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GLOBAL DYNAMICAL PROPERTIES OF A SYSTEM OF QUADRATIC-RATIONAL DIFFERENCE EQUATIONS WITH ARBITRARY DELAY

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Dedicated to Professor Mustafa Kulenović on the occasion of his 70th birthday

ABSTRACT. In this paper, we investigate the global dynamics of the following system of quadratic-higher order difference equations:

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-m}^2}$$

where A and B are positive numbers and the initial values are positive numbers. We first examine the existence of bounded solutions of the system. Additionally, we study the stability analysis of the of solutions of the system. We also analyze the rate of convergence and oscillation behavior of the solutions of the system.

1. INTRODUCTION

Over the last 20 years, the analysis of the behavior of the solutions of the system of difference equations has attracted the attention of many researchers. This interest is especially related to applications of difference equations or their systems. Applications of difference equations (or systems) are widely used in many sciences such as biology, ecology and economics etc. On the other hand, many mathematicians have studied the different behaviors of solutions of difference equations and systems. These are in particular boundedness, periodicity, stability, and oscillation of solutions of difference equations or systems. There are many examples in the literature investigating the behavior of solutions of difference equations, see [1]-[16]. Additionally, there are many papers related to our study as follows:

In [12], Abualrub and Aloqeili discussed the global behavior of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, y_{n+1} = A + \frac{x_n}{x_{n-k}},$$

where $A > 0$ and the initial values are arbitrary positive numbers. Additionally, the author handled the oscillatory behavior of the solutions of considered system.

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In [14], Kulenović and M. Nurkanović handled the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+y_n}, y_{n+1}=\frac{d+y_n}{e+x_n},$$

where the parameters a, b, d, and e are positive numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In [15] Kulenović, M. Nurkanović and Yakubu studied the global asymptotic behavior of solutions of the density-dependent discrete-time *SI* epidemic model with the variables S_n and I_n representing the populations of susceptibles and infectives at time $n=0,1,...$, respectively. This model is

$$S_{n+1}=ae^{-\beta I_n}S_n+B, I_{n+1}=a(1-e^{-\beta I_n})S_n+bI_n,$$

where $0<a$, $b<1$, $\beta, B>0$, $S_0\geq 0$ and $I_0\geq 0$.

In [13], Bao investigated the local stability and oscillation of positive solutions of the system of difference equations

$$x_{n+1}=A+\frac{x_{n-1}^p}{y_n^p}, y_{n+1}=A+\frac{y_{n-1}^p}{x_n^p},$$

where $A>0$, $p>1$ and the initial values are positive numbers.

In [6], Hadžiabdić, Kulenović and Pilav dealt with the global dynamics of the system of difference equations

$$x_{n+1}=\frac{b_1x_n^2}{A_1+y_n^2}, y_{n+1}=\frac{a_2+c_2y_n^2}{x_n^2},$$

where the parameters b_1,a_2,A_1,c_2 are positive numbers and the initial conditions x_0 is a positive number, and y_0 is an arbitrary non-negative number.

In [16], Burgić, Kulenović and M. Nurkanović dealth with the global stability properties and asymptotic behavior of solutions of the system of difference equations

$$x_{n+1}=\frac{x_n}{a+y_n^2}, y_{n+1}=\frac{y_n}{b+x_n^2},$$

where the parameters a and b are positive numbers, and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In [3], Bešo, Kalabušić, Mujić, and Pilav considered the global asymptotic stability of solutions of following second order difference equation

$$x_{n+1}=\gamma+\delta\frac{x_n}{x_{n-1}^2},$$

where γ, δ are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

In [10], Taşdemir studied the periodicity, boundedness, semi-cycles, global asymptotically stability and rate of convergence of solutions of the following higher order difference equation

$$x_{n+1} = A + B\frac{x_n}{x_{n-m}^2},$$

where A and B are positive numbers and the initial values are positive numbers.

In [11], Taşdemir dealt with the dynamical behaviors of solutions of the following system of second order difference equations with quadratic terms

$$x_{n+1} = A + B\frac{y_n}{y_{n-1}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-1}^2},$$

where A and B are positive numbers and the initial values are positive numbers. The author handled the boundedness, oscillation, stability, periodicity, and rate of convergence of the solutions of the system considered.

With the above studies in mind, we consider the following system of higher order difference equations with quadratic terms

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-m}^2} \tag{1.1}$$

where the parameters A, B are positive numbers, and the initial conditions are positive numbers and $m \in \{2, 3, \cdots\}$. Particularly, we handle the boundedness, global asymptotic stability, semicycles and rate of convergence of solutions of system (1.1).

Here, we do not prefer to present detailed preliminaries as our references include many detailed information about the theory of difference equations and their systems. Readers can reach these results from references (Please see [1, 5, 7, 8]). We now give two theorems that play an important role for our results.

Let us consider the following system of difference equations:

$$E_{n+1} = (A + B(n))E_n, \tag{1.2}$$

where E_n is a $k-$dimensional vector, $A \in C^{k\times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0, \tag{1.3}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm that is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Theorem 1.1 (Perron's Theorem, [9])**.** *Assume that condition (1.3) holds. If E_n is a solution of (1.2), then either $E_n = 0$ for all $n \to \infty$, or*

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|},$$

or

$$\lim_{n\to\infty}\frac{\|E_{n+1}\|}{\|E_n\|},$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 1.2 ([4]). *Let $n \in N_{n_0}^{+}$ and $g(n,u,v)$ be a nondecreasing function in u and v for any fixed n. Suppose that for $n \geq n_0$, the inequalities*

$$y_{n+1} \leq g(n, y_n, y_{n-1})$$
$$u_{n+1} \geq g(n, u_n, u_{n-1})$$

hold. Then

$$y_{n_0-1} \leq u_{n_0-1}, y_{n_0} \leq u_{n_0}$$

implies that

$$y_n \leq u_n, n \geq n_0.$$

2. Boundedness of System (2.1)

In this section, we handle the existence of bounded solutions of system (1.1). Thus, we find that if $p \in (0,1)$ holds, then system (2.1) has an invariant interval. We also discover that if $p \geq 1$ and m is even, then every solution of system (2.1) is bounded from below and above.

First, we consider the following substitutions for system (1.1):

$$t_n = \frac{x_n}{A}, z_n = \frac{y_n}{A}.$$

Therefore, we have the following system of higher order difference equations, which is easier to work with:

$$t_{n+1} = 1 + p\frac{z_n}{z_{n-m}^2}, z_{n+1} = 1 + p\frac{t_n}{t_{n-m}^2}, n = 0,1,2,... \tag{2.1}$$

where $p = \frac{B}{A^2} > 0$. From here, we handle the system (2.1).

The system (2.1) has a unique positive equilibrium point such that

$$(\bar{t},\bar{z}) = \left(\frac{1+\sqrt{1+4p}}{2}, \frac{1+\sqrt{1+4p}}{2}\right),$$

where $p > 0$.

Here, we determine the boundedness character of the solutions of system (2.1).

Firstly, let $p > 0$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (2.1). It is easy to see that

$$t_n > 1, \tag{2.2}$$

and

$$z_n > 1, \tag{2.3}$$

for $n \geq 1$.

Theorem 2.1. *Let $p \in (0,1)$. Then, every solution of system (2.1) is bounded from below and above.*

Proof. Let $p \in (0,1)$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (2.1). From (2.2), (2.3) and system (2.1), we have

$$t_{n+1} < 1 + p + p^2 t_{n-1}, n \geq 0. \tag{2.4}$$

According to this, $\{u_n\}$ satisfy the following

$$u_{n+1} = 1 + p + p^2 u_{n-1}, \tag{2.5}$$

for $n \geq 1$, such that

$$u_0 = t_0, u_1 = t_1.$$

Thus, we have $t_n \leq u_n$ for $n = 0, 1, \ldots$. Hence, the solution u_n of the difference equation (2.5) is

$$u_n = \frac{1}{1-p} + p^n C_1 + (-p)^n C_2, \tag{2.6}$$

where C_1 and C_2 are constants to be determined. By Theorem 1.2 we have that

$$t_n \leq u_n, n > 1. \tag{2.7}$$

Since $u_0 = t_0$ and $u_1 = t_1$, from (2.2), (2.3), (2.6) and (2.7) we get

$$1 < t_n \leq \frac{1}{1-p} + p^n C_1 + (-p)^n C_2,$$

where

$$C_1 = \frac{1}{2p}\left(pt_0 + t_1 - \frac{1+p}{1-p}\right),$$
$$C_2 = \frac{1}{2p}\left(pt_0 - t_1 + 1\right).$$

Likewise, we have that

$$1 < z_n \leq \frac{1}{1-p} + p^n C_3 + (-p)^n C_4,$$

where

$$C_3 = \frac{1}{2p}\left(pz_0 + z_1 - \frac{1+p}{1-p}\right),$$
$$C_4 = \frac{1}{2p}\left(pz_0 - z_1 + 1\right).$$

□

Remark 2.1. Note that the boundedness of the solution in Theorem 2.1 is not uniform for all solutions but depends on the initial conditions.

Theorem 2.2. *Let $p \geq 1$ and m is even. Then every solution of system (2.1) is bounded from above and below as follows:*

$$1 < t_n < 1 + p(1+p)^m,$$

and

$$1 < z_n < 1 + p(1+p)^m,$$

for $n \geq 2m+2$.

Proof. Let $p \geq 1$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (2.1). From system (2.1), we have

$$t_n = 1 + p\frac{z_{n-1}}{z_{n-m-1}^2} = 1 + \frac{p}{z_{n-m-1}}\frac{z_{n-1}}{z_{n-m-1}}.$$

Assume that m is even. Then, we obtain from system (2.1)

$$t_n = 1 + \frac{p}{z_{n-m-1}}\left(\prod_{i=1}^{\frac{m}{2}}\frac{z_{n-2i+1}}{z_{n-2i-1}}\right). \tag{2.8}$$

Moreover, we get from system (2.1)

$$z_{n-2i+1} = 1 + p\frac{t_{n-2i}}{t_{n-m-2i}^2},$$

$$\frac{z_{n-2i+1}}{t_{n-2i}} = \frac{1}{t_{n-2i}} + \frac{p}{t_{n-m-2i}^2}, \tag{2.9}$$

and

$$t_{n-2i} = 1 + p\frac{z_{n-2i-1}}{z_{n-m-2i-1}^2},$$

$$\frac{t_{n-2i}}{z_{n-2i-1}} = \frac{1}{z_{n-2i-1}} + \frac{p}{z_{n-m-2i-1}^2}, \tag{2.10}$$

for $i = 1, 2, \cdots, \frac{m}{2}$. Thus, multiplying (2.9) and (2.10), we get the following:

$$\frac{z_{n-2i+1}}{z_{n-2i-1}} = \left(\frac{1}{t_{n-2i}} + \frac{p}{t_{n-m-2i}^2}\right)\left(\frac{1}{z_{n-2i-1}} + \frac{p}{z_{n-m-2i-1}^2}\right). \tag{2.11}$$

Additionally, we obtain from (2.2), (2.3) and (2.11)

$$\frac{z_{n-2i+1}}{z_{n-2i-1}} < (1+p)^2,$$

for $i = 1, 2, \cdots, \frac{m}{2}$. Therefore, we have

$$\prod_{i=1}^{\frac{m}{2}}\frac{z_{n-2i+1}}{z_{n-2i-1}} < (1+p)^m. \tag{2.12}$$

So, we have from (2.2), (2.3), (2.8) and (2.12)

$$\begin{aligned} t_n &= 1+\frac{p}{z_{n-m-1}}\left(\prod_{i=1}^{\frac{m}{2}}\frac{z_{n-2i+1}}{z_{n-2i-1}}\right) \\ &< 1+p\,(1+p)^m, \end{aligned}$$

for $n \geq 2m+2$. With similar calculations, we obtain

$$z_n < 1+p\,(1+p)^m,$$

for $n \geq 2m+2$. □

3. Global Asymptotic Stability of System (2.1)

Here, we overcome the stability of solutions of system (2.1).
Now, we take the following transformation:

$$(t_n, t_{n-1}, \cdots, t_{n-m}, z_n, z_{n-1}, \cdots, z_{n-m}) \to (f, f_1, \cdots, f_m, g, g_1, \cdots, g_m)$$

where $f = 1+p\frac{z_n}{z_{n-m}^2}, f_1 = t_n, \cdots f_m = t_{n-m-1}, g = 1+p\frac{t_n}{t_{n-m}^2}, g_1 = z_n, \cdots, g_m = z_{n-m-1}$. Hence, we obtain the following jacobian matrix about unique positive equilibrium point $(\bar{t}, \bar{z})$:

$$B(\bar{t},\bar{z}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)}.$$

Theorem 3.1. *The unique positive equilibrium point $(\bar{t}, \bar{z})$ of system (2.1) is locally asymptotically stable for $p < \frac{3}{4}$.*

Proof. The linearized system of system (2.1) about the unique positive equilibrium point is given by $X_{N+1} = B(\bar{t}, \bar{z})X_N$, where

$$X_N = \begin{pmatrix} t_n \\ \vdots \\ t_{n-m} \\ z_n \\ \vdots \\ z_{n-m} \end{pmatrix},$$

$$E=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & c & 0 & \cdots & 0 & -2c \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & \cdots & 0 & -2c & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)},$$

and $c=\frac{p}{\bar{z}^2}=\frac{p}{\bar{t}^2}$. Let $\lambda_1,\lambda_2,\cdots,\lambda_{2m+2}$ denote the $2m+2$ eigenvalues of matrix E. Let $D=diag\left(d_1,d_2,\cdots,d_{2m+2}\right)$ be a diagonal matrix such that

$$d_1=d_{m+2}=1, d_k=d_{m+1+k}=1-k\varepsilon, 2\leq k\leq m+1,$$

and

$$0<\varepsilon<\frac{3c-1}{(m+1)(c-1)}.$$

Clearly, D is an invertible matrix. Computing the matrix DED^{-1}, we get that

$$DED^{-1}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{cd_1}{d_{m+2}} & 0 & \cdots & 0 & \frac{-2cd_1}{d_{2m+2}} \\ \frac{d_2}{d_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{d_{m+1}}{d_m} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{cd_{m+2}}{d_1} & 0 & \cdots & 0 & \frac{-2cd_{m+2}}{d_{m+1}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{d_{m+3}}{d_{m+2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{d_{2m+2}}{d_{2m+1}} & 0 \end{pmatrix}.$$

From the following inequalities

$$1=d_1>d_2>\cdots>d_m>d_{m+1}>0,$$
$$1=d_{m+2}>d_{m+3}>\cdots>d_{2m+1}>d_{2m+2}>0,$$

which implies that

$$d_2d_1^{-1}<1, d_3d_2^{-1}<1,\cdots,d_{m+1}d_m^{-1}<1,$$
$$d_{m+3}d_{m+2}^{-1}<1, d_{m+4}d_{m+3}^{-1}<1,\cdots,d_{2m+2}d_{2m+1}^{-1}<1.$$

Moreover, we have that

$$cd_1 d_{m+2}^{-1} + 2cd_1 d_{2m+2}^{-1} = c\left(1 + \frac{2}{1-(m+1)\varepsilon}\right) < 1,$$
$$cd_{m+2} d_1^{-1} + 2cd_{m+2} d_{m+1}^{-1} = c\left(1 + \frac{2}{1-(m+1)\varepsilon}\right) < 1.$$

It is a well-known fact that E has the same eigenvalues as DED^{-1}. Thus, we get

$$\begin{aligned}\max_{1\le k\le 2m+2} |\lambda_k| &= \left\|DED^{-1}\right\| \\ &= \max\left\{\begin{array}{c} d_2 d_1^{-1}, \cdots, d_{m+1} d_m^{-1}, \\ d_{m+3} d_{m+2}^{-1}, \cdots, d_{2m+2} d_{2m+1}^{-1}, \\ cd_1 d_{m+2}^{-1} + 2cd_1 d_{2m+2}^{-1}, \\ cd_{m+2} d_1^{-1} + 2cd_{m+2} d_{m+1}^{-1} \end{array}\right\} \\ &< 1.\end{aligned}$$

So, the positive equilibrium point $(\bar{t}, \bar{z})$ of system (2.1) is locally asymptotically stable for $0 < p < \frac{3}{4}$. □

Theorem 3.2. *Suppose that* $0 < p < \frac{1}{2}$. *Then the positive equilibrium point of system (2.1) is globally asymptotically stable.*

Proof. From (2.2) and (2.3), we know that

$$\begin{aligned} 1 &\le l_1 = \liminf_{n\to\infty} t_n, \\ 1 &\le l_2 = \liminf_{n\to\infty} z_n, \\ 1 &\le L_1 = \limsup_{n\to\infty} t_n, \\ 1 &\le L_2 = \limsup_{n\to\infty} z_n. \end{aligned}$$

Thus, we have the following for system (2.1)

$$L_1 \le 1 + p\frac{L_2}{l_2^2}, l_1 \ge 1 + p\frac{l_2}{L_2^2},$$
$$L_2 \le 1 + p\frac{L_1}{l_1^2}, l_2 \ge 1 + p\frac{l_1}{L_1^2}.$$

Therefore, we obtain

$$L_1 + p\frac{l_1}{L_1} \le L_1 l_2 \le l_2 + p\frac{L_2}{l_2},$$
$$L_2 + p\frac{l_2}{L_2} \le L_2 l_1 \le l_1 + p\frac{L_1}{l_1}.$$

Hence, we get that

$$L_1+p\frac{l_1}{L_1}+L_2+p\frac{l_2}{L_2}\leq l_2+p\frac{L_2}{l_2}+l_1+p\frac{L_1}{l_1},$$

i.e.,

$$\Leftrightarrow (L_1-l_1)\left(1-p\left(\frac{1}{l_1}+\frac{1}{L_1}\right)\right)+(L_2-l_2)\left(1-p\left(\frac{1}{l_2}+\frac{1}{L_2}\right)\right)\leq 0. \tag{3.1}$$

From $l_1,L_1,l_2,L_2>1$, we have

$$\frac{1}{l_1}+\frac{1}{L_1}\leq 2,$$

and

$$\frac{1}{l_2}+\frac{1}{L_2}\leq 2.$$

Hence, we get

$$1-p\left(\frac{1}{l_1}+\frac{1}{L_1}\right)\geq 1-2p,$$
$$1-p\left(\frac{1}{l_2}+\frac{1}{L_2}\right)\geq 1-2p.$$

Meanwhile, we know that $L_1\geq l_1$ and $L_2\geq l_2$. Therefore, if $1-2p>0$, then from (3.1) we obtain

$$L_1-l_1+L_2-l_2\leq 0.$$

So, $L_1=l_1$ and $L_2=l_2$. □

4. Rate of Convergence and Oscillation of System (2.1)

First, we handle the rate of convergence of system (2.1). Then, we investigate the oscillatory behavior of solutions of system (2.1).

Theorem 4.1. *Assume that $0<p<\frac{1}{2}$ and $\{(t_n,z_n)\}_{n=-m}^{\infty}$ be a solution of the system (2.1) such that $\lim_{n\to\infty}t_n=\bar{t}$ and $\lim_{n\to\infty}z_n=\bar{z}$. Then, the error vector*

$$E_n=\begin{pmatrix} e_n^1\\ \vdots\\ e_{n-m}^1\\ e_n^2\\ \vdots\\ e_{n-m}^2\end{pmatrix}=\begin{pmatrix} t_n-\bar{t}\\ \vdots\\ t_{n-m}-\bar{t}\\ z_n-\bar{z}\\ \vdots\\ z_{n-m}-\bar{z}\end{pmatrix},$$

of every solution of system (2.1) accomplishes all of the asymptotic relations as follows:

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|} = |\lambda F_J(\bar{t},\bar{z})|,$$
$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_J(\bar{t},\bar{z})|,$$

where $\lambda F_J(\bar{t},\bar{z})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{t},\bar{z})$.

Proof. We firstly set to determine the error terms,

$$t_{n+1} - \bar{t} = \sum_{i=0}^{m} A_i (t_{n-i} - \bar{t}) + \sum_{i=0}^{m} B_i (z_{n-i} - \bar{z}),$$
$$z_{n+1} - \bar{z} = \sum_{i=0}^{m} C_i (t_{n-i} - \bar{t}) + \sum_{i=0}^{m} D_i (z_{n-i} - \bar{z}),$$

and

$$e_n^1 = t_n - \bar{t}, e_n^2 = z_n - \bar{z}.$$

Hence, we obtain

$$e_{n+1}^1 = \sum_{i=0}^{m} A_i e_{n-i}^1 + \sum_{i=0}^{m} B_i e_{n-i}^2,$$
$$e_{n+1}^2 = \sum_{i=0}^{m} C_i e_{n-i}^1 + \sum_{i=0}^{m} D_i e_{n-i}^2,$$

where $A_i = 0$ and $D_i = 0$ for $i = 0,1,\cdots,m$,

$$B_0 = \frac{p}{z_{n-m}^2}, B_i = 0, i \in \{1,2,\cdots,m-1\}, B_m = \frac{-p(\bar{z}+z_{n-m})}{\bar{z}z_{n-m}^2},$$
$$C_0 = \frac{p}{t_{n-m}^2}, C_i = 0, i \in \{1,2,\cdots,m-1\}, C_m = \frac{-p(\bar{t}+t_{n-m})}{\bar{t}t_{n-m}^2}.$$

Taking the limits, we have $\lim_{n\to\infty} A_i = \lim_{n\to\infty} D_i = 0$ for $i \in \{0,1,\cdots,m\}$ and $\lim_{n\to\infty} B_i = \lim_{n\to\infty} C_i = 0$ for $i \in \{1,\cdots,m-1\}$. Moreover, we obtain that

$$\lim_{n\to\infty} B_0 = \frac{p}{\bar{z}^2}, \lim_{n\to\infty} B_m = \frac{-2p}{\bar{z}^2},$$
$$\lim_{n\to\infty} C_0 = \frac{p}{\bar{t}^2}, \lim_{n\to\infty} C_m = \frac{-2p}{\bar{t}^2}.$$

Hence,

$$B_0 = \frac{p}{\bar{z}^2} + a_n, B_m = \frac{-2p}{\bar{z}^2} + b_n,$$
$$C_0 = \frac{p}{\bar{t}^2} + c_n, C_m = \frac{-2p}{\bar{t}^2} + d_n,$$

where

$$a_n \to 0, b_n \to 0, c_n \to 0, d_n \to 0$$

as $n \to \infty$. Thus, we get the system of the form (1.2)

$$E_{n+1} = (A + B(n)) E_n,$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$B(n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & a_n & 0 & \cdots & 0 & b_n \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c_n & 0 & \cdots & 0 & d_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and $\|B(n)\| \to 0$ as $n \to \infty$. Therefore, we can write the limiting system of error terms about the equilibrium point $(\bar{t}, \bar{z})$ as follows:

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ \vdots \\ e^1_{n-m+1} \\ e^2_{n+1} \\ e^2_n \\ \vdots \\ e^2_{n-m+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ \vdots \\ e^1_{n-m} \\ e^2_n \\ e^2_{n-1} \\ \vdots \\ e^2_{n-m} \end{pmatrix},$$

which is same as the linearized system of the system (2.1) about equilibrium point $(\bar{t}, \bar{z})$. □

Now, we study the semicycles of solutions of system (2.1).

Theorem 4.2. *Suppose that* $\{(t_n, z_n)\}_{n=-m}^{\infty}$ *is a positive solution of system (2.1) and* $p > 0$*. Then, either* $\{(t_n, z_n)\}_{n=-m}^{\infty}$ *the solution of system (2.1) has a single semicycle or* $\{(t_n, z_n)\}_{n=-m}^{\infty}$ *the solution of system (2.1) has semicycles with at most* m *terms.*

Proof. Let $\{(t_n, z_n)\}_{n=-m}^{\infty}$ the solution of system (2.1) have at least two semicycles. Hence, there exists $N \geq 0$ such that either

$$t_N, z_{N+1} < \bar{t} = \bar{z} < t_{N+1}, z_N$$

or

$$t_{N+1}, z_N < \bar{t} = \bar{z} < t_N, z_{N+1}.$$

Firstly, we assume that $t_N, z_{N+1} < \bar{t} = \bar{z} < t_{N+1}, z_N$. Moreover, we suppose that the positive semicycle have m terms and it begins with the term (t_{N+1}, z_{N+1}). Thus, we obtain the following

$$t_N < \bar{t} = \bar{z} < t_{N+m},$$
$$z_{N+m} < \bar{t} = \bar{z} < z_N.$$

From this, we get

$$t_{N+m+1} = 1 + p\frac{z_{N+m}}{z_N^2} < \bar{t} = \bar{z},$$
$$z_{N+m+1} = 1 + p\frac{t_{N+m}}{t_N^2} > \bar{t} = \bar{z}.$$

□

5. Numerical Examples

Now, we present two examples that support our theoretical outcomes.

Example 5.1. *With* $m = 3$ *and* $p = 0.49$*, we handle system (2.1). Therefore, we have the following system of difference equations*

$$t_{n+1} = 1 + 0.49\frac{z_n}{z_{n-3}^2}, z_{n+1} = 1 + 0.49\frac{t_n}{t_{n-3}^2}. \tag{5.1}$$

Moreover, we take the following initial values $t_{-3} = 6$*,* $t_{-2} = 1$*,* $t_{-1} = 0.8$*,* $t_0 = 4$*,* $z_{-3} = 0.4$*,* $z_{-2} = 5$*,* $z_{-1} = 3$ *and* $z_0 = 10$ *for system (5.1). According to Theorem 2.1, system (5.1) is bounded from above and below. Moreover, the positive equilibrium point* $(\bar{t}, \bar{z}) = (1.36, 1.36)$ *of system (5.1) is globally asymptotically stable (see Figure 1).*

Example 5.2. *With* $m = 2$ *and* $p = 1.5$*, we consider system (2.1). Then, we get the third order system of difference equations such that*

$$t_{n+1} = 1 + 1.5\frac{z_n}{z_{n-2}^2}, z_{n+1} = 1 + 1.5\frac{t_n}{t_{n-2}^2}. \tag{5.2}$$

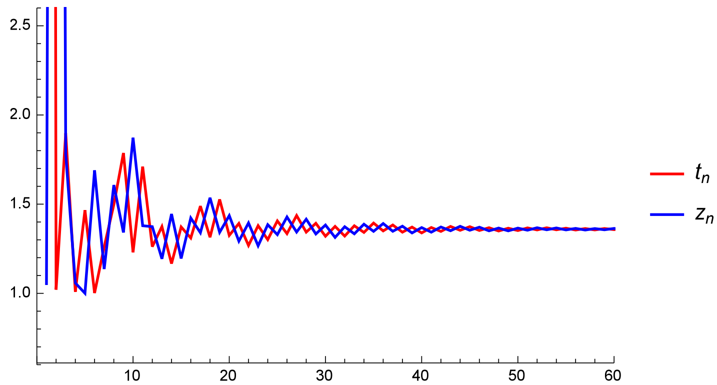

FIGURE 1. *Plot of system (2.1) with $m = 3$ and $p = 0.49$.*

Now, we handle the system (5.2) with the following initial values $t_{-2} = 3$, $t_{-1} = 4$, $t_0 = 0.6$, $z_{-2} = 2$, $z_{-1} = 0.4$ and $z_0 = 3$. Hence, the positive solutions of system (5.2) oscillate about the unique positive equilibrium point $(\bar{t}, \bar{z}) = (1.82, 1.82)$. Also, according to Theorem 2.2, every solution of system (5.2) is bounded from below and above (see Figure 2).

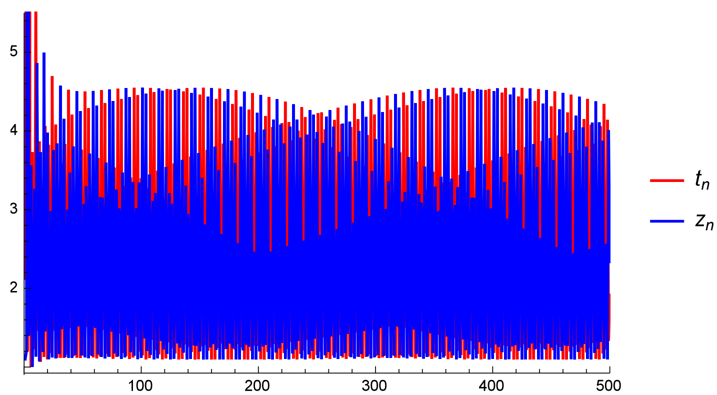

FIGURE 2. *Plot of system (2.1) with $m = 2$ and $p = 1.5$.*

6. CONCLUSIONS

In this paper, we handled the dynamics of system (2.1). First, we found the unique positive equilibrium point of system (2.1). We also investigated the boundedness of solutions of system (2.1) in detail. Moreover, we dealt with the local and global stability of system (2.1). Hence, we acquired that every solution of system (2.1) converges to the unique positive equilibrium point when $0 < p < \frac{1}{2}$ holds. In addition to this, we studied the rate of convergence and oscillation behaviors of solutions of system (2.1). Finally, we presented three numerical examples to verify our theoretical results.

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