}(J_T(E_0)) = 0$ then $|\operatorname{tr}(J_T(E_0))| < 1+\det(J_T(E_0))$ and $\det(J_T(E_0)) < 1$. By applying Theorem 1.3 the equilibrium $E_0(0,0)$ of the system (1.1) is locally asymptotically stable. □

Theorem 3.2. *Let $b=0$, $a>0$, $A>0$, $B>0$. Then the equilibrium points of system (1.1) satisfy the following statements:*

(i) *$E_0(0,0)$ is locally asymptotically stable, $E_1\left(0,\frac{\sqrt{B}}{B}\right)$ and $E_2\left(0,-\frac{\sqrt{B}}{B}\right)$ are the saddle points.*

(ii) *If $27A^2B > 4a^3$, then $E_3\left(\frac{\sqrt{a}}{a},\bar{y}\right)$ and $E_4\left(-\frac{\sqrt{a}}{a},-\bar{y}\right)$ are the repellers, where $\bar{y}$ is unique negative solution of Eq.(2.3).*

(iii) *If $27A^2B < 4a^3$, then $E_5\left(\frac{\sqrt{a}}{a},\bar{y}_1\right)$, $E_7\left(\frac{\sqrt{a}}{a},\bar{y}_3\right)$, $E_8\left(-\frac{\sqrt{a}}{a},-\bar{y}_3\right)$, $E_{10}\left(-\frac{\sqrt{a}}{a},-\bar{y}_1\right)$ are the repellers and $E_6\left(\frac{\sqrt{a}}{a},\bar{y}_2\right)$, $E_9\left(-\frac{\sqrt{a}}{a},-\bar{y}_2\right)$ are the* *saddle points, where* $\bar{y}_1$, $\bar{y}_2$ *and* $\bar{y}_3$ *are three different real solutions of Eq.*(2.3) $\bar{y}_1 \in \left(-\infty, -\frac{1}{\sqrt{3B}}\right)$, $\bar{y}_2 \in \left(0, \frac{1}{\sqrt{3B}}\right)$, $\bar{y}_3 \in \left(\frac{1}{\sqrt{3B}}, +\infty\right)$.

(iv) *If* $27A^2B = 4a^3$ *then* $E_{11}\left(\frac{\sqrt{a}}{a}, \bar{y}\right)$, $E_{13}\left(-\frac{\sqrt{a}}{a}, -\bar{y}\right)$ *are the repellers and* $E_{12}\left(\frac{\sqrt{a}}{a}, \frac{1}{\sqrt{3B}}\right)$, $E_{14}\left(-\frac{\sqrt{a}}{a}, -\frac{1}{\sqrt{3B}}\right)$ *are the nonhyperbolic points, where* $\bar{y}$ *is the negative solution of Eq.*(2.3).

Proof. Indeed,

(i) In view of Theorem 1.3 $E_0(0,0)$ is locally asymptotically stable. Since $\det(J_T(E_{1,2})) = 0$ and $\mathrm{tr}(J_T(E_{1,2})) = 3$ then $|\mathrm{tr}(J_T(E_{1,2}))| > |1 + \det(J_T(E_{1,2}))|$ so equilibriums E_1 and E_2 are the saddle points.

(ii) Clearly, $\det(J_T(E_{3,4})) = 9\bar{y}^2B$ and $\mathrm{tr}(J_T(E_{3,4})) = 3\left(1 + B\bar{y}^2\right)$ then $\mathrm{tr}(J_T(E_{3,4})) - \det(J_T(E_{3,4})) = 3\left(1 - 2B\bar{y}^2\right)$. Since $\bar{y}$ is the unique solution of Eq.(2.3) and $|\bar{y}| > \frac{1}{\sqrt{3B}}$ (see Figure 1) then $3\left(1 - 2B\bar{y}^2\right) < 1$. Now $|\det(J_T(E_{3,4}))| > 1$ and $|\mathrm{tr}(J_T(E_{3,4}))| < |1 + \det(J_T(E_{3,4}))|$. By applying Theorem 1.3 equilibriums E_3 and E_4 are the repellers.

(iii) Obviously, for all points E_i we obtaine $\det(J_T(E_i)) = 9\bar{y}_i^2B$ and $\mathrm{tr}(J_T(E_i)) = 3\left(1 + B\bar{y}_i^2\right)$, where $i = \overline{5,10}$ and $\bar{y}_i$ is real solution of Eq.(2.3). It is easy to see that y coordinate of equilibrium points E_5, E_7, E_8, E_{10} satisfies inequality $\left|\bar{y}_j\right| > \frac{1}{\sqrt{3B}}$ (see Figure 2), so according to (ii) we get $\left|\det(J_T(E_j))\right| > 1$ and $\left|\mathrm{tr}(J_T(E_j))\right| < \left|1 + \det(J_T(E_j))\right|$ where $j \in \{5,7,8,10\}$. Thus, by using Theorem 1.3 equilibriums E_5, E_7, E_8, E_{10} are the repellers. On the other side, y coordinate of equilibrium points E_6 and E_9 satisfies inequality $|\bar{y}_k| < \frac{1}{\sqrt{3B}}$, $\bar{y}_k \in \{\bar{y}_6, \bar{y}_9\}$, where are $\mathrm{tr}(J_T(E_{6,9})) - \det(J_T(E_{6,9})) = 3\left(1 - 2B\bar{y}_k^2\right) > 1$ and $|\mathrm{tr}(J_T(E_{1,2}))| > |1 + \det(J_T(E_{1,2}))|$. Hence, by applying Theorem 1.3 equilibriums E_6 and E_9 are the saddle points.

(iv) In this case we have $\det(J_T(E_{12,14})) = 3$ and $\mathrm{tr}(J_T(E_{12,14})) = 4$ which implies $|\mathrm{tr}(J_T(E_{3,4}))| = |\det(J_T(E_{3,4})) + 1|$ and from Theorem 1.3 yields equilibriums E_{12} and E_{14} are the nonhyperbolic points. For equilibriums E_{11} and E_{13} we obtaine $\mathrm{tr}(J_T(E_{11,13})) - \det(J_T(E_{11,13})) = 3\left(1 - 2B\bar{y}^2\right)$. Since $\bar{y}$ is the negative solution of Eq.(2.3) that satifies $|\bar{y}| > \frac{1}{\sqrt{3B}}$ (see Figure 3) then $3\left(1 - 2B\bar{y}^2\right) < 1$. That implies the following $|\mathrm{tr}(J_T(E_{11,13}))| < |1 + \det(J_T(E_{11,13}))|$ and $|\det(J_T(E_{11,13}))| > 1$. By applying Theorem 1.3 equilibriums E_{11} and E_{13} are the repellers. □

Theorem 3.3. *Let* $a = 0$, $b > 0$, $A > 0$, $B > 0$. *Let* $\bar{y}_-$ *and* $\bar{y}_+$ *are the symmetric solutions of equation* $b^3A\bar{y}^8 + B\bar{y}^2 = 1$. *Then equilibrium points* $E_0(0,0)$, $E_1\left(b\bar{y}_-^3, \bar{y}_-\right)$, $E_2\left(b\bar{y}_+^3, \bar{y}_+\right)$ *of the system* (1.1) *satisfy the following statements:* E_0 *is locally asymptotically stable and*

(i) *if* $16Ab^3 > 27B^4$, *then* E_1 *and* E_2 *are repellers,*

(ii) *if* $16Ab^3 < 27B^4$*, then* E_1 *and* E_2 *are the saddle points,*
(iii) *If* $16b^3A = 27B^4$*, then* E_1 *and* E_2 *are the nonhyperbolic points.*

Proof. By applying Theorem 1.3 the equilibrium $E_0(0,0)$ of the system (1.1) is locally asymptotically stable. One can find that $\det(J_T(E_{1,2})) = -9b^3A\overline{y}^8 = 9(B\overline{y}^2 - 1)$ and $\operatorname{tr}(J_T(E_{1,2})) = 3B\overline{y}^2$ where $\overline{y} \in \{\overline{y}_-, \overline{y}_+\}$. Furthermore $Ab^3\overline{y}^8 + B\overline{y}^2 = 1$ implies $B\overline{y}^2 \in (0,1)$. Now

(i) If $B\overline{y}^2 \in \left(0, \frac{2}{3}\right)$ then $|\det(J_T(E_{1,2}))| = 9(1 - B\overline{y}^2) > 9\left(1 - \frac{2}{3}\right) = 3 > 1$ and $|1 + \det(J_T(E_{1,2}))| = 8 - 9B\overline{y}^2 = 12\left(\frac{2}{3} - B\overline{y}^2\right) + 3B\overline{y}^2 > 3B\overline{y}^2 = |\operatorname{tr}(J_T(E_{1,2}))|$. Thus, by applying Theorem 1.3 equilibriums E_1 and E_2 are repellers. It is remain to us to show $B\overline{y}^2 \in \left(0, \frac{2}{3}\right)$ if and only if $16Ab^3 > 27B^4$. Indeed, if $B\overline{y}^2 < \frac{2}{3}$, then
$$1 = Ab^3\overline{y}^8 + B\overline{y}^2 = \frac{Ab^3}{B^4}\left(B\overline{y}^2\right)^4 + B\overline{y}^2 < \frac{16Ab^3}{81B^4} + \frac{2}{3} \Rightarrow 16Ab^3 > 27B^4.$$
Now, if $Ab^3 > \frac{27}{16}B^4$, then
$$\begin{aligned} 0 = Ab^3\overline{y}^8 + B\overline{y}^2 - 1 &> \frac{27}{16}B^4\overline{y}^8 + B\overline{y}^2 - 1 \\ &= \frac{27}{16}\left(B\overline{y}^2\right)^4 + B\overline{y}^2 - 1 = \frac{1}{16}(3z-2)\left(9z^3 + 6z^2 + 4z + 8\right), \end{aligned}$$
which yields $z \in \left(0, \frac{2}{3}\right)$, where we set $z = B\overline{y}^2$.

(ii) It easy to show that $|1 + \det(J_T(E_{1,2}))| = \left|9B\overline{y}^2 - 8\right| < 3B\overline{y}^2 = |\operatorname{tr}(J_T(E_{1,2}))|$ for all $B\overline{y}^2 \in \left(\frac{2}{3}, 1\right)$. Therefore by applying Theorem 1.3 equilibriums E_1 and E_2 are the saddle points. By using the method shown in (i) one can prove that $B\overline{y}^2 \in \left(\frac{2}{3}, 1\right)$ if and only if $16Ab^3 < 27B^4$.

(iii) If $B\overline{y}^2 = \frac{2}{3}$ then $|1 + \det(J_T(E_{1,2}))| = 2$ and $|\operatorname{tr}(J_T(E_{1,2}))| = 2$ by applying Theorem 1.3 equilibriums E_1 and E_2 are the nonhyperbolic points. Clearly, $B\overline{y}^2 = \frac{2}{3}$ if and only if $16b^3A = 27B^4$. □

Theorem 3.4. *Let* $a > 0$*,* $b > 0$*,* $A > 0$*,* $B > 0$ *and* $aB = bA$*. The points* $E_0(0,0)$*,* $E_1\left(-\frac{a}{\sqrt{a^3+A^2B}}, -\frac{A}{\sqrt{a^3+A^2B}}\right)$*,* $E_2\left(\frac{a}{\sqrt{a^3+A^2B}}, \frac{A}{\sqrt{a^3+A^2B}}\right)$ *are equilibrium points of the system* (1.1) *and the following statement holds:* E_0 *is locally asymptotically stable and* E_1*,* E_2 *are the saddle points.*

Proof. By applying Theorem 1.3 equilibrium $E_0(0,0)$ of the system (1.1) is locally asymptotically stable. Furthermore, $\det(J_T(E_{1,2})) = 0$, $\operatorname{tr}(J_T(E_{1,2})) = 3$ which implies $|\operatorname{tr}(J_T(E_{1,2}))| > |1 + \det(J_T(E_{1,2}))|$ so equilibriums E_1 and E_2 are the saddle points. □

Theorem 3.5. *Let* $a > 0$*,* $b > 0$*,* $A > 0$*,* $B > 0$ *and* $aB \neq bA$*. The equilibrium point* $E_0(0,0)$ *of the system* (1.1) *is locally asymptotically stable.*

(i) *If* $bA - aB > 0$, *then the equilibrium points* $E_1(\overline{x}_-, \overline{y}_-)$ *and* $E_2(\overline{x}_+, \overline{y}_+)$ *of the system* (1.1), *where* $\overline{x}_-$ *and* $\overline{x}_+$ *are two symmetric solutions of equation* (2.7) *satisfy the following:*
 (i_1) *if* $tr(J_T(E_{1,2})) \in (0,2)$, *then* E_1 *and* E_2 *are repellers,*
 (i_2) *if* $tr(J_T(E_{1,2})) = 2$, *then* E_1 *and* E_2 *are nonhyperbolics,*
 (i_3) *if* $tr(J_T(E_{1,2})) \in (2,3)$, *then* E_1 *and* E_2 *are saddle points.*

(ii) *If* $bA - aB < 0$, *then number of equilibrium points* $E(\overline{x}, \overline{y})$ *of the system* (1.1) *is given by the table* (1). *The following statements hold:*
 (ii_1) *E is not locally asymptotically stable,*
 (ii_2) *if* $tr(J_T(E)) \in (4, +\infty)$, *then E is a repeller,*
 (ii_3) *if* $tr(J_T(E)) = 4$, *then E is nonhyperbolic,*
 (ii_4) *if* $tr(J_T(E)) \in (3,4)$, *then E is a saddle point.*

Proof. By applying Theorem 1.3 the equilibrium $E_0(0,0)$ of the system (1.1) is locally asymptotically stable.

(i) Straightforward calculation implies $\det(J_T(E_{1,2})) = -9\overline{x}^2\overline{y}^2(bA - aB) < 0$ and $\operatorname{tr}(J_T(E_{1,2})) = 3\left(a\overline{x}^2 + B\overline{y}^2\right) > 0$, where $(\overline{x}, \overline{y}) \in \{E_1, E_2\}$. In this case we have $|\overline{x}| < \frac{1}{\sqrt{a}}$ and

$$a\overline{x}^3 + b\overline{y}^3 = \overline{x} \Leftrightarrow b\frac{\overline{y}^3}{\overline{x}} = 1 - a\overline{x}^2,$$
$$A\overline{x}^3 + B\overline{y}^3 = \overline{y} \Leftrightarrow A\frac{\overline{x}^3}{\overline{y}} = 1 - B\overline{y}^2.$$

By multiplying the last two relations after an easy calculation we have

$$\left(a\overline{x}^2 + B\overline{y}^2\right) + (bA - aB)\overline{x}^2\overline{y}^2 = 1. \tag{3.3}$$

Since $bA - aB > 0$ then $a\overline{x}^2 + B\overline{y}^2 \in (0,1)$ so we have $1 > a\overline{x}^2 + B\overline{y}^2 \geq 2\sqrt{aB}|\overline{xy}|$. Hence, $\overline{x}^2\overline{y}^2 < \frac{1}{4aB}$. Set $d = \det(J_T(E_{1,2}))$ and $t = \operatorname{tr}(J_T(E_{1,2}))$. From (3.3) yields

$$d = 3t - 9, \ t \in (0,3). \tag{3.4}$$

Theorem 1.3 and (3.4) immediately imply that equilibriums E_1 and E_2 are:
 (i_1) repellers if $t < |3t - 8| \wedge 9 - 3t > 1$ if and only if $t \in (0,2)$.
 (i_2) nonhyperbolic points if $t = |3t - 8|$ if and only if $t = 2 \vee t = 4$. From $d = 1$ and $|t| \leq 2$ we get $t = \frac{10}{3}$. From (3.4) yields $t = 2$.
 (i_3) saddle points if $t > |3t - 8|$ if and only if $t \in (2,4)$, therefore $t \in (2,3)$

(ii) In this case we have $d = \det(J_T(E)) = -9\overline{x}^2\overline{y}^2(bA - aB) > 0$, $t = \operatorname{tr}(J_T(E)) = 3\left(a\overline{x}^2 + B\overline{y}^2\right) > 0$ and by applying (3.3) we get $0 > (bA - aB)\overline{x}^2\overline{y}^2 = 1 - \left(a\overline{x}^2 + B\overline{y}^2\right) \Rightarrow a\overline{x}^2 + B\overline{y}^2 > 1$ so

$$d = 3t - 9 > 0, \ t \in (3, +\infty). \tag{3.5}$$

From Theorem 1.3 and (3.5) we get that the equilibrium is:
 (ii_1) locally asymptotically stable if $t < 3t - 8 \wedge 3t - 9 < 1$ if and only if $t > 4 \wedge t < \frac{10}{3}$, which is impossible,

(ii_2) repeller if $t < 3t-8 \wedge 3t-9 > 1$ if and only if $t \in (4,+\infty)$,
(ii_3) nonhyperbolic point if $t = 3t-8$ if and only if $t = 4$.
(ii_4) saddle point if $t > 3t-8$ if and only if $t < 4$, therefore $t \in (3,4)$. □

One can give a geometric interpretation of Theorem 3.5:

Theorem 3.6. *Let $a > 0$, $b > 0$, $A > 0$, $B > 0$ and $aB \neq bA$. Let $ax^3 + by^3 = x$ and $Ax^3 + By^3 = y$ be curves and let the point E be an intersection point of the given curves. Set $r(x,y) = 3\left(ax^2 + By^2\right)$ ($r(x,y) = k$, $k > 0$, is an ellipse).*

(i) *If $bA - aB > 0$, then E is the equilibrium point of system (1.1).*
   (i_1) *If $E \in \{(x,y) : 0 < r(x,y) < 2\}$, then E is a repeller.*
   (i_2) *If $E \in \{(x,y) : r(x,y) = 2\}$, then E is nonhyperbolic.*
   (i_3) *If $E \in \{(x,y) : 2 < r(x,y) < 3\}$, then E is a saddle point.*

(ii) *If $bA - aB < 0$, then the number of equilibrium points $E(\overline{x},\overline{y})$ of the system (1.1) is given by the table (1). The following statements hold:*
   (ii_1) *E is not locally asymptotically stable,*
   (ii_2) *if $E \in \{(x,y) : r(x,y) > 4\}$, then E is a repeller,*
   (ii_3) *if $E \in \{(x,y) : r(x,y) = 4\}$ then E is nonhyperbolic,*
   (ii_4) *if $E \in \{(x,y) : 3 < r(x,y) < 4\}$, then E is a saddle point.*

4. Existence of Prime Period-two Solutions

Let $\ldots,\Phi,\Psi,\Phi,\Psi,\ldots$ be a two cycle of the system (1.1). Let T be the function defined by (2.1). Then (Φ,Ψ) is fixed point of T^2, the second iterete of T. Now

$$\begin{aligned} T^2(x,y) &= T\left(T(x,y)\right) = T\left(ax^3+by^3, Ax^3+By^3\right) \\ &= \left(a\left(ax^3+by^3\right)^3 + b\left(Ax^3+By^3\right)^3, A\left(ax^3+by^3\right)^3 + B\left(Ax^3+By^3\right)^3\right), \end{aligned} \tag{4.1}$$

and period-two solutions $\ldots,\Phi,\Psi,\Phi,\Psi,\ldots$ satisfies the system:

$$\begin{aligned} a\left(a\Phi^3+b\Psi^3\right)^3 + b\left(A\Phi^3+B\Psi^3\right)^3 &= \Phi, \\ A\left(a\Phi^3+b\Psi^3\right)^3 + B\left(A\Phi^3+B\Psi^3\right)^3 &= \Psi. \end{aligned} \tag{4.2}$$

4.1. case $b = 0$, $a > 0$, $A > 0$, $B > 0$

In this case system (4.2) becomes

$$\begin{aligned} a^4\Phi^9 &= \Phi, \\ A\left(a\Phi^3\right)^3 + B\left(A\Phi^3+B\Psi^3\right)^3 &= \Psi. \end{aligned}$$

Now, from $a^4\Phi^9 = \Phi$ we have $\Phi\left(\sqrt{a}\Phi - 1\right)\left(\sqrt{a}\Phi + 1\right)\left(a\Phi^2+1\right)\left(a^2\Phi^4+1\right) = 0$ so $\Phi_0 = 0$, $\Phi_1 = \frac{1}{\sqrt{a}}$ and $\Phi_2 = -\frac{1}{\sqrt{a}}$. For $\Phi_0 = 0$ we obtain $B^4\Psi^9 = \Psi$ which implies $\Psi_0 = 0$, $\Psi_1 = \frac{1}{\sqrt{B}}$ and $\Psi_2 = -\frac{1}{\sqrt{B}}$. Hence, $(\Phi,\Psi) \in \left\{(0,0), \left(0,\frac{1}{\sqrt{B}}\right), \left(0,\frac{-1}{\sqrt{B}}\right)\right\}$ and all these solutions are the equilibrium points of the system (1.1). For $\Phi_1 = \frac{1}{\sqrt{a}}$

we have $\frac{A}{a\sqrt{a}}+B\left(\frac{A}{a\sqrt{a}}+B\Psi^3\right)^3=\Psi$ and after straight forward calculation we obtain
$\left(\frac{A}{a\sqrt{a}}+B\Psi^3-\Psi\right)\left(1+\frac{A^2B}{a^3}+\frac{\sqrt{a}AB}{a^2}\Psi+B\Psi^2+\frac{2\sqrt{a}AB^2}{a^2}\Psi^3+B^2\Psi^4+B^3\Psi^6\right)$
$=0$. Since every solution of $\frac{A}{a\sqrt{a}}+B\Psi^3-\Psi=0$ is a solution of Eq.(2.3), then any period-two solution is a solution of equation

$$q(\Psi)=1+\frac{A^2B}{a^3}+\frac{\sqrt{a}AB}{a^2}\Psi+B\Psi^2+\frac{2\sqrt{a}AB^2}{a^2}\Psi^3+B^2\Psi^4+B^3\Psi^6=0. \quad (4.3)$$

Let $q'(\Psi)$ be the first derivative of $q(\Psi)$ that is

$$q'(x)=\frac{\sqrt{a}AB}{a^2}+2B\Psi+\frac{6\sqrt{a}AB^2}{a^2}\Psi^2+4B^2\Psi^3+6B^3\Psi^5.$$

Theorem 4.1. *Let $q(\Psi)$ be the polynomial defined by* (4.3). *If $a>0$, $A>0$, $B>0$ then $q(\Psi)>0$ for all Ψ.*

Proof. The Sylvester matrix $\mathrm{Syl}(q,q')$ of $q(\Psi)$ and $q'(\Psi)$ is the matrix of form

$$\begin{pmatrix}
B^3 & 0 & B^2 & \frac{2\sqrt{a}AB^2}{a^2} & B & \frac{\sqrt{a}AB}{a^2} & 1+\frac{A^2B}{a^3} & 0 & 0 & 0 & 0 & 0\\
0 & 6B^3 & 0 & 4B^2 & \frac{6\sqrt{a}AB^2}{a^2} & 2B & \frac{\sqrt{a}AB}{a^2} & 0 & 0 & 0 & 0 & 0\\
0 & B^3 & 0 & B^2 & \frac{2\sqrt{a}AB^2}{a^2} & B & \frac{\sqrt{a}AB}{a^2} & 1+\frac{A^2B}{a^3} & 0 & 0 & 0 & 0\\
0 & 0 & 6B^3 & 0 & 4B^2 & \frac{6\sqrt{a}AB^2}{a^2} & 2B & \frac{\sqrt{a}AB}{a^2} & 0 & 0 & 0 & 0\\
0 & 0 & B^3 & 0 & B^2 & \frac{2\sqrt{a}AB^2}{a^2} & B & \frac{\sqrt{a}AB}{a^2} & 1+\frac{A^2B}{a^3} & 0 & 0 & 0\\
0 & 0 & 0 & 6B^3 & 0 & 4B^2 & \frac{6\sqrt{a}AB^2}{a^2} & 2B & \frac{\sqrt{a}AB}{a^2} & 0 & 0 & 0\\
0 & 0 & 0 & B^3 & 0 & B^2 & \frac{2\sqrt{a}AB^2}{a^2} & B & \frac{\sqrt{a}AB}{a^2} & 1+\frac{A^2B}{a^3} & 0 & 0\\
0 & 0 & 0 & 0 & 6B^3 & 0 & 4B^2 & \frac{6\sqrt{a}AB^2}{a^2} & 2B & \frac{\sqrt{a}AB}{a^2} & 0 & 0\\
0 & 0 & 0 & 0 & B^3 & 0 & B^2 & \frac{2\sqrt{a}AB^2}{a^2} & B & \frac{\sqrt{a}AB}{a^2} & 1+\frac{A^2B}{a^3} & 0\\
0 & 0 & 0 & 0 & 0 & 6B^3 & 0 & 4B^2 & \frac{6\sqrt{a}AB^2}{a^2} & 2B & \frac{\sqrt{a}AB}{a^2} & 0\\
0 & 0 & 0 & 0 & 0 & B^3 & 0 & B^2 & \frac{2\sqrt{a}AB^2}{a^2} & B & \frac{\sqrt{a}AB}{a^2} & 1+\frac{A^2B}{a^3}\\
0 & 0 & 0 & 0 & 0 & 0 & 6B^3 & 0 & 4B^2 & \frac{6\sqrt{a}AB^2}{a^2} & 2B & \frac{\sqrt{a}AB}{a^2}
\end{pmatrix}$$

Let D_k be determinant of the submatrix of $\mathrm{Syl}(q,q')$ formed by the first $2k$ rows and the first $2k$ columns $k=1,2,\ldots,6$. Hence,

$$D_1=6B^6>0,$$

$$D_2=-12B^{11}<0,$$

$$D_3=8\frac{B^{15}}{a^3}\left(4a^3-27A^2B\right),$$

$$D_4=972\frac{A^2B^{19}}{a^3}>0,$$

$$D_5 = 162\frac{A^2B^{21}}{a^6}\left(32a^3+27A^2B\right) > 0,$$

$$D_6 = -\frac{B^{21}\left(16a^3+27A^2B\right)\left(32a^3+27A^2B\right)^2}{a^9} < 0,$$

and

$$\begin{aligned}\{\text{sign}(D_1),\text{sign}(D_2),\text{sign}(D_3),\text{sign}(D_4),\text{sign}(D_5),\text{sign}(D_6)\}\\ = \{1,-1,\text{sign}(D_3),1,1,-1\}.\end{aligned} \tag{4.4}$$

Now, by applying Theorem 1.4 the polynomial $q(\Psi)$ has no real roots if and only if $\text{sign}(D_3) = 1$ or $\text{sign}(D_3) = -1$ which yields $4a^3 - 27A^2B \neq 0$. If $\text{sign}(D_3) = 0$ then (4.4) yields $\{1,-1,0,1,1,-1\}$ and the revised sign list is $\{1,-1,1,1,1,-1\}$ so Theorem 1.4 implies the polynomial $q(\Psi)$ has no real roots. ☐

In view of Theorem 4.1 it is clear that:

Theorem 4.2. *If* $b = 0$*,* $a > 0$*,* $A > 0$*,* $B > 0$*, then the system* (1.1) *has no minimal period-two solution.*

4.2. **case** $a = 0$**,** $b > 0$**,** $A > 0$**,** $B > 0$

In this case system (4.2) becomes

$$b\left(A\Phi^3+B\Psi^3\right)^3 = \Phi,$$
$$Ab^3\Psi^9 + B\left(A\Phi^3+B\Psi^3\right)^3 = \Psi.$$

If $\Psi = 0$, then $Ab^3\Psi^9 + B\left(A\Phi^3+B\Psi^3\right)^3 = \Psi$ if and only if $A^3B\Phi^9 = 0 \Rightarrow \Phi = 0$, which is impossible. Since we have the symmetry of the first and third quadrant and the second and fourth quadrant it is enough to observe case where $\Phi \in \mathbb{R}$ and $\Psi > 0$. Also, if $\Phi \geq 0$, then $\left(A\Phi^3+B\Psi^3\right) \geq 0$ which implies $Ab^3\Psi^9 \leq \Psi \Rightarrow 1 - Ab^3\Psi^8 \geq 0$. Now after a quick calculation we have $Ab^3\Psi^9 + B\frac{\Phi}{b} = \Psi$, hence

$$\Phi = \frac{b}{B}\Psi\left(1-Ab^3\Psi^8\right). \tag{4.5}$$

From $b\left(A\Phi^3+B\Psi^3\right)^3 = \Phi$ we get

$$\left(\frac{Ab^3}{B^3}\Psi^3\left(1-Ab^3\Psi^8\right)^3 + B\Psi^3\right)^3 = \frac{\Psi\left(1-Ab^3\Psi^8\right)}{B}$$

and

$$\left(\frac{Ab^3}{B^3}\left(1-Ab^3\Psi^8\right)^3 + B\right)^3 = \frac{1-Ab^3\Psi^8}{B\Psi^8}.$$

If we set $\Psi^8 = t > 0$, then

$$\left(\frac{Ab^3}{B^3}\left(1-Ab^3t\right)^3 + B\right)^3 = \frac{1-Ab^3t}{Bt}.$$

Let $u(t)$ be polynomial defined by

$$u(t) = \frac{t}{B^8}\left(Ab^3\left(1-Ab^3t\right)^3 + B^4\right)^3 + Ab^3t - 1. \tag{4.6}$$

One can show that

$$u(t) = -\frac{1}{B^8}v(t)\,w(t),$$

where

$$v(t) = 1 - \left(4Ab^3 + B^4\right)t + 6A^2b^6t^2 - 4A^3b^9t^3 + A^4b^{12}t^4, \tag{4.7}$$

and

$$\begin{aligned} w(t) = B^8 - \left(A^3b^9 + 3A^2b^6B^4\right)t + \left(5A^4b^{12} + 5A^3b^9B^4\right)t^2 - \\ - \left(10A^5b^{15} + 2A^4b^{12}B^4\right)t^3 + 10A^6b^{18}t^4 - 5A^7b^{21}t^5 + A^8b^{24}t^6. \end{aligned} \tag{4.8}$$

Now, let Ψ be solution of Eq.(2.5), more precisely, $b^3A\Psi^8 + B\Psi^2 = 1$ if and only if $t = \frac{1-B\sqrt[4]{t}}{Ab^3}$. After a straight forward calculation we get

$$v\left(\frac{1-B\sqrt[4]{t}}{Ab^3}\right) = \frac{B^4}{Ab^3}\left(Ab^3t + B\sqrt[4]{t} - 1\right) = 0. \tag{4.9}$$

This yields the positive zero of polynomial $v(t)$ implies equlibrium points E_1 and E_2 (including symmetry) of the system (1.1), except point $E_0(0,0)$. Let D_k be determinant of the submatrix of $\mathrm{Syl}(v, v_t')_{8\times 8}$ formed by the first $2k$ rows and the first $2k$ columns $k = 1, 2, \ldots, 4$. Clearly,

$$v'(t) = -\left(4Ab^3 + B^4\right) + 12A^2b^6t - 12A^3b^9t^2 + 4A^4b^{12}t^3,$$

and

$$\mathrm{Syl}(v, v_t') =$$

$$\begin{pmatrix} A^4b^{12} & -4A^3b^9 & 6A^2b^6 & -\left(4Ab^3+B^4\right) & 1 & 0 & 0 & 0 \\ 0 & 4A^4b^{12} & -12A^3b^9 & +12A^2b^6 & -\left(4Ab^3+B^4\right) & 0 & 0 & 0 \\ 0 & A^4b^{12} & -4A^3b^9 & 6A^2b^6 & -\left(4Ab^3+B^4\right) & 1 & 0 & 0 \\ 0 & 0 & 4A^4b^{12} & -12A^3b^9 & +12A^2b^6 & -\left(4Ab^3+B^4\right) & 0 & 0 \\ 0 & 0 & A^4b^{12} & -4A^3b^9 & 6A^2b^6 & -\left(4Ab^3+B^4\right) & 1 & 0 \\ 0 & 0 & 0 & 4A^4b^{12} & -12A^3b^9 & +12A^2b^6 & -\left(4Ab^3+B^4\right) & 0 \\ 0 & 0 & 0 & A^4b^{12} & -4A^3b^9 & 6A^2b^6 & -\left(4Ab^3+B^4\right) & 1 \\ 0 & 0 & 0 & 0 & 4A^4b^{12} & -12A^3b^9 & +12A^2b^6 & -\left(4Ab^3+B^4\right) \end{pmatrix}$$

After straight forward calculation we obtain

$$\begin{aligned} D_1 &= 4b^{24}A^8 > 0, \\ D_2 &= 0, \\ D_3 &= -36b^{48}A^{16}B^8 < 0, \\ D_4 &= -b^{48}A^{16}B^{12}\left(256b^3A + 27B^4\right) < 0. \end{aligned}$$

The sign list of the discriminant sequence D_i of $v(t)$ is $\{1,0,-1,-1\}$ and the revised sign list is $\{1,-1,-1,-1\}$. By applying Theorem 1.4 we get

sign changes v	real zeros $\deg v(t)-2v$	distinct real zeros $l-2v$
1	$4-2\times 1=2$	$4-2\times 1=2$

TABLE 2

In view of Table 2 and Theorems 1.5 and 1.6 we obtain that $v(t)=0$ has exactly two positive zeros. Since, one positive zero implies equlibrium points of the system (1.1), except point $E_0(0,0)$, then the other one implies period-two points including symmetry. Furthemore, $v(0)=1$, $v\left(\frac{1}{Ab^3}\right)=-\frac{B^4}{Ab^3}<0$ and $v(+\infty)=+\infty$ this yields $v(t)=0$ has two positive zeros in $\left(0,\frac{1}{Ab^3}\right)$ and $\left(\frac{1}{Ab^3},+\infty\right)$. From (4.5) we have $\operatorname{sgn}(\Phi)=\operatorname{sgn}\left(\frac{1}{Ab^3}-t\right)$, so if $t\in\left(\frac{1}{Ab^3},+\infty\right)$, then $\operatorname{sgn}(\Phi)=-1$ which implies there exists period-two solution in second and fourth qudrant $P_1\in Q_2(E_0)$ and $P_2\in Q_4(E_0)$ (including symmetry). Moreover, If $16Ab^3\le 27B^4$, then $\frac{1}{Ab^3}<\frac{1}{Ab^3}\left(1+\frac{2\sqrt[3]{2}}{3}\right)=\alpha$ and

$$v(\alpha)=\frac{2\sqrt[3]{2}\left(16Ab^3-27B^4\right)-81B^4}{81Ab^3}<0,$$

so $t_0\in(\alpha,+\infty)$, otherwise if $16Ab^3>27B^4$, then

$$v\left(\frac{3}{Ab^3}\right)=\frac{16Ab^3-3B^4}{Ab^3}>\frac{24B^4}{Ab^3}>0$$

which yields $t_0\in\left(\frac{1}{Ab^3},\frac{3}{Ab^3}\right)$.

Let E_k be the determinant of the submatrix of $\mathrm{Syl}(w,w'_t)_{12\times 12}$ formed by the first $2k$ rows and the first $2k$ columns $k=1,2,\ldots,6$. Since

$$\begin{aligned} w'(t)=&-\left(A^3b^9+3A^2b^6B^4\right)+2\left(5A^4b^{12}+5A^3b^9B^4\right)t-3\left(10A^5b^{15}+2A^4b^{12}B^4\right)t^2\\ &+40A^6b^{18}t^3-25A^7b^{21}t^4+6A^8b^{24}t^5, \end{aligned}$$

then after straight forward calculation we obtain:

$$\begin{aligned} E_1&=6b^{48}A^{12}>0,\\ E_2&=5b^{90}A^{30}>0,\\ E_3&=8A^{40}b^{120}B^4\left(5Ab^3-27B^4\right),\\ E_4&=-180A^{50}b^{150}B^8<0,\\ E_5&=80A^{56}b^{168}B^{12}\left(4Ab^3+27B^4\right)>0,\\ E_6&=A^{58}b^{174}B^{20}\left(4Ab^3+27B^4\right)^2\left(16Ab^3-27B^4\right). \end{aligned}$$

The sign list of the discriminant sequence of $w(t)$ is $\{1,1,\operatorname{sign}(E_3),-1,1,\operatorname{sign}(E_6)\}$. Obviously, $\operatorname{sign}(E_3)$ does not affect the number of sign changes in the sign list of the discriminant sequence of $w(t)$:

sign (E_3)	discriminant sequence of $w(t)$	revised sign list	sign changes v to sign(E_6)
-1	$\{1,1,-1,-1,1,\operatorname{sign}(E_6)\}$	$\{1,1,-1,-1,1,\operatorname{sign}(E_6)\}$	2
0	$\{1,1,0,-1,1,\operatorname{sign}(E_6)\}$	$\{1,1,-1,-1,1,\operatorname{sign}(E_6)\}$	2
1	$\{1,1,1,-1,1,\operatorname{sign}(E_6)\}$	$\{1,1,1,-1,1,\operatorname{sign}(E_6)\}$	2

In view of Theorem 1.4 we get the following table:

sign(E_6)	sign changes v	real zeros $\deg w-2v$	distinct real zeros $l-2v$
-1	3	$6-2\times 3=0$	$6-2\times 3=0$
0	2	$6-2\times 2=2$	$5-2\times 2=1$
1	2	$6-2\times 2=2$	$6-2\times 2=2$

TABLE 3

Let us consider the case when $\operatorname{sign}(E_6)=0$ if and only if $16Ab^3=27B^4$. Then polynomial $w(t)$ becomes

$$w(t)=\frac{B^8\left(16-81B^4t\right)^2h(t)}{4294967296},$$

where

$$h(t)=16777216-54079488B^4t+105815808B^8t^2- \\ -110539728B^{12}t^3+43046721B^{16}t^4.$$

In view of Table 3 if $16Ab^3=27B^4$, then $w(t)$ has one real zero so $h(t)\neq 0$, for all $t\in\mathbb{R}$, which implies $t=\frac{16}{81B^4}$ is only zero of $w(t)$. Since $\Psi^8=t$, then

$$Ab^3\Psi^8+B\Psi^2=\frac{27B^4}{16}\frac{16}{81B^4}+B\sqrt[4]{\frac{16}{81B^4}}=1.$$

By applying (2.5) and Theorem 3.3 (Φ,Ψ) is not period-two solution. From Theorems 1.5 and 1.6 we obtain that $w(t)=0$ has even number of positive zeros, so if $\operatorname{sign}(E_6)=1$ if and only if $16Ab^3>27B^4$, then $w(t)=0$ has exactly two positive zeros which implies four period-two solutions (including symmetry). Furthemore,

$$\begin{aligned}
&w(0)=B^8>0\\
&w\left(\frac{1}{3Ab^3}\right)=-\frac{1}{729}\left(16Ab^3-27B^4\right)\left(2Ab^3+27B^4\right)<0,\\
&w\left(\frac{1}{Ab^3}\right)=B^8>0.
\end{aligned}$$

This yields that $w(t)=0$ has two positive zeros in $\left(0,\frac{1}{3Ab^3}\right)$ and $\left(\frac{1}{3Ab^3},\frac{1}{Ab^3}\right)$. From (4.5) we have $\operatorname{sgn}(\Phi)=\operatorname{sgn}\left(\frac{1}{Ab^3}-t\right)$, so if $t\in\left(0,\frac{1}{Ab^3}\right)$, then $\operatorname{sgn}(\Phi)=1$ which implies there exists 2 period-two solution in first qudrant $P_3,P_5\in Q_1(E_0)$ ($P_4,P_6\in Q_3(E_0)$ including symmetry).

All this leads to the following theorem:

Theorem 4.3. *Assume that $a = 0$, $b > 0$, $A > 0$, $B > 0$.*

(i) *If $16Ab^3 > 27B^4$, then the system (1.1) has six period-two solutions $\{P_i(\Phi_i, \Psi_i)\}_{i=1}^{6}$ with $P_1 \in Q_2(E_0)$, $P_2 \in Q_4(E_0)$, $P_3, P_5 \in Q_1(E_0)$ and $P_4, P_6 \in Q_3(E_0)$.*

(ii) *If $16Ab^3 \le 27B^4$, then the system (1.1) has two period-two solutions $P_1 \in Q_2(E_0)$ and $P_2 \in Q_4(E_0)$.*

4.3. **case $a > 0$, $b > 0$, $A > 0$, $B > 0$ and $aB = bA$**

Let $\frac{a}{A} = \frac{b}{B} = k > 0$. The period-two solutions (Φ, Ψ) satisfy the system:

$$Ak\left(Ak\Phi^3 + Bk\Psi^3\right)^3 + Bk\left(A\Phi^3 + B\Psi^3\right)^3 = \Phi,$$
$$A\left(Ak\Phi^3 + Bk\Psi^3\right)^3 + B\left(A\Phi^3 + B\Psi^3\right)^3 = \Psi.$$

By multiplying the second equation with k, after subtracting the equations we obtain $\Phi = k\Psi$. Now, by substitution $\Phi = k\Psi$ in the second equation of the system given above, we have

$$\Psi\left(\Psi\sqrt{Ak^3+B} - 1\right)\left(\Psi\sqrt{Ak^3+B} + 1\right)\left(\Psi\sqrt{Ak^3+B} - 1\right)\cdot$$
$$\cdot\left(\Psi^2\left(Ak^3+B\right) + 1\right)\left(\Psi^4\left(Ak^3+B\right)^2 + 1\right) = 0$$

and $\Psi = 0$ or $\Psi = \frac{\pm 1}{\sqrt{Ak^3+B}} = \frac{\pm A}{\sqrt{a^3+A^2B}}$. Clearly, the following theorem holds:

Theorem 4.4. *If $a > 0$, $b > 0$, $A > 0$, $B > 0$ and $aB = bA$, then the system (1.1) has no minimal period-two solution.*

Since the case $a = 0$, $b > 0$, $A > 0$, $B > 0$ leads us to a very cumbersome calculation the case $a > 0$, $b > 0$, $A > 0$, $B > 0$ and $aB \neq bA$ will be omitted from our consideration.

5. Local Stability of Prime Period-two Solutions

In view of Theorem 4.2 the system (1.1) has no minimal period-two solution, thus we will consider only the case $a = 0, b > 0, A > 0, B > 0$. The Jacobian matrix of map T^2, where T is given by (2.1) for $a = 0$, evaluated at point (Φ, Ψ) is

$$J_{T^2}(\Phi, \Psi) = \begin{pmatrix} 9Ab\Phi^2\left(A\Phi^3+B\Psi^3\right)^2 & 9Bb\Psi^2\left(A\Phi^3+B\Psi^3\right)^2 \\ 9AB\Phi^2\left(A\Phi^3+B\Psi^3\right)^2 & 9Ab^3\Psi^8 + 9B^2\Psi^2\left(A\Phi^3+B\Psi^3\right)^2 \end{pmatrix} \quad (5.1)$$

The determinant and trace of (5.1) are

$$\mathcal{D} = \det\left(J_{T^2}(\Phi, \Psi)\right) = 81A^2b^4\Phi^2\Psi^8\left(A\Phi^3+B\Psi^3\right)^2 > 0,$$

and

$$\mathcal{S} = \operatorname{tr}\left(J_{T^2}(\Phi, \Psi)\right) = 9Ab^3\Psi^8 + 9\left(Ab\Phi^2 + B^2\Psi^2\right)\left(A\Phi^3+B\Psi^3\right)^2 > 0.$$

By using the fact $\Phi = \frac{b}{B}\Psi\left(1-Ab^3\Psi^8\right)$ and seting $\Psi^8 = t > 0$ we get

$$\begin{aligned}
\mathcal{D}(t) &= \frac{81A^2b^6}{B^8}t^2\left(1-Ab^3t\right)^2\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)^2,\\
\mathcal{S}(t) &= \frac{9t}{B^8}\left(Ab^3B^8+B^4\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)^2\right)\\
&\quad+\frac{9t}{B^8}\left(Ab^3\left(1-Ab^3t\right)^2\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)^2\right).
\end{aligned}$$

Moreover, from (4.6) if $u(t)=0$, we have

$$u(t) = \frac{t}{B^8}\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)^3+Ab^3t-1=0,$$

and

$$\frac{t}{B^8}\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)^2 = \frac{1-Ab^3t}{B^4+Ab^3\left(1-Ab^3t\right)^3},$$

so

$$\begin{aligned}
\mathcal{D}(t) &= \frac{81A^2b^6t\left(1-Ab^3t\right)^3}{B^4+Ab^3\left(1-Ab^3t\right)^3},\\
\mathcal{S}(t) &= 9Ab^3t+9\left(1-Ab^3t\right)\frac{B^4+Ab^3\left(1-Ab^3t\right)^2}{B^4+Ab^3\left(1-Ab^3t\right)^3},
\end{aligned}$$

and

$$9\left(\mathcal{S}(t)-\mathcal{D}(t)-1\right) = 8\left(9-\mathcal{D}(t)\right). \tag{5.2}$$

The following lemma holds:

Lemma 5.1. *Let (Φ,Ψ) be a minimal period-two solution of the system (1.1), then any period-two solution is hyperbolic, a repeler or a saddle point.*

(a) *If $16Ab^3 > 27B^4$, then the system (1.1) has six period-two solutions $\{P_i(\Phi_i,\Psi_i)\}_{i=1}^6$ with $P_1 \in Q_2(E_0)$ and $P_2 \in Q_4(E_0)$ are repellers, $P_3,P_5 \in Q_1(E_0)$ and $P_4,P_6 \in Q_3(E_0)$ are saddle points.*

(b) *If $16Ab^3 \le 27B^4$, then the system (1.1) has two period-two solutions $P_1 \in Q_2(E_0)$ and $P_2 \in Q_4(E_0)$ and they are repellers.*

Proof. Assume that t_0 is a positive zero of $u(t)$ which implies period-two solution. In view of Theorem 4.3 the system (1.1) always has minimal period-two solution. Let us consider the case when $\mathcal{D}(t_0) \le 1$. If $t_0 \in \left(0,\frac{1}{Ab^3}\right)$ if and only if $1-Ab^3t_0 \in (0,1)$, then $\left(1-Ab^3t_0\right)^3 < \left(1-Ab^3t_0\right)^2$ and

$$\begin{aligned}\mathcal{S}(t_0) &= 9Ab^3t_0 + 9\left(1-Ab^3t_0\right)\frac{B^4+Ab^3\left(1-Ab^3t_0\right)^2}{B^4+Ab^3\left(1-Ab^3t_0\right)^3} \\ &> 9Ab^3t_0 + 9\left(1-Ab^3t_0\right)\frac{B^4+Ab^3\left(1-Ab^3t_0\right)^2}{B^4+Ab^3\left(1-Ab^3t_0\right)^2} \\ &= 9Ab^3t_0 + 9\left(1-Ab^3t_0\right) = 9.\end{aligned}$$

Now, $\mathcal{S}(t_0) > 9 = 1+8 > 1+\mathcal{D}(t_0)$ which implies if the period-two solution exists it is a saddle point. We have seen that if $16Ab^3 > 27B^4$, $w(t_0) = 0$ and $t_0 \in \left(0, \frac{1}{Ab^3}\right)$, where $w(t)$ is defined by (4.8), then there exist 2 period-two solutions in first qudrant $P_3, P_5 \in Q_1(E_0)$ ($P_4, P_6 \in Q_3(E_0)$ including symmetry). Hence, if exist, period-two solutions $P_3, P_5 \in Q_1(E_0)$ are saddle points. Since $\mathcal{D}(t) > 0$ for all $t > 0$, then $\operatorname{sgn}\left(1-Ab^3t\right) = \operatorname{sgn}\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)$. If $t_0 > \frac{1}{Ab^3}$ if and only if $1-Ab^3t_0 < 0$, then

$$B^4+Ab^3\left(1-Ab^3t_0\right)^3 < 0$$

and from $\mathcal{D}(t_0) \le 1$ we get

$$\begin{aligned}B^4+Ab^3\left(1-Ab^3t_0\right)^3 &\le 81A^2b^6t_0\left(1-Ab^3t_0\right)^3. \\ B^4 &\le 81A^2b^6t_0\left(1-Ab^3t_0\right)^3 - Ab^3\left(1-Ab^3t_0\right)^3 \\ 0 < B^4 &\le Ab^3\left(1-Ab^3t_0\right)^3\left(81Ab^3t_0-1\right) \Rightarrow \\ 81Ab^3t_0 - 1 < 0 &\Leftrightarrow t_0 < \frac{1}{81Ab^3}\end{aligned}$$

Thus $\frac{1}{Ab^3} < t_0 < \frac{1}{81Ab^3}$, which is impossible. Furthermore, we have obtained that if $v(t_0) = 0$ and $t_0 \in \left(\frac{1}{Ab^3}, +\infty\right)$, where $v(t)$ is defined by (4.7), then there exist period-two solution in the second and fourth qudrant $P_1 \in Q_2(E_0)$ and $P_2 \in Q_4(E_0)$. Now, if $t_0 > \frac{1}{Ab^3}$ if and only if $1-Ab^3t_0 < 0$, then $9Ab^3t_0 - 1 > 0$ and

$$\begin{aligned}Ab^3\left(1-Ab^3t_0\right)^3\left(9Ab^3t_0-1\right) &< 0 < B^4, \\ 9A^2b^6t_0\left(1-Ab^3t_0\right)^3 &< B^4+Ab^3\left(1-Ab^3t_0\right)^3.\end{aligned}$$

Since $\operatorname{sgn}\left(1-Ab^3t\right) = \operatorname{sgn}\left(B^4+Ab^3\left(1-Ab^3t\right)^3\right)$, so it holds

$$B^4+Ab^3\left(1-Ab^3t_0\right)^3 < 0.$$

Hence

$$\frac{9A^2b^6t_0\left(1-Ab^3t_0\right)^3}{B^4+Ab^3\left(1-Ab^3t_0\right)^3} > 1 \Leftrightarrow \mathcal{D}(t_0) > 9.$$

From (5.2) yields $\mathcal{S}(t_0) - \mathcal{D}(t_0) - 1 < 0$. By applying Theorem 1.3, the minimal period-two solution $\{P_1, P_2\}$ is repeller. □

6. Global Behavior

6.1. case $b=0$, $a>0$, $A>0$, $B>0$

Lemma 6.1. *Assume that $b=0$, $a>0$, $A>0$, $B>0$. Let T be the function defined by (2.1). The following statements hold:*

(i) *The sets $\mathcal{S}=\left\{(0,y): y\in\left(-\frac{\sqrt{B}}{B},\frac{\sqrt{B}}{B}\right)\right\}$, $\mathcal{S}^u=\left\{(0,y): y>\frac{\sqrt{B}}{B}\right\}$, $\mathcal{S}^d=\left\{(0,y): y<-\frac{\sqrt{B}}{B}\right\}$ are invariant sets under the function T.*

(ii) *If $27A^2B>4a^3$, then the sets $\mathcal{S}_1=\left\{\left(\frac{\sqrt{a}}{a},y\right): y\in\mathbb{R}\right\}$ and $\mathcal{S}_2=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y\in\mathbb{R}\right\}$ are invariant sets under function T,*

(iii) *If $27A^2B<4a^3$, then the sets $\mathcal{S}_+=\left\{\left(\frac{\sqrt{a}}{a},y\right): y\in(\overline{y}_1,\overline{y}_3)\right\}$, $\mathcal{S}_+^u=\left\{\left(\frac{\sqrt{a}}{a},y\right): y>\overline{y}_3\right\}$, $\mathcal{S}_+^d=\left\{\left(\frac{\sqrt{a}}{a},y\right): y<\overline{y}_1\right\}$ are invariant sets under the function T and $\overline{y}_1<\overline{y}_2<\overline{y}_3$ are three different real solutions of Eq.(2.3).*

(iv) *If $27A^2B<4a^3$, then the sets $\mathcal{S}_-=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y\in(\overline{y}_1^*,\overline{y}_3^*)\right\}$, $\mathcal{S}_-^u=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y>\overline{y}_3^*\right\}$, $\mathcal{S}_-^d=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y<\overline{y}_1^*\right\}$ are invariant sets under function T and $\overline{y}_1^*<\overline{y}_2^*<\overline{y}_3^*$ are three different real solutions of Eq.(2.4).*

(v) *If $27A^2B=4a^3$, then the sets $\mathcal{S}_+=\left\{\left(\frac{\sqrt{a}}{a},y\right): y\in\left(\overline{y},\frac{1}{\sqrt{3B}}\right)\right\}$, $\mathcal{S}_+^u=\left\{\left(\frac{\sqrt{a}}{a},y\right): y>\frac{1}{\sqrt{3B}}\right\}$ and $\mathcal{S}_+^d=\left\{\left(\frac{\sqrt{a}}{a},y\right): y<\overline{y}\right\}$ are invariant sets under the function T and $\overline{y}$ is the only negative solution of Eq.(2.3).*

(vi) *If $27A^2B=4a^3$, then the sets $\mathcal{S}_-=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y\in\left(-\frac{1}{\sqrt{3B}},\overline{y}^*\right)\right\}$, $\mathcal{S}_-^u=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y>\overline{y}^*\right\}$ and $\mathcal{S}_-^d=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y<-\frac{1}{\sqrt{3B}}\right\}$ are invariant sets under the function T and $\overline{y}^*$ is the only positive solution of Eq.(2.4).*

Proof. Indeed,

(i) If $x=0$ then $T(0,y)=\left(0,By^3\right)$. Obviously, By^3 is increasing function and $-\frac{\sqrt{B}}{B}$, 0 and $\frac{\sqrt{B}}{B}$ are the fixed points of the function By^3. Now, if $y\in\left(-\frac{\sqrt{B}}{B},\frac{\sqrt{B}}{B}\right)$ then $By^3\in\left(-\frac{\sqrt{B}}{B},\frac{\sqrt{B}}{B}\right)$ and $T(0,y)\in\mathcal{S}$. Similarly, if $y>\frac{\sqrt{B}}{B}$ then $By^3>\frac{\sqrt{B}}{B}$ and $T(0,y)\in\mathcal{S}^u$. If $y<-\frac{\sqrt{B}}{B}$ then $By^3<-\frac{\sqrt{B}}{B}$ and $T(0,y)\in\mathcal{S}^d$.

(ii) If $x=\frac{\sqrt{a}}{a}$ then $T\left(\frac{\sqrt{a}}{a},y\right)=\left(\frac{\sqrt{a}}{a},\frac{A\sqrt{a}}{a^2}+By^3\right)\in\mathcal{S}_1$ and if $x=-\frac{\sqrt{a}}{a}$ then $T\left(-\frac{\sqrt{a}}{a},y\right)=\left(\frac{\sqrt{a}}{a},\frac{A\sqrt{a}}{a^2}+By^3\right)\in\mathcal{S}_2$.

(iii) If $27A^2B<4a^3$, then Eq.(2.3) has three different real solutions of Eq.(2.3) $\overline{y}_1\in\left(-\infty,-\frac{1}{\sqrt{3B}}\right)$, $\overline{y}_2\in\left(0,\frac{1}{\sqrt{3B}}\right)$, $\overline{y}_3\in\left(\frac{1}{\sqrt{3B}},+\infty\right)$. If $x=\frac{\sqrt{a}}{a}$ and $\alpha\in(\overline{y}_1,\overline{y}_3)$, then $T\left(\frac{\sqrt{a}}{a},\alpha\right)=\left(\frac{\sqrt{a}}{a},\frac{A\sqrt{a}}{a^2}+B\alpha^3\right)$. Set $u(y)=By^3+\frac{A\sqrt{a}}{a^2}$. Clearly, $u(y)$ is increasing function and $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$ are the fixed points of the function $u(y)$. If $\alpha\in(\bar{y}_1,\bar{y}_2)$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}\in(u(\bar{y}_1),u(\bar{y}_2))=(\bar{y}_1,\bar{y}_2)\subset(\bar{y}_1,\bar{y}_3)$. If $\alpha\in(\bar{y}_2,\bar{y}_3)$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}\in(u(\bar{y}_2),u(\bar{y}_3))=(\bar{y}_2,\bar{y}_3)\subset(\bar{y}_1,\bar{y}_3)$. Hence, $T\left(\frac{\sqrt{a}}{a},u(\alpha)\right)\in\mathcal{S}_+$. If $\alpha\in(\bar{y}_3,+\infty)$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}>u(\bar{y}_3)=\bar{y}_3$ and $u(\alpha)\in(\bar{y}_3,+\infty)$. Hence, $T\left(\frac{\sqrt{a}}{a},u(\alpha)\right)\in\mathcal{S}_+^u$. If $\alpha\in(-\infty,\bar{y}_1)$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}<u(\bar{y}_1)=\bar{y}_1$ and $u(\alpha)\in(-\infty,\bar{y}_1)$. Hence, $T\left(\frac{\sqrt{a}}{a},u(\alpha)\right)\in\mathcal{S}_+^d$.

(iv) This statement follows from the symmetry of the first and third quadrant and the second and fourth quadrant and Lemma 6.1 (iii).

(v) If $27A^2B=4a^3$, then Eq.(2.3) has two different real solutions $\bar{y}\in\left(-\infty,-\frac{1}{\sqrt{3B}}\right)$ and $\bar{y}_2=\frac{1}{\sqrt{3B}}$. If $x=\frac{\sqrt{a}}{a}$ and $\alpha\in\left(\bar{y},\frac{1}{\sqrt{3B}}\right)$, then $T\left(\frac{\sqrt{a}}{a},\alpha\right)=\left(\frac{\sqrt{a}}{a},\frac{A\sqrt{a}}{a^2}+B\alpha^3\right)$. Set $u(y)=By^3+\frac{A\sqrt{a}}{a^2}$. Clearly, $u(y)$ is increasing function and $\bar{y}$ and $\bar{y}_2$ are the fixed points of the function $u(y)$. If $\alpha\in\left(\bar{y},\frac{1}{\sqrt{3B}}\right)$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}\in\left(u(\bar{y}),u\left(\frac{1}{\sqrt{3B}}\right)\right)=\left(\bar{y},\frac{1}{\sqrt{3B}}\right)$. Hence, $T\left(\frac{\sqrt{a}}{a},u(\alpha)\right)\in\mathcal{S}_+$. If $\alpha\in\left(\frac{1}{\sqrt{3B}},+\infty\right)$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}>u\left(\frac{1}{\sqrt{3B}}\right)=\frac{1}{\sqrt{3B}}$. Hence, $T\left(\frac{\sqrt{a}}{a},u(\alpha)\right)\in\mathcal{S}_+^u$. If $\alpha\in(-\infty,\bar{y})$, then $u(\alpha)=B\alpha^3+\frac{A\sqrt{a}}{a^2}<u(\bar{y})=\bar{y}$. Hence, $T\left(\frac{\sqrt{a}}{a},u(\alpha)\right)\in\mathcal{S}_+^d$.

(vi) This statement is following from the symmetry of the first and third quadrant and the second and fourth quadrant and Lemma 6.1 (v). □

Let $\mathcal{B}(E_0)$ be the basin of attraction of E_0 and $\mathcal{B}(\infty,\infty)$ be the basin of attraction of (∞,∞). The following lemma is true.

Lemma 6.2. *Let T be the function defined by* (2.1). *For the nonhyperbolic point $E_{12}\left(\frac{\sqrt{a}}{a},\frac{1}{\sqrt{3B}}\right)$ the following statements hold:*

(i) *If $Q_1(E_{12})=\left\{(x,y):x>\frac{\sqrt{a}}{a}\wedge y>\frac{1}{\sqrt{3B}}\right\}$, then $int(Q_1(E_{12}))\subset\mathcal{B}(\infty,\infty)$.*

(ii) *If $Q_3(E_{12})=\left\{(x,y):0<x<\frac{\sqrt{a}}{a}\wedge 0<y<\frac{1}{\sqrt{3B}}\right\}$, then $int(Q_3(E_{12}))\subset\mathcal{B}(E_0)$.*

Since we have the symmetry of the first and third quadrant a similar statement holds for nonhyperbolic point E_{14}.

Proof. Assume that $(x_0,y_0)\in int(Q_3(E_{12}))$. By Theorem 6 in [10] there exists $(\widetilde{x}_0,\widetilde{y}_0)\in int(Q_3(E_{12}))$ such that $(x_0,y_0)\preccurlyeq_{ne}(\widetilde{x}_0,\widetilde{y}_0)$, $E_0\preccurlyeq_{ne}(\widetilde{x}_0,\widetilde{y}_0)$ and $T(\widetilde{x}_0,\widetilde{y}_0)\preccurlyeq_{ne}(\widetilde{x}_0,\widetilde{y}_0)$. By monotonicity of T we obtain $T^{i+1}(\widetilde{x}_0,\widetilde{y}_0)\preccurlyeq_{ne}T^i(\widetilde{x}_0,\widetilde{y}_0)\preccurlyeq_{ne}E_{12}$ which implies $T^n(\widetilde{x}_0,\widetilde{y}_0)\to E_0$ as $n\to\infty$. Similarly, one can prove $int(Q_1(E_{12}))\subset\mathcal{B}(\infty,\infty)$. □

Also, one can show analogue statements for nonhyperbolic point E_{14}.

The following theorem describes global behavior of the system (1.1) when $b=0$, $a>0$, $A>0$, $B>0$.

Theorem 6.1. *Assume that $b=0$, $a>0$, $A>0$, $B>0$. Then $E_0(0,0)$ is locally asymptotically stable, $E_1\left(0,\frac{\sqrt{B}}{B}\right)$ and $E_2\left(0,-\frac{\sqrt{B}}{B}\right)$ are the saddle points. In this case there exist continuous curves $\mathcal{W}^s(E_1)$, $\mathcal{W}^u(E_1)$ and $\mathcal{W}^s(E_2)$, $\mathcal{W}^u(E_1)$, where $\mathcal{W}^s(E_1)$ is passing trough the point E_1 and $\mathcal{W}^s(E_2)$ is passing trough the point E_2 and they are the graphs of decreasing functions, $\mathcal{W}^u(E_1)=\left\{(0,y): y>\frac{\sqrt{B}}{B}\right\}$ and $\mathcal{W}^u(E_2)=\left\{(0,y): y<-\frac{\sqrt{B}}{B}\right\}$. The basin of attraction of the point E_1 is $\mathcal{B}(E_1)=\mathcal{W}^s(E_1)$ and the point E_2 is $\mathcal{B}(E_2)=\mathcal{W}^s(E_2)$.*

(i) *If $27A^2B>4a^3$ then there exist equilibrium points $E_3\left(\frac{\sqrt{a}}{a},\bar{y}\right)$ and $E_4\left(-\frac{\sqrt{a}}{a},-\bar{y}\right)$ and they are the repellers, where $\bar{y}$ is the unique negative solution of Eq.(2.3). The points E_3 and E_4 are the endpoints of the curves $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$. The region between $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$ is invariant and the basin of attraction $\mathcal{B}(E_0)$ is precisely the region between $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$. For every (x_0,y_0) where $|x_0|<\frac{\sqrt{a}}{a}$ such that point (x_0,y_0) is above $\mathcal{W}^s(E_1)$ every solution is asymptotic to $\mathcal{W}^u(E_1)=\left\{(0,y): y\in\left(0,\frac{\sqrt{B}}{B}\right)\right\}\cup\mathcal{S}^u$. For every (x_0,y_0) where $|x_0|<\frac{\sqrt{a}}{a}$ such that point (x_0,y_0) is below $\mathcal{W}^s(E_2)$ every solution is asymptotic to $\mathcal{W}^u(E_2)=\left\{(0,y): y\in\left(-\frac{\sqrt{B}}{B},0\right)\right\}\cup\mathcal{S}^d$. For every (x_0,y_0) where $|x_0|>\frac{\sqrt{a}}{a}$ every solution $\{(x_n,y_n)\}$ goes to the point at infinity. (see Figure 7)*

(ii) *If $27A^2B<4a^3$, the following holds for equilibrium points: $E_5\left(\frac{\sqrt{a}}{a},\bar{y}_1\right)$ $E_7\left(\frac{\sqrt{a}}{a},\bar{y}_3\right)$, $E_8\left(-\frac{\sqrt{a}}{a},-\bar{y}_3\right)$, $E_{10}\left(-\frac{\sqrt{a}}{a},-\bar{y}_1\right)$ are repellers and $E_6\left(\frac{\sqrt{a}}{a},\bar{y}_2\right)$, $E_9\left(-\frac{\sqrt{a}}{a},-\bar{y}_2\right)$ are saddle points, where $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$ are three different real solutions of Eq.(2.3) $\bar{y}_1=\bar{y}\in\left(-\infty,-\frac{1}{\sqrt{3B}}\right)$, $\bar{y}_2\in\left(0,\frac{1}{\sqrt{3B}}\right)$, $\bar{y}_3\in\left(\frac{1}{\sqrt{3B}},+\infty\right)$. In this case there exist continuous curves $\mathcal{W}^s(E_6)$, $\mathcal{W}^u(E_6)$ and $\mathcal{W}^s(E_9)$, $\mathcal{W}^u(E_9)$, where $\mathcal{W}^u(E_6)$ is passing trough the point E_6 and $\mathcal{W}^u(E_9)$ is passing trough the point E_9 and they are the graphs of increasing functions which starting at E_0,*

$$\mathcal{W}^s(E_6)=\left\{\left(\frac{\sqrt{a}}{a},y\right): y\in(\bar{y}_1,\bar{y}_3)\right\},$$

$$\mathcal{W}^s(E_9)=\left\{\left(-\frac{\sqrt{a}}{a},y\right): y\in(-\bar{y}_3,-\bar{y}_1)\right\}.$$

The region between $\mathcal{W}^s(E_1)$, $\mathcal{W}^s(E_2)$, $\mathcal{W}^s(E_6)$ and $\mathcal{W}^s(E_9)$ is invariant and it is the basin of attraction $\mathcal{B}(E_0)$. For every (x_0,y_0) where $|x_0| < \frac{\sqrt{a}}{a}$ such that point (x_0,y_0) is above $\mathcal{W}^s(E_1)$ every solution is asymptotic to $\mathcal{W}^u(E_1)$. For every (x_0,y_0) where $|x_0| < \frac{\sqrt{a}}{a}$ such that point (x_0,y_0) is below $\mathcal{W}^s(E_2)$ every solution is asymptotic to $\mathcal{W}^u(E_2)$. For every (x_0,y_0) where $|x_0| > \frac{\sqrt{a}}{a}$ every solution $\{(x_n,y_n)\}$ goes to the point at infinity. (see Figure 8)

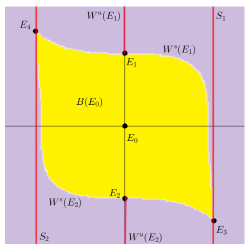

FIGURE 7

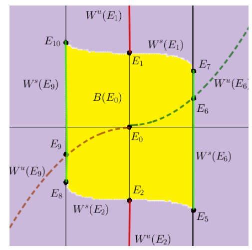

FIGURE 8

(iii) *If $27A^2B = 4a^3$ then there exist equilibrium points $E_{11}\left(\frac{\sqrt{a}}{a}, \bar{y}\right)$, $E_{13}\left(-\frac{\sqrt{a}}{a}, -\bar{y}\right)$ are the repellers and $E_{12}\left(\frac{\sqrt{a}}{a}, \frac{1}{\sqrt{3B}}\right)$, $E_{14}\left(-\frac{\sqrt{a}}{a}, -\frac{1}{\sqrt{3B}}\right)$ are the nonhyperbolic points, where $\bar{y}$ is the negative solution of Eq.(2.3). The points E_{12} and E_{13} are the endpoints of the curves $\mathcal{W}^s(E_1)$ and points E_{11} and E_{14} are the endpoints of the curves $\mathcal{W}^s(E_2)$. In this case there exist continuous curves $\mathcal{W}^u(E_{12})$, $\mathcal{W}^s(E_{12})$ and $\mathcal{W}^s(E_{14})$, $\mathcal{W}^u(E_{14})$, where $\mathcal{W}^u(E_{12})$ is passing trough the point E_{12} and $\mathcal{W}^u(E_{14})$ is passing trough the point E_{14} and they are the graphs of increasing functions which starting at E_0,*

$$\mathcal{W}^s(E_{12}) = \left\{\left(\frac{\sqrt{a}}{a}, y\right) : y \in \left(\bar{y}, \frac{1}{\sqrt{3B}}\right)\right\},$$
$$\mathcal{W}^s(E_{14}) = \left\{\left(-\frac{\sqrt{a}}{a}, y\right) : y \in \left(-\frac{1}{\sqrt{3B}}, -\bar{y}\right)\right\}.$$

The region between $\mathcal{W}^s(E_1)$, $\mathcal{W}^s(E_2)$, $\mathcal{W}^s(E_{12})$ and $\mathcal{W}^s(E_{14})$ is invariant and it is the basin of attraction $\mathcal{B}(E_0)$. For every (x_0,y_0) where $|x_0| < \frac{\sqrt{a}}{a}$ such that point (x_0,y_0) is above $\mathcal{W}^s(E_1)$ every solution is asymptotic to $\mathcal{W}^u(E_1)$. For every (x_0,y_0) where $|x_0| < \frac{\sqrt{a}}{a}$ such that point (x_0,y_0) is below $\mathcal{W}^s(E_2)$ every solution is asymptotic to $\mathcal{W}^u(E_2)$. For every (x_0,y_0) where $|x_0| > \frac{\sqrt{a}}{a}$ every solution $\{(x_n,y_n)\}$ goes to the point at infinity. (see Figure 9)

Proof. Existence and local stability of all equilibrium points follows from Theorems 2.1 and 3.2. In view of Theorem 4.2 system (1.1) has no minimal period-two solution. All conditions of Theorem 1.2 are satisfied with respect to two saddle equilibrium points (period-two solution clearly does not exist), which guarantee

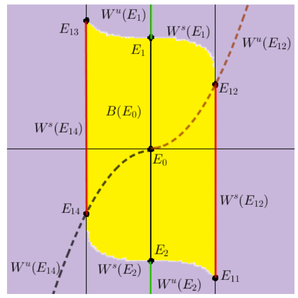

FIGURE 9

the existence of two stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$. The basin of attraction $\mathcal{B}(E_0)$ of E_0 is the region between the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$, where $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ are the graphs of a strictly decreasing continuous functions of the first coordinate on an interval. The basins of attraction $\mathcal{B}(E_1)=\mathcal{W}^s(E_1)$ and $\mathcal{B}(E_2)=\mathcal{W}^s(E_2)$ are exactly the global stable manifolds of E_1 and E_2. By applying Theorem 1.1 there exist two unstable manifolds $\mathcal{W}^u(E_1)$ and $\mathcal{W}^u(E_2)$ passing trough the points E_1 and E_2, respectively. From Lemma 6.1 (i) the sets $\mathcal{S}$, $\mathcal{S}^u$, $\mathcal{S}^d$ are invariant sets under the function T, which implies $\mathcal{W}^u(E_1)=\left\{(0,y):y\in\left(0,\frac{\sqrt{B}}{B}\right)\right\}\cup\mathcal{S}^u$ and $\mathcal{W}^u(E_2)=\left\{(0,y):y\in\left(-\frac{\sqrt{B}}{B},0\right)\right\}\cup\mathcal{S}^d$.

(i) Let T be the function defined by (2.1), then corresponding function f and g are increasing in both variables,which implies $Q_1(E_3)$, $Q_3(E_3)$, $Q_1(E_4)$ and $Q_3(E_4)$ are invariant sets under function T. This yields $\mathcal{B}(E_0)\subset Q_4(E_3)\cap Q_2(E_4)$ and set $\mathcal{B}(E_0)$ is bounded. In view of Lemma 6.1 (ii) sets $\mathcal{S}_1=\left\{\left(\frac{\sqrt{a}}{a},y\right):y\in\mathbb{R}\right\}$ and $\mathcal{S}_2=\left\{\left(-\frac{\sqrt{a}}{a},y\right):y\in\mathbb{R}\right\}$ are invariant sets under function T. Since the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$ are decreasing continuous functions and $\mathcal{B}(E_0)=\{(x,y):\exists y_u,y_l: y_l<y<y_u,\ (x,y_l)\in\mathcal{W}^s(E_1),(x,y_u)\in\mathcal{W}^s(E_2)\}$, then the points E_3 and E_4 are endpoints of the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$. The rest of the proof follows from Theorems 1.1 and 1.2.

(ii) Similar to the previous case we obtain that the points E_7, E_{10} are endpoints of the global stable manifold $\mathcal{W}^s(E_1)$ and E_5, E_8 are endpoints of the global stable manifold $\mathcal{W}^s(E_2)$. In view of Lemma 6.1 (iii) and (iv) sets $\mathcal{S}_+=\left\{\left(\frac{\sqrt{a}}{a},y\right):y\in(\bar{y}_1,\bar{y}_3)\right\}$, $\mathcal{S}_+^u=\left\{\left(\frac{\sqrt{a}}{a},y\right):y>\bar{y}_3\right\}$, $\mathcal{S}_+^d=\left\{\left(\frac{\sqrt{a}}{a},y\right):y<\bar{y}_1\right\}$, $\mathcal{S}_-=\left\{\left(-\frac{\sqrt{a}}{a},y\right):y\in(\bar{y}_1^*,\bar{y}_3^*)\right\}$, $\mathcal{S}_-^u=\left\{\left(-\frac{\sqrt{a}}{a},y\right):y>\bar{y}_3^*\right\}$, $\mathcal{S}_-^d=\left\{\left(-\frac{\sqrt{a}}{a},y\right):y<\bar{y}_1^*\right\}$ are invariant sets under the function T, thus $\mathcal{W}^s(E_6)=\left\{\left(\frac{\sqrt{a}}{a},y\right):y\in(\bar{y}_1,\bar{y}_3)\right\}$ and $\mathcal{W}^s(E_9)=\left\{\left(-\frac{\sqrt{a}}{a},y\right):y\in(-\bar{y}_3,-\bar{y}_1)\right\}$. The rest of the proof follows from Theorems 1.1 and 1.2.

(iii) By using Lemma 6.2 the rest of the proof of this case is similar to the proof of (i) and (ii) and will be omitted. □

6.2. **case** $a=0$**,** $b>0$**,** $A>0$**,** $B>0$

Let $\mathcal{U}_1$ denote the boundary of $\mathcal{B}(E_0)$ considered as a subset of $Q_2(E_2)$ in the first quadrant and $\mathcal{U}_2$ denote the boundary of $\mathcal{B}(E_0)$ considered as a subset of $Q_4(E_2)$ in the first quadrant. It is easy to see that $E_2 \in \mathcal{U}_1, \mathcal{U}_2$. The proof of the following lemma for a cooperative map is the same as the proof of Claims 1 and 2 in [7] for a competitive map, so we skip it.

Lemma 6.3. *Assume that* $a=0$*,* $b>0$*,* $A>0$*,* $B>0$*. Let* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *be the sets defined as above, then:*

(a) *If* $(x_0,y_0) \in \mathcal{B}(E_0)$*, then* $(x_1,y_1) \in \mathcal{B}(E_0)$ *for all* $(x_1,y_1) \preccurlyeq_{ne} (x_0,y_0)$.
(b) *If* $(x_0,y_0) \in \mathcal{U}_1 \cup \mathcal{U}_2$*, then* $(x_1,y_1) \in int(\mathcal{B}(E_0))$ *for all* $(x_1,y_1) \preccurlyeq_{ne} (x_0,y_0)$.
(c) $\mathcal{U}_1 \cap Q_2(E_2) \neq \emptyset$ *and* $\mathcal{U}_2 \cap Q_4(E_2) \neq \emptyset$.
(d) $T(\mathcal{U}_1 \cup \mathcal{U}_2) \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$.
(e) $(x_1,y_1),(x_0,y_0) \in \mathcal{U}_1 \cup \mathcal{U}_2$ *implies* $(x_0,y_0) \preccurlyeq_{se} (x_1,y_1)$ *or* $(x_1,y_1) \preccurlyeq_{se} (x_0,y_0)$.
(f) $\mathcal{U}_1 \cup \mathcal{U}_2$ *is the graph of continuous strictly decreasing function.*

Lemma 6.4. *Assume that* $a=0$*,* $b>0$*,* $A>0$*,* $B>0$ *and* $16Ab^3>27B^4$*. The minimal period-two solution* $\{P_3,P_5\}$ *is a saddle point, such that* $P_3 \preceq_{se} E_2 \preceq_{se} P_5$.

Proof. By applying Theorem 3.3 (i) equilibrium point E_2 is a repeller and by Lemma 5.1 all period-two solutions are hyperbolic and $\{P_3,P_5\}$ is a saddle point. In view of Lemma 6.3 we see that $(\mathcal{U}_1 \cup \mathcal{U}_2, \preccurlyeq_{se})$ is a totally ordered set, which is invariant under T. If $(x_0,y_0) \in (\mathcal{U}_1 \cup \mathcal{U}_2) \setminus \{E_2\}$, then $\{T^{2n}(x_0,y_0)\}$ is eventually componentwise monotone. Since $\mathcal{U}_1 \cup \mathcal{U}_2$ is the graph of continuous strictly decreasing function, there exists a minimal period-two solution $\{(\Phi,\Psi),(\Psi,\Phi)\}$ such that $T^{2n}(x_0,y_0) \to (\Phi,\Psi)$ as $n\to\infty$, so $\{P_3(\Phi,\Psi),P_5(\Psi,\Phi)\}$. Since $\mathcal{U}_1 \cup \mathcal{U}_2 \subset \partial\mathcal{B}(E_0)$ is a closed set, we see that $\{(\Phi,\Psi),(\Psi,\Phi)\} \in (\mathcal{U}_1 \cup \mathcal{U}_2) \setminus \{E_2\}$. Hence,

$$P_3 \preceq_{se} E_2 \preceq_{se} P_5 \text{ and } P_6 \preceq_{se} E_1 \preceq_{se} P_4.$$ □

Lemma 6.5. *Assume that* $a=0$*,* $b>0$*,* $A>0$*,* $B>0$*. The minimal period-two solution* $\{P_1,P_2\}$ *is a repeller.*

(a) *If* $16Ab^3 \leq 27B^4$*, then* $P_1 \preceq_{se} E_1 \preceq_{se} P_2$*,* $P_1 \preceq_{se} E_2 \preceq_{se} P_2$.
(b) *If* $16Ab^3 > 27B^4$*, then* $P_1 \preceq_{se} P_3 \preceq_{se} E_2 \preceq_{se} P_5 \preceq_{se} P_2$ *and* $P_1 \preceq_{se} P_6 \preceq_{se} E_1 \preceq_{se} P_4 \preceq_{se} P_2$.

Proof. In view of Lemma 5.1 the minimal period-two solution $\{P_1,P_2\}$ is a repeller. By applying Lemma 6.4 we we already have $P_3 \preceq_{se} E_2 \preceq_{se} P_5$ and $P_6 \preceq_{se} E_1 \preceq_{se} P_4$.

(a) Let T^2 be the function defined by (4.1), then corresponding function f and g are increasing in both variables,which implies $Q_1(P_1)$, $Q_3(P_1)$, $Q_1(P_2)$ and $Q_3(P_2)$ are invariant sets under function T^2. This yields $\mathcal{B}(E_0) \subset Q_4(P_1) \cap$

$Q_2(P_2)$ and set $\mathcal{B}(E_0)$ is bounded. The boundary of set $\mathcal{B}(E_0)$ are the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$ and they are decreasing continuous functions with
$\mathcal{B}(E_0) = \{(x,y) : \exists y_u, y_l : \ y_l < y < y_u, \ (x,y_l) \in \ \mathcal{W}^s(E_1), (x,y_u) \in \ \mathcal{W}^s(E_2)\}$, then the points P_1 and P_2 are endpoints of the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_2)$, which yields $P_1 \preceq_{se} E_1 \preceq_{se} P_2$ and $P_1 \preceq_{se} E_2 \preceq_{se} P_2$.

(b) By Lemma 5.1, the period-two solutions $\{P_3, P_5\}$ and $\{P_3, P_5\}$ are saddle points. Since $\mathcal{B}(E_0) \subset Q_4(E_3) \cap Q_2(E_4)$, then set $\mathcal{B}(E_0)$ is bounded and the boundary of set $\mathcal{B}(E_0)$ is $\mathcal{U}_1 \cup \ \mathcal{U}_2 \cup \{E_1, E_2\}$, where are $\mathcal{U}_1 = \mathcal{W}^s(P_3) \cup \mathcal{W}^s(P_5)$ and $\mathcal{U}_2 = \mathcal{W}^s(P_4) \cup \mathcal{W}^s(P_6)$ the global stable manifolds of points P_1, P_2, P_3 and P_4. Clearly, $\mathcal{U}_1$ and $\mathcal{U}_2$ are decreasing continuous functions with $\mathcal{B}(E_0) = \{(x,y) : \exists y_u, y_l : y_l < y < y_u, \ (x,y_l) \in \mathcal{U}_1 \cup \{E_2\}, (x,y_u) \in \mathcal{U}_2 \cup \{E_1\}\}$, which implies then the points P_1 and P_2 are endpoints of the sets $\mathcal{U}_1$ and $\mathcal{U}_2$. Hence,

$$P_1 \preceq_{se} P_3 \preceq_{se} E_2 \preceq_{se} P_5 \preceq_{se} P_2,$$
$$P_1 \preceq_{se} P_6 \preceq_{se} E_1 \preceq_{se} P_4 \preceq_{se} P_2. \qquad \square$$

The following theorem describes the global behavior of the system (1.1) when $a = 0, b > 0, A > 0, B > 0$.

Theorem 6.2. *Assume that $a = 0$, $a > 0$, $A > 0$, $B > 0$. Then the system (1.1) has exactly three equilibrium points $E_0(0,0)$, $E_1\left(b\overline{y}_-^3, \overline{y}_-\right)$ and $E_2\left(b\overline{y}_+^3, \overline{y}_+\right)$ where $\overline{y}_-$ and $\overline{y}_+$ are symmetric solutions of Eq.(2.5).The equilibrium point $E_0(0,0)$ is locally asymptotically stable.*

(i) *If $16Ab^3 > 27B^4$, then E_1 and E_2 are repellers and there exist six minimal period-two solutions. The set $int(Q_2(E_2)) \cup int(Q_4(E_2))$ contains even number minimal period-two solutions $P_1(\Phi_1, \Psi_1)$, $P_2(\Psi_1, \Phi_1)$, $P_3(\Phi_2, \Psi_2)$ and $P_5(\Phi_3, \Psi_3)$ such that $P_1 \preceq_{se} P_3 \preceq_{se} E_2 \preceq_{se} P_5 \preceq_{se} P_2$. The period-two points P_3 and P_5 are the saddle points and the period-two points P_1 and P_2 are the repellers. The global stable manifold $\mathcal{W}^s(P_3)$ trough the point P_3 is the graph of a continuous strictly decreasing function with endpoints at P_1, E_2 and the global stable manifold $\mathcal{W}^s(P_5)$ trough the point P_5 is the graph of a continuous strictly decreasing function with endpoints and E_2, P_2. Further, the set $int(Q_2(E_1)) \cup int(Q_4(E_1))$ contains even number minimal period-two solutions $P_1(\Phi_1, \Psi_1)$, $P_2(\Psi_1, \Phi_1)$, $P_6(\Psi_3, \Phi_3)$ and $P_4(\Psi_2, \Phi_2)$ such that $P_1 \preceq_{se} P_6 \preceq_{se} E_1 \preceq_{se} P_4 \preceq_{se} P_2$. The period-two points P_4 and P_6 are the saddles. The global stable manifold $\mathcal{W}^s(P_4)$ trough the point P_4 is the graph of a continuous strictly decreasing function with endpoints at P_1, E_1 and the global stable manifold $\mathcal{W}^s(P_6)$ trough the point P_6 is the graph of a continuous strictly decreasing function with endpoints and E_1, P_1. Also, $\mathcal{B}((P_3, P_5)) = \mathcal{W}^s(P_3) \cup \mathcal{W}^s(P_5)$ and $\mathcal{B}((P_4, P_6)) = \mathcal{W}^s(P_4) \cup \mathcal{W}^s(P_6)$. The region between $\mathcal{W}^s(P_3) \cup \mathcal{W}^s(P_5)$ and $\mathcal{W}^s(P_4) \cup \mathcal{W}^s(P_6)$ is invariant and the basin* *of attraction* $\mathcal{B}(E_0)$ *is precisely the region between* $\mathcal{W}^s(P_3) \cup \mathcal{W}^s(P_5)$ *and* $\mathcal{W}^s(P_4) \cup \mathcal{W}^s(P_6)$. *The global unstable manifolds of* $\{P_3, P_4, P_5, P_6\}$ *are* $\mathcal{W}^u(P_3)$, $\mathcal{W}^u(P_4)$, $\mathcal{W}^u(P_5)$, $\mathcal{W}^u(P_6)$, *respectively, are the graphs of continuous strictly increasing functions with endpoints at* E_0 *and the point at infinity. (see Figure 10)*

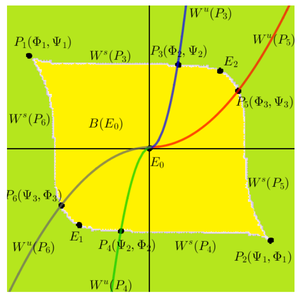

FIGURE 10

(ii) *If* $16Ab^3 < 27B^4$, *then* E_1 *and* E_2 *are the saddle points and there exist two minimal period-two solutions of* (1.1) $P_1(\Phi < 0, \Psi > 0)$ *and* $P_2(\Psi, \Phi)$ *and they are repellers. In this case there exist four continuous curves* $\mathcal{W}^s(E_1)$, $\mathcal{W}^s(E_2)$, $\mathcal{W}^u(E_1)$ *and* $\mathcal{W}^u(E_2)$. *The graph of* $\mathcal{W}^s(E_1)$ *is passing through the point* E_1 *and the graph of* $\mathcal{W}^s(E_2)$ *is passing through the point* E_2 *and they are graphs of decreasing functions. The points* P_1 *and* P_2 *are the endpoints of the curves* $\mathcal{W}^s(E_1)$ *and* $\mathcal{W}^s(E_2)$.*The curves* $\mathcal{W}^u(E_1)$ *and* $\mathcal{W}^u(E_2)$ *are the graphs of increasing functions and are starting at* $E_0(0,0)$. *The region between* $\mathcal{W}^s(E_1)$ *and* $\mathcal{W}^s(E_2)$ *is invariant and the basin of attraction* $\mathcal{B}(E_0)$ *is precisely the region between* $\mathcal{W}^s(E_1)$ *and* $\mathcal{W}^s(E_2)$. *Every solution* $\{(x_n, y_n)\}$ *which starts outside of* $\mathcal{W}^s(E_1) \cup \mathcal{W}^s(E_2)$ *converges to the point at infinity (see Figure 11).*

(iii) *If* $16b^3A = 27B^4$, *then* E_1 *and* E_2 *are the nonhyperbolic points and there exist two minimal period-two solutions of the system* (1.1) $P_1(\Phi < 0, \Psi > 0)$ *and* $P_2(\Psi, \Phi)$ *and they are repellers. In this case there exist two continuous curves* $\mathcal{W}^s(E_1)$, $\mathcal{W}^s(E_2)$. *The graph of* $\mathcal{W}^s(E_1)$ *is passing through the point* E_1 *and the graph of* $\mathcal{W}^s(E_2)$ *is passing through the point* E_2 *and they are graphs of decreasing functions. The points* P_1 *and* P_2 *are the endpoints of the curves* $\mathcal{W}^s(E_1)$ *and* $\mathcal{W}^s(E_2)$. *The region between* $\mathcal{W}^s(E_1)$ *and* $\mathcal{W}^s(E_2)$ *is invariant and the basin of attraction* $\mathcal{B}(E_0)$ *is precisely the region between* $\mathcal{W}^s(E_1)$ *and* $\mathcal{W}^s(E_2)$. *Every solution* $\{(x_n, y_n)\}$ *which starts outside of* $\mathcal{W}^s(E_1) \cup \mathcal{W}^s(E_2)$ *converges to the point at infinity (see Figure 12).*

Proof. The existence and local stability of all equilibrium points follows from Theorems 2.1, 2.2, 3.2 and 3.3. The existence and local stability of all period-two solution(s) follows from Theorems 4.2, 4.3 and Lemma 5.1. The theory of monotone

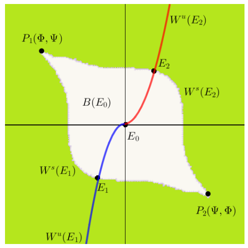

FIGURE 11

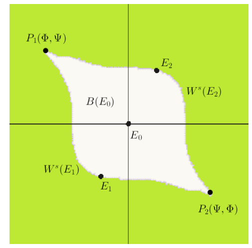

FIGURE 12

maps, and in particular cooperative maps, guarantee the existence and uniqueness of the stable and unstable manifolds for the saddle (nonhyperbolic) fixed points and periodic points, more precisely, the existence of mentioned curves with the described properties is guaranteed by Theorems 1 and 4 of [10] applied to the map T^2 given by (4.1). By applying Lemma 6.5 the set $\mathcal{B}(E_0)$ is bounded and the points P_1 and P_2 are endpoints of the boundary of set $\mathcal{B}(E_0)$ with respect to $\preceq_{se}$order for all fixed points of T^2. The global result and the rest of the proof follow from Theorem 1.2. □

ACKNOWLEDGEMENTS

The authors would like to express their sincere thanks to anonymous reviewers on their valuable and constructive suggestions that contributed to this paper in clarity and form.

REFERENCES

[1] J. Bektešević, V. Hadžiabdić, M. Mehuljić, N. Mujić, *The Global Behavior of a Quadratic Difference Equation*, Filomat, (32:18), 6203–6210. (2018).

[2] J. Bektešević, M. R. S. Kulenović, E. Pilav, *Global Dynamics of Quadratic Second Order Difference Equation in the First Quadrant*, Appl. Math. Comput., 227 (2014), 50–65.

[3] J. Bektešević, M. R. S. Kulenović, E. Pilav, *Global Dynamics of Cubic Second Order Difference Equation in the First Quadrant*, Adv. Differ. Equ., (2015), 2015:176.

[4] J. Bektešević, M. R. S. Kulenović, E. Pilav, *Asymptotic Approximation of the Stable and Unstable Manifolds of the Fixed Points of a Two-dimensional Cubic Map*, Int. J. Difference Equ., Vol 10 No 1, (2015), 39–58.

[5] J. Bektešević, M. R. S. Kulenović, E. Pilav, *Asymptotic Approximation of the Stable and Unstable Manifolds of the Fixed Points of a Two-dimensional Quadratic Map*, J. Computational Analysis and Applications, Vol 21 No1 (2016), 35–51

[6] A. Brett and M. R. S. Kulenović, *Basins of Attraction of Equilibrium Points of Monotone Difference Equations*, Sarajevo J. Math. 5(18) (2009), 211–233.

[7] V. Hadžiabdić, M. R. S. Kulenović, E. Pilav, *Dynamics of a Competitive System of Rational Difference Equations with Quadratic Terms*, Adv. Differ. Equ. 2014, 301 (2014).

[8] S. Janson, *Resultant and discriminant of polynomials*, (2010). http://www2.math.uu.se/˜svante/papers/.

[9] F. Kirwan, *Complex Algebraic Curves*, London Mathematical Society Student Texts. Cambridge University Press, Cambridge (1992).

[10] M. R. S. Kulenović and O. Merino, *Invariant Manifolds for Competitive Discrete Systems in the Plane*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 20 (2010), no. 8, 2471–2486.
[11] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman & Hall/CRC, Boca Raton (2002).
[12] M. R. S. Kulenović and O. Merino, *Global Bifurcation for Competitive Systems in the Plane*, Discrete Contin. Dyn. Syst. B, 12 (2009), 133–149.
[13] M. R. S. Kulenović and G. Ladas, *Dynamics of Second-order rational difference equations with open problems and conjectures*, Chapman & Hall/CRC, Boca Raton, FL, 2002.
[14] H. L. Smith, *Planar Competitive and Cooperative Difference Equations*, J. Difference Equ. Appl. 3(1998), 335–357.
[15] R.J. Walker, *Algebraic Curves*, Princeton University Press, Princeton (1950).
[16] L. Yang, X. Hou, Z. Zeng, *A Complete Discrimination System for Polynomials*, Science in China (Series E), Vol 39 (1996), No 6, 628-646.

(Received: November 05, 2021)
(Revised: January 22, 2022)

Jasmin Bektešević
e-mail: *bektesevic@mef.unsa.ba*
and
Vahidin Hadžiabdić
e-mail: *hadziabdic@mef.unsa.ba*
and
Midhat Mehuljić
e-mail: *mehuljic@mef.unsa.ba*
Faculty of Mechanical Engineering
Vilsonovo šetalšte 9
71000 Sarajevo
and
Naida Mujić
University of Sarajevo
Faculty of Electrical Engineering
Zmaja od Bosne bb,
71000 Sarajevo
e-mail: *nmujic@etf.unsa.ba*