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A STUDY ON FIBONACCI FUNCTIONS AND GAUSSIAN FIBONACCI FUNCTIONS

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Dedicated to Professor Mustafa Kulenović on the occasion of his 70th birthday

ABSTRACT. In this paper, we define Gaussian Fibonacci functions and investigate them on the set of real numbers $\mathbb{R}$, *i.e.*, functions $f_G : \mathbb{R} \to \mathbb{C}$ such that for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $f_G(x+n) = f(x+n) + if(x+n-1)$ where $f : \mathbb{R} \to \mathbb{R}$ is a Fibonacci function which is given as $f(x+2) = f(x+1) + f(x)$ for all $x \in \mathbb{R}$. Then the concept of Gaussian Fibonacci functions by using the concept of f-even and f-odd functions is developed. Also, we present linear sum formulas of Gaussian Fibonacci functions. Moreover, it is shown that if f_G is a Gaussian Fibonacci function with Fibonacci function f, then $\lim_{x\to\infty} \frac{f_G(x+1)}{f_G(x)} = \alpha$ and $\lim_{x\to\infty} \frac{f_G(x)}{f(x)} = 1 + (\alpha - 1)\, i$, where α is the positive real root of equation $x^2 - x - 1 = 0$ for which $\alpha > 1$. Furthermore, matrix formulations of Fibonacci functions and Gaussian Fibonacci functions are given. We also present linear sum formulas and matrix formulations of Fibonacci functions which have not been studied in the literature.

1. INTRODUCTION

A function f defined on the real numbers $\mathbb{R}$ is said to be a k-step Fibonacci function if it satisfies the following relation

$$f(x+k) = f(x+k-1) + f(x+k-2) + f(x+k-3) + \ldots + f(x)$$

for all $x \in \mathbb{R}$. See Sriponpaew and Sassanapitax [13], and Wolfram [18] for more information on k-step Fibonacci functions. We can consider some special cases of k-step Fibonacci functions.

A function f defined on the real numbers $\mathbb{R}$ is said to be a Fibonacci function if it satisfies the following relation

$$f(x+2) = f(x+1) + f(x)$$

for all $x \in \mathbb{R}$. First and foremost, Elmore [2], Parker [8] and Spickerman [12] discovered useful properties of the Fibonacci functions (a short review on Fibonacci functions will be given in this section below). Later, many renowned researchers

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such as Fergy and Rabago [3], Han, et al. [5], Sharma [9], Sroysang [14], and Gandhi [4], have devoted their study to the analysis of many properties of the Fibonacci function.

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tribonacci function if it satisfies the following relatio

$$f(x+3) = f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$. Some references on Tribonacci functions are Arolkar [1], Magnani [6], Parizi [7], Sharma [10] and Soykan, et al. [15].

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tetranacci function if it satisfies the following relation

$$f(x+4) = f(x+3) + f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$. See Sharma [11] for more information on Tetranacci functions.

Before giving a short review on Fibonacci functions, we recall the definition of a Fibonacci sequence. A Fibonacci sequence $\{F_n\}_{n\geq 0} = \{V_n(F_0, F_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$F_n = F_{n-1} + F_{n-2},$$

with the initial values $F_0 = 0, F_1 = 1$.

Next, we present the first few values of the Fibonacci numbers with positive and negative subscripts:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233
F_{-n}	....	1	-1	2	-3	5	-8	13	-21	34	-55	89	-144	233

If we let $u_0 = 0, u_1 = 1$, then we consider the full (bilateral) Fibonacci sequence $\{u_n\}_{n=-\infty}^{\infty}$: $..., 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, ...$, i.e. $F_{-n} = (-1)^{n+1}F_n$ (see, for example, Soykan [16, Corollary 3.4.]) for $n > 0$ and $u_n = F_n$, the nth Fibonacci number.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function if it satisfies the formula

$$f(x+2) = f(x+1) + f(x) \tag{1.1}$$

for all $x \in \mathbb{R}$ or equivalently

$$f(x) = f(x-1) + f(x-2)$$

for all $x \in \mathbb{R}$.

Note that for a Fibonacci function f, we have

$$f(x+n) = F_n f(x+1) + F_{n-1} f(x) \tag{1.2}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

We next present the Binet's formula of f.

Lemma 1.1. *[18, p.141] The Binet's formula of a Fibonacci function f is*

$$f(x) = \frac{(f(1) - f(0)\beta)\alpha^x - (f(1) - f(0)\alpha)\beta^x}{\sqrt{5}}.$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of characteristic equation

$$t^2 - t - 1 = 0$$

of the equation (1.1).

Next, we list some examples of Fibonacci functions.

Example 1.1.

(a): *[5, Example 2.1.]*

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \alpha^x$$

is a Fibonacci function, where α is a positive root of equation $t^2 - t - 1 = 0$ and α is greater than one and given as

$$\alpha = \frac{1+\sqrt{5}}{2}.$$

(b): *[5, Example 2.2., Example 2.4.] Let $\{u_n\}_{n=-\infty}^{\infty}$ and $\{v_n\}_{n=-\infty}^{\infty}$ be full Fibonacci sequences and define a function $f(x)$ by $f(x) = u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor} t$, where $t = x - \lfloor x \rfloor \in (0,1)$ and $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer function (floor function). Then*

$$f(x+2) = f(x+1) + f(x)$$

so f is a Fibonacci function.
If we let $v_{\lfloor x \rfloor} := u_{(\lfloor x \rfloor - 1)}$ then f is a Fibonacci function.

(c): *[5, Proposition 2.3] Let f be a Fibonacci function and define $g(x) = f(x+t)$ for any $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then g is also a Fibonacci function. If $f(x) = \alpha^x$ which is a Fibonacci function, then $g(x) = \alpha^{x+t} = \alpha^t f(x)$ is a Fibonacci function.*

We now present the concepts of f-even and f-odd functions which were defined by Han, et al. [5] in 2012.

Definition 1.1. *Suppose that $a(x)$ is a real-valued function of a real variable such that if $a(x)h(x) = 0$ and $h(x)$ is continuous. Then we get $h(x) \equiv 0$. The map $a(x)$ is called an f-even function if $a(x+1) = a(x)$ and f-odd function if $a(x+1) = -a(x)$ for all $x \in \mathbb{R}$.*

We present an f-even and an f-odd function.

Example 1.2.

(a): *If $a(x) = x - \lfloor x \rfloor$ then $a(x)$ is an f-even function.*
(b): *If $a(x) = \sin(\pi x)$ then $a(x)$ is an f-odd function.*

Solution.

(a): If $a(x) = x - \lfloor x \rfloor$, then $a(x)h(x) = 0$ implies $h(x) \equiv 0$ if $x \notin \mathbb{Z}$. By continuity of $h(x)$, it follows that $h(n) = \lim_{x \to n} h(x) = 0$ for any integer $n \in \mathbb{Z}$ and therefore $h(x) \equiv 0$. Since

$$a(x+1) = (x+1) - \lfloor x+1 \rfloor = (x+1) - (\lfloor x \rfloor + 1) = x - \lfloor x \rfloor = a(x),$$

we see that $a(x)$ is an f-even function.

(b): If $a(x) = \sin(\pi x)$, then $a(x)h(x) \equiv 0$ implies that $h(x) = 0$ if $x \neq n\pi$ for any integer $n \in \mathbb{Z}$. Since $h(x)$ is continuous, it follows that $h(n\pi) = \lim_{x \to n\pi} h(x) = 0$ for $n \in \mathbb{Z}$, and therefore, $h(x) \equiv 0$. Since

$$a(x+1) = \sin(\pi x + \pi) = \sin(\pi x)\cos(\pi) = -\sin(\pi x) = -a(x),$$

we see that $a(x)$ is an f-odd function.

The following theorem is given in [5, Theorem 3.4.].

Theorem 1.1. *Assume that $f(x) = a(x)g(x)$ is a function, where $a(x)$ is an f-even function and $g(x)$ is a continuous function. Then $f(x)$ is a Fibonacci function if and only if $g(x)$ is a Fibonacci function.*

The following theorem shows that the limit of the quotient of a Fibonacci function exists.

Theorem 1.2. *If $f(x)$ is a Fibonacci function, then the limit of the quotient $\frac{f(x+1)}{f(x)}$ exists ([5, Theorem 4.1.]) and $\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \alpha$ ([5, Corollary 4.2.]).*

Theorem 1.3. *If $f(x)$ is a Fibonacci function, then, for $0 \leq k, m \in \mathbb{N}$ we have*

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x+m)} = \alpha^{k-m}. \tag{1.3}$$

Proof. If $0 \leq k, m \leq 1$, then (1.3) is true (Theorem 1.2). We give the proof in three stages:

Stage I:

$$\lim_{x \to \infty} \frac{f(x+2)}{f(x)} = \alpha^2.$$

Stage II: for $2 \leq k \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x)} = \alpha^k.$$

Stage III: for $2 \leq k, m \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x+m)} = \alpha^{k-m}.$$

Proof of Stage I:

Given $x \in \mathbb{R}$, there exist $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y + n$. Then, using the formula

$$f(x+n) = F_n f(x+1) + F_{n-1} f(x),$$

we get

$$\begin{aligned}\frac{f(x+2)}{f(x)} = \frac{f(y+n+2)}{f(y+n)} &= \frac{F_{n+2}f(y+1)+F_{n+2-1}f(y)}{F_n f(y+1)+F_{n-1}f(y)}\\ &= \frac{\frac{F_{n+2}}{F_{n-1}}\frac{f(y+1)}{f(y)}+\frac{F_{n+1}}{F_{n-1}}}{\frac{F_n}{F_{n-1}}\frac{f(y+1)}{f(y)}+1}.\end{aligned}$$

Since

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \lim_{y\to\infty}\frac{f(y+1)}{f(y)} = \alpha$$

and

$$\lim_{n\to\infty}\frac{F_{n+p}}{F_{n+q}} = \alpha^{p-q}, \quad p,q\in\mathbb{Z}$$

and

$$\lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \lim_{y\to\infty}\frac{f(y+2)}{f(y)} = u \quad \text{(say)}$$

we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{f(x+2)}{f(x)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{\frac{F_{n+2}}{F_{n-1}}f(y+1)+\frac{F_{n+1}}{F_{n-1}}f(y)}{\frac{F_n}{F_{n-1}}f(y+1)+f(y)}\\ &= \frac{\alpha^3\alpha+\alpha^2}{\alpha\alpha+1} = \alpha^2\end{aligned}$$

and so

$$\lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \alpha^2.$$

This completes the proof of Stage I.

Proof of Stage II:

Given $x\in\mathbb{R}$, there exists $y\in\mathbb{R}$ and $n\in\mathbb{N}$ such that $x=y+n$. Then, by using Stage I, we get

$$\begin{aligned}\frac{f(x+k)}{f(x)} = \frac{f(y+n+k)}{f(y+n)} &= \frac{F_{n+k}f(y+1)+F_{n+k-1}f(y)}{F_n f(y+1)+F_{n-1}f(y)}\\ &= \frac{F_{n+k-1}}{F_{n-1}}\,\frac{\frac{F_{n+k}}{F_{n+k-1}}\frac{f(y+1)}{f(y)}+1}{\frac{F_n}{F_{n-1}}\frac{f(y+1)}{f(y)}+1}\end{aligned}$$

and so

$$\lim_{x\to\infty}\frac{f(x+k)}{f(x)} = \lim_{y\to\infty}\lim_{n\to\infty}\frac{F_{n+k-1}}{F_{n-1}}\,\frac{\frac{F_{n+k}}{F_{n+k-1}}\frac{f(y+1)}{f(y)}+1}{\frac{F_n}{F_{n-1}}\frac{f(y+1)}{f(y)}+1} = \alpha^k\frac{\alpha\alpha+1}{\alpha\alpha+1} = \alpha^k$$

which completes the proof of Stage II.

Proof of Stage III:

By using Stage II, we obtain

$$\lim_{x\to\infty}\frac{f(x)}{f(x+m)}=\frac{1}{\lim_{x\to\infty}\frac{f(x+m)}{f(x)}}=\frac{1}{\alpha^m}=\alpha^{-m}.$$

Now, it follows that

$$\lim_{x\to\infty}\frac{f(x+k)}{f(x+m)}=\lim_{x\to\infty}\frac{f(x+k)}{f(x)}\lim_{x\to\infty}\frac{f(x)}{f(x+m)}=\alpha^k\alpha^{-m}=\alpha^{k-m}$$

which completes the proof of Stage III. □

2. Gaussian Fibonacci Function

Gaussian Fibonacci numbers $\{GF_n\}_{n\geq 0}=\{GF_n(GF_0,GF_1)\}_{n\geq 0}$ are defined by

$$GF_n=GF_{n-1}+GF_{n-2},$$

with the initial conditions $GF_0=i$ and $GF_1=1$. Note that

$$GF_n=F_n+iF_{n-1}.$$

The first few values of Gaussian Fibonacci numbers with positive and negative subscript are given in the following table.

n	0	1	2	3	4	5	6	7
GF_n	i	1	$1+i$	$2+i$	$3+2i$	$5+3i$	$8+5i$	$13+8i$
GF_{-n}		$1-i$	$-1+2i$	$2-3i$	$-3+5i$	$5-8i$	$-8+13i$	$13-21i$

The full Gaussian Fibonacci sequence, where $Gu_n=GF_n$ are n^{th} Gaussian Fibonacci numbers, is: $\ldots,5-8i,-3+5i,2-3i,-1+2i,1-i,i,1,1+i,2+i,3+2i,5+3i,\ldots$.

Definition 2.1. *A Gaussian function f_G on the real numbers $\mathbb{R}$ is said to be a Gaussian Fibonacci function if it satisfies the formula*

$$f_G(x+n)=GF_nf(x+1)+GF_{n-1}f(x) \tag{2.1}$$

for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$ where f is a Fibonacci function.

To emphasize which Fibonacci function is used we can say that f_G is a Gaussian Fibonacci function with the Fibonacci function f.

The following theorem gives an equivalent characterization of a Gaussian Fibonacci function.

Theorem 2.1. *A Gaussian function f_G on the real numbers $\mathbb{R}$ is a Gaussian Fibonacci function if and only if*

$$f_G(x+n)=f(x+n)+if(x+n-1) \tag{2.2}$$

for $x\in\mathbb{R}$, $n\in\mathbb{Z}$ where f is a Fibonacci function.

Proof. (:⇒) Assume that f_G is a Gaussian Fibonacci function, i.e., f_G satisfies (2.1). Then

$$\begin{aligned} f_G(x+n) &= GF_n f(x+1) + GF_{n-1} f(x) \\ &= (F_n + iF_{n-1}) f(x+1) + (F_{n-1} + iF_{n-2}) f(x) \\ &= (F_n f(x+1) + F_{n-1} f(x)) + i(F_{n-1} f(x+1) + F_{n-2} f(x)) \\ &= f(x+n) + i f(x+n-1) \end{aligned}$$

since

$$GF_n = F_n + iF_{n-1}$$

and

$$f(x+n) = F_n f(x+1) + F_{n-1} f(x).$$

(⇐:) If we suppose that (2.2) holds then we obtain

$$\begin{aligned} f_G(x+n) &= f(x+n) + i f(x+n-1) \\ &= (F_n f(x+1) + F_{n-1} f(x)) + i(F_{n-1} f(x+1) + F_{n-2} f(x)) \\ &= (F_n + iF_{n-1}) f(x+1) + (F_{n-1} + iF_{n-2}) f(x) \\ &= GF_n f(x+1) + GF_{n-1} f(x). \end{aligned}$$

□

Remark 2.1. Using the Binet's formula of a Fibonacci function f (see Lemma 1.1) and (2.1) or equivalenly (2.2), the Binet's formula of a Gaussian Fibonacci function can be found.

Now, we present an example of a Fibonacci function.

Example 2.1. *The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \alpha^x$, considered in Example 1.1, is a Fibonacci function. Then*

$$f_G(x+n) = f(x+n) + i f(x+n-1) = \alpha^{x+n} + i\alpha^{x+n-1} = (1 + i\alpha^{-1})\alpha^{x+n}$$

is a Gaussian Fibonacci function.

The following example shows that using the floor function, a Fibonacci function and a Gaussian Fibonacci function can be obtained.

Example 2.2. *Let $\{Gu_n\}_{n=-\infty}^{\infty}$ and $\{Gv_n\}_{n=-\infty}^{\infty}$, be full (bilateral) Gaussian Fibonacci sequences. We define a function f_G by $f_G(x+n) = Gu_{\lfloor x \rfloor + n} + Gv_{\lfloor x \rfloor + n} t = Gu_{\lfloor x+n \rfloor} + Gv_{\lfloor x+n \rfloor} t$ and $f(x) = u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor} t$, where $t = x - \lfloor x \rfloor \in (0,1)$ and $x \in \mathbb{R}$. Then, f is a Fibonacci function and f_G is a Gaussian Fibonacci function.*

Solution.

Since

$$\begin{aligned} f(x) &= u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor} t \\ f(x+1) &= u_{\lfloor x+1 \rfloor} + v_{\lfloor x+1 \rfloor} t = u_{\lfloor x \rfloor + 1} + v_{\lfloor x \rfloor + 1} t \\ f(x+2) &= u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor} t = u_{\lfloor x \rfloor + 2} + v_{\lfloor x \rfloor + 2} t \end{aligned}$$

and

$$\begin{aligned} f(x+1)+f(x) &= (u_{\lfloor x\rfloor+1}+v_{\lfloor x\rfloor+1}t)+(u_{\lfloor x\rfloor}+v_{\lfloor x\rfloor}t) \\ &= (u_{\lfloor x\rfloor+1}+u_{\lfloor x\rfloor})+(v_{\lfloor x\rfloor+1}+v_{\lfloor x\rfloor})t \\ &= u_{\lfloor x\rfloor+2}+v_{\lfloor x\rfloor+2}t = u_{\lfloor x+2\rfloor}+v_{\lfloor x+2\rfloor}t \\ &= f(x+2) \end{aligned}$$

f is a Fibonacci function and since

$$\begin{aligned} Gu_{\lfloor x\rfloor+n} &= u_{\lfloor x\rfloor+n}+iu_{\lfloor x\rfloor+n-1}, \\ Gv_{\lfloor x\rfloor+n} &= v_{\lfloor x\rfloor+n}+iv_{\lfloor x\rfloor+n-1}, \end{aligned}$$

we get

$$\begin{aligned} f_G(x+n) &= Gu_{\lfloor x\rfloor+n}+Gv_{\lfloor x\rfloor+n}t \\ &= (u_{\lfloor x\rfloor+n}+iu_{\lfloor x\rfloor+n-1})+(v_{\lfloor x\rfloor+n}+iv_{\lfloor x\rfloor+n-1})t \\ &= (u_{\lfloor x\rfloor+n}+v_{\lfloor x\rfloor+n}t)+(u_{\lfloor x\rfloor+n-1}+v_{\lfloor x\rfloor+n-1}t)i \\ &= f(x+n)+if(x+n-1). \end{aligned}$$

Therefore, f_G is a Gaussian Fibonacci function.

Lemma 2.1. *Let f_G be a Gaussian Fibonacci function, i.e., $f_G(x+n)=f(x+n)+if(x+n-1)$ for $x\in\mathbb{R}$, $n\in\mathbb{Z}$ where f is a Fibonacci function. We define $g_G(x+n)=f_G(x+t+n)$ and $g(x)=f(x+t)$ for any $x\in\mathbb{R}$ where $t\in\mathbb{R}$. Then g is a Fibonacci function and g_G is a Gaussian Fibonacci function.*

Proof. Let $x\in\mathbb{R}$. Since f_G is a Gaussian Fibonacci function and f is a Fibonacci function, it follows that

$$\begin{aligned} g(x+2) &= f(x+2+t)=f(x+t+2) \\ &= f(x+t+1)+f(x+t) \\ &= g(x+1)+g(x) \end{aligned}$$

which shows that g is a Fibonacci function and

$$\begin{aligned} g_G(x+n) &= f_G(x+t+n) \\ &= f(x+t+n)+if(x+t+n-1) \\ &= g(x+n)+ig(x+n-1) \end{aligned}$$

which shows that g_G is a Gaussian Fibonacci function. □

Lemma 2.2. *Let $\{u_n\}$ and $\{Gu_n\}$ be the full Fibonacci and Gaussian Fibonacci sequences, respectively. Then*

$$\begin{aligned} Gu_{\lfloor x\rfloor+n} &= GF_n u_{\lfloor x\rfloor+1}+GF_{n-1}u_{\lfloor x\rfloor}, \\ Gu_{\lfloor x\rfloor+n-1} &= GF_n u_{\lfloor x\rfloor}+GF_{n-1}u_{\lfloor x\rfloor-1}. \end{aligned}$$

Proof. The functions $f_G(x+n) = Gu_{\lfloor x\rfloor+n} + Gv_{\lfloor x\rfloor+n}t$ and $f(x) = u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t$ where $t = x - \lfloor x\rfloor \in (0,1)$ and $x \in \mathbb{R}$, considered in Example 2.2, are Gaussian Fibonacci and Fibonacci functions, respectively. So, if we let $v_{\lfloor x\rfloor} = u_{\lfloor x\rfloor-1}$, $Gv_{\lfloor x\rfloor+n} = Gu_{\lfloor x\rfloor+n-1}$, then $f(x)$ and $f_G(x)$ are Fibonacci function and Gaussian Fibonacci function, respectively. Note that

$$\begin{aligned} f(x) &= u_{\lfloor x\rfloor} + v_{\lfloor x\rfloor}t = u_{\lfloor x\rfloor} + u_{\lfloor x\rfloor-1}t \\ f(x+1) &= u_{\lfloor x+1\rfloor} + v_{\lfloor x+1\rfloor}t = u_{\lfloor x\rfloor+1} + v_{\lfloor x\rfloor+1}t = u_{\lfloor x\rfloor+1} + u_{\lfloor x\rfloor}t. \end{aligned}$$

Then we obtain

$$\begin{aligned} Gu_{\lfloor x\rfloor+n} + Gu_{\lfloor x\rfloor+n-1}t &= Gu_{\lfloor x\rfloor+n} + Gv_{\lfloor x\rfloor+n}t = f_G(x+n) \\ &= GF_n f(x+1) + GF_{n-1}f(x) \\ &= GF_n(u_{\lfloor x\rfloor+1} + u_{\lfloor x\rfloor}t) + GF_{n-1}(u_{\lfloor x\rfloor} + u_{\lfloor x\rfloor-1}t) \\ &= GF_n u_{\lfloor x\rfloor+1} + GF_n u_{\lfloor x\rfloor}t + GF_{n-1}u_{\lfloor x\rfloor} + GF_{n-1}u_{\lfloor x\rfloor-1}t \\ &= GF_n u_{\lfloor x\rfloor+1} + GF_{n-1}u_{\lfloor x\rfloor} + GF_n u_{\lfloor x\rfloor}t + GF_{n-1}u_{\lfloor x\rfloor-1}t \\ &= GF_n u_{\lfloor x\rfloor+1} + GF_{n-1}u_{\lfloor x\rfloor} + (GF_n u_{\lfloor x\rfloor} + GF_{n-1}u_{\lfloor x\rfloor-1})t. \end{aligned}$$

This completes the proof. □

By taking $\{u_n\} = \{F_n\}$ in the last theorem, we have the following corollary.

Corollary 2.1. *For $x \in \mathbb{R}$, we have the following formulas:*

$$\begin{aligned} GF_{\lfloor x\rfloor+n} &= GF_nF_{\lfloor x\rfloor+1} + GF_{n-1}F_{\lfloor x\rfloor}, \\ GF_{\lfloor x\rfloor+n-1} &= GF_nF_{\lfloor x\rfloor} + GF_{n-1}F_{\lfloor x\rfloor-1}. \end{aligned}$$

By taking $\lfloor x\rfloor = m \in \mathbb{Z}$ in the last corollary, we see that for all integers m,n we have

$$\begin{aligned} GF_{m+n} &= GF_nF_{m+1} + GF_{n-1}F_m, \\ GF_{m+n-1} &= GF_nF_m + GF_{n-1}F_{m-1}. \end{aligned}$$

Theorem 2.2. *Let $f_G(x) = a(x)g_G(x)$ be a function, $g(x)$ and $f(x) = a(x)g(x)$ be Fibonacci functions, where $a(x)$ is an f-even function, and suppose that $g_G(x)$ and $g(x)$ are continuous functions. Then $f_G(x)$ is a Gaussian Fibonacci function with Fibonacci function $f(x)$ if and only if $g_G(x)$ is a Gaussian Fibonacci function with Fibonacci function $g(x)$.*

Proof. By definition of the function f_G and since $a(x)$ is an f-even function, we have

$$f_G(x+n) = a(x+n)g_G(x+n) = a(x)g_G(x+n). \tag{2.3}$$

Suppose that f_G is a Gaussian Fibonacci function. Then, since $a(x)$ is an f-even function, we obtain

$$\begin{aligned}
f_G(x+n) &= f(x+n)+if(x+n-1) \\
&= a(x+n)g(x+n)+ia(x+n-1)g(x+n-1) \\
&= a(x)g(x+n)+ia(x)g(x+n-1) \\
&= a(x)(g(x+n)+ig(x+n-1)). \qquad (2.4)
\end{aligned}$$

From the equations (2.3) and (2.4), we get

$$a(x)(g_G(x+n)-g(x+n)-ig(x+n-1)) \equiv 0$$

and so

$$g_G(x+n)-g(x+n)-ig(x+n-1) \equiv 0$$

i.e.,

$$g_G(x+n) = g(x+n)+ig(x+n-1).$$

Therefore, g_G is a Gaussian Fibonacci function.
On the other hand, if g_G is a Gaussian Fibonacci function, then

$$g_G(x+n) = g(x+n)+ig(x+n-1). \qquad (2.5)$$

Since $f(x) = a(x)g(x)$ and $a(x)$ is an f-even function, we obtain

$$f(x+n) = a(x+n)g(x+n) = a(x)g(x+n),$$
$$f(x+n-1) = a(x+n-1)g(x+n-1) = a(x)g(x+n-1).$$

Then, since $f_G(x) = a(x)g_G(x)$ and $a(x)$ is an f-even function, the equation (2.5) implies that

$$\begin{aligned}
f_G(x+n) &= a(x+n)g_G(x+n) = a(x)g_G(x+n) \\
&= a(x)(g(x+n)+ig(x+n-1)) \\
&= a(x)g(x+n)+ia(x)g(x+n-1) \\
&= f(x+n)+if(x+n-1).
\end{aligned}$$

Hence, f_G is a Gaussian Fibonacci function. □

3. Sums of Fibonacci and Gaussian Fibonacci Functions

In this section, we discuss the sums of the terms of a Fibonacci function and a Gaussian Fibonacci function. The following corollary gives linear sum formulas of Fibonacci numbers.

Corollary 3.1. *For $n \geq 0$, Fibonacci numbers have the following property:*

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1.$$

Proof. For a proof see, for example, Soykan [17, Corollary 2.2 (a)]. □

The following theorem gives linear sum formulas of Fibonacci functions.

Theorem 3.1. *Suppose that f is a Fibonacci function. Then for all $x \in \mathbb{R}$ and $n \geq 0$, the following sum formula holds:*

$$\sum_{k=0}^{n} f(x+k) = f(x+n+2) - f(x+1).$$

Proof. We use corollary 3.1. Since

$$\sum_{k=0}^{n} F_{k-1} = F_{n+1}$$

and

$$f(x+n) = F_n f(x+1) + F_{n-1} f(x),$$

we obtain

$$\begin{aligned}\sum_{k=0}^{n} f(x+k) &= f(x+1)\sum_{k=0}^{n} F_k + f(x)\sum_{k=0}^{n} F_{k-1} \\ &= f(x+1)(F_{n+2}-1) + f(x)F_{n+1} \\ &= (F_{n+2}f(x+1) + F_{n+1}f(x)) - f(x+1) \\ &= f(x+n+2) - f(x+1). \qquad \square\end{aligned}$$

Note that if we consider the Fibonacci function

$$f(x) = \alpha^x$$

then, using Theorem 3.1, we have the sum formula

$$\sum_{k=0}^{n} \alpha^{x+k} = \alpha^{x+n+2} - \alpha^{x+1}$$

for all $x \in \mathbb{R}$ and $n \geq 0$.

The following corollary gives linear sum formulas of Gaussian Fibonacci numbers.

Corollary 3.2. *For $n \geq 0$ we have the following formula:*

$$\sum_{k=0}^{n} GF_k = GF_{n+2} - 1.$$

Proof. It is given in Soykan [17, Corollary 4.2 (a)]. $\square$

The following theorem gives linear sum formulas of Gaussian Fibonacci functions.

Theorem 3.2. *Suppose that f_G is a Gaussian Fibonacci function with Fibonacci function f. Then for all $x \in \mathbb{R}$ and $n \geq 0$ the following sum formula holds:*

$$\sum_{k=0}^{n} f_G(x+k) = f_G(x+n+2) - f(x+1) - i f(x).$$

Proof. We use corollary 3.2. Since

$$\sum_{k=0}^{n} GT_k = GF_{n+2} - 1$$
$$\sum_{k=0}^{n} GT_{k-1} = GF_{n+1} - i$$

and

$$f_G(x+n) = GF_n f(x+1) + GF_{n-1} f(x),$$

we get

$$\begin{aligned}\sum_{k=0}^{n} f_G(x+k) &= f(x+1)\sum_{k=0}^{n} GF_k + f(x)\sum_{k=0}^{n} GF_{k-1} \\ &= f(x+1)(GF_{n+2}-1) + f(x)(GF_{n+1}-i) \\ &= (GF_{n+2}f(x+1) + GF_{n+1}f(x)) - f(x+1) - if(x) \\ &= f_G(x+n+2) - f(x+1) - if(x).\end{aligned}$$

□

Note that if we consider the Fibonacci function

$$f(x) = \alpha^x$$

and the Gaussian Fibonacci function

$$f_G(x+n) = (1+i\alpha^{-1})\alpha^{x+n}$$

then, using Theorem 3.2, we have the sum formula

$$\sum_{k=0}^{n} (1+i\alpha^{-1})\alpha^{x+k} = (1+i\alpha^{-1})\alpha^{x+n+2} - \alpha^{x+1} - i\alpha^x$$

for all $x \in \mathbb{R}$ and $n \geq 0$.

4. Ratio of Gaussian Fibonacci Functions

In this section, we discuss the limit of the quotient of a Gaussian Fibonacci function. Note that since

$$\lim_{n\to\infty} \frac{F_{n+p}}{F_{n+q}} = \alpha^{p-q}, \quad p, q \in \mathbb{Z}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \frac{GF_{n+p}}{GF_{n+q}} &= \lim_{n\to\infty} \frac{F_{n+p} + iF_{n+p-1}}{F_{n+q} + iF_{n+q-1}} = \lim_{n\to\infty} \frac{\frac{F_{n+p}}{F_{n+q}} + i\frac{F_{n+p-1}}{F_{n+q}}}{\frac{F_{n+q}}{F_{n+q}} + i\frac{F_{n+q-1}}{F_{n+q}}} \\ &= \frac{\alpha^{p-q} + i\alpha^{p-1-q}}{1+i\alpha^{-1}}\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{GF_{n+p}}{F_{n+q}} &= \lim_{n\to\infty} \frac{F_{n+p} + iF_{n+p-1}}{F_{n+q}} = \lim_{n\to\infty} \frac{F_{n+p}}{F_{n+q}} + i\lim_{n\to\infty} \frac{F_{n+p-1}}{F_{n+q}} \\ &= \alpha^{p-q} + i\alpha^{p-1-q}.\end{aligned}$$

Theorem 4.1. *If f_G is a Gaussian Fibonacci function, then the limit of quotient*

$$\frac{f_G(x+k)}{f_G(x+m)}$$

exists and is given by

$$\lim_{x\to\infty} \frac{f_G(x+k)}{f_G(x+m)} = \alpha^{k-m}$$

for all $k,m \in \mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Fibonacci function with Fibonacci function f. Note that from Theorem 1.2, the limit of quotients $\frac{f(x+1)}{f(x)}$ exists and $\lim\limits_{x\to\infty} \frac{f(x+1)}{f(x)} = \alpha$. We use the formula, by definition,

$$f_G(x+n) = GF_n f(x+1) + GF_{n-1} f(x).$$

Given $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y+n$. Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{f_G(x+k)}{f_G(x+m)} &= \lim_{y\to\infty}\lim_{n\to\infty} \frac{f_G(y+n+k)}{f_G(y+n+m)} = \lim_{x\to\infty}\lim_{n\to\infty} \frac{f_G(x+n+k)}{f_G(x+n+m)} \\
&= \lim_{x\to\infty}\lim_{n\to\infty} \frac{GF_{n+k} f(x+1) + GF_{n+k-1} f(x)}{GF_{n+m} f(x+1) + GF_{n+m-1} f(x)} \\
&= \lim_{x\to\infty}\lim_{n\to\infty} \frac{\dfrac{GF_{n+k}}{GF_{n+m-1}} \dfrac{f(x+1)}{f(x)} + \dfrac{GF_{n+k-1}}{GF_{n+m-1}}}{\dfrac{GF_{n+m}}{GF_{n+m-1}} \dfrac{f(x+1)}{f(x)} + 1}.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{F_{n+p}}{F_{n+q}} &= \alpha^{p-q}, \quad p,q \in \mathbb{Z}, \\
\lim_{n\to\infty} \frac{GF_{n+p}}{GF_{n+q}} &= \frac{\alpha^{p-q} + i\alpha^{p-1-q}}{1+i\alpha^{-1}}, \quad p,q \in \mathbb{Z}, \\
\lim_{x\to\infty} \frac{f(x+1)}{f(x)} &= \alpha,
\end{aligned}$$

it follows that

$$\lim_{x\to\infty} \frac{f_G(x+k)}{f_G(x+m)} = \alpha^{k-m}.$$ □

Note that if we consider the Fibonacci function

$$f(x) = \alpha^x$$

and the Gaussian Fibonacci function

$$f_G(x+n) = (1+i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$\lim_{x\to\infty} \frac{f_G(x+k)}{f_G(x+m)} = \lim_{x\to\infty} \frac{(1+i\alpha^{-1})\alpha^{x+k}}{(1+i\alpha^{-1})\alpha^{x+m}} = \lim_{x\to\infty} \frac{1}{\alpha^{m+x}}\alpha^{k+x} = \alpha^{k-m}.$$

Also, it follows from Theorem 4.1, that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)}=\alpha^{k-m}.$$

Corollary 4.1. *If f_G is a Gaussian Fibonacci function, then*

$$\lim_{x\to\infty}\frac{f_G(x+1)}{f_G(x)}=\alpha.$$

Proof. Take $k=1,m=0$ in Theorem 4.1. □

Theorem 4.2. *If f_G is a Gaussian Fibonacci function with Fibonacci function f, then the limit of quotient*

$$\frac{f_G(x+k)}{f(x)}$$

exists and is given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=\alpha GF_k+GF_{k-1}$$

for all $k\in\mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Fibonacci function with Fibonacci function f. Note that from Theorem 1.2, the limit of quotients $\frac{f(x+1)}{f(x)}$ exists. Using the formula, by definition,

$$f_G(x+n)=GF_nf(x+1)+GF_{n-1}f(x).$$

we get

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}&=\lim_{x\to\infty}\frac{GF_kf(x+1)+GF_{k-1}f(x)}{f(x)}\\&=\lim_{x\to\infty}GF_k\frac{f(x+1)}{f(x)}+GF_{k-1}.\end{aligned}$$

Hence, since the limit of quotient $\frac{f(x+1)}{f(x)}$ exists, $\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}$ exists and

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=\alpha GF_k+GF_{k-1}.$$ □

Note that if we consider the Fibonacci function

$$f(x)=\alpha^x$$

and the Gaussian Fibonacci function

$$f_G(x+n)=(1+i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}&=\lim_{x\to\infty}\frac{(1+i\alpha^{-1})\alpha^{x+k}}{\alpha^x}=\lim_{x\to\infty}(1+i\alpha^{-1})\alpha^k\\&=(1+i\alpha^{-1})\alpha^k.\end{aligned}\tag{4.1}$$

Also, from Theorem 4.2, we know that

$$\lim_{x\to\infty} \frac{f_G(x+k)}{f(x)} = \alpha GF_k + GF_{k-1}. \tag{4.2}$$

Therefore, comparing (4.1) and (4.2), we obtain

$$\alpha GF_k + GF_{k-1} = (1+i\alpha^{-1})\alpha^k$$

for $k \in \mathbb{Z}$.

Corollary 4.2. *If f_G is a Gaussian Fibonacci function with Fibonacci function f, then*

$$\lim_{x\to\infty} \frac{f_G(x)}{f(x)} = 1 + (\alpha - 1)\, i,$$
$$\lim_{x\to\infty} \frac{f_G(x+1)}{f(x)} = \alpha + i,$$
$$\lim_{x\to\infty} \frac{f_G(x+2)}{f(x)} = \alpha + 1 + \alpha i.$$

Proof. Take $k = 0, 1, 2$ in Theorem 4.2, respectively. □

We can generalize Theorem 4.2 as follows.

Theorem 4.3. *If f_G is a Gaussian Fibonacci function with Fibonacci function f, then the limit of the quotient*

$$\frac{f_G(x+k)}{f(x+m)}$$

exists and is given by

$$\lim_{x\to\infty} \frac{f_G(x+k)}{f(x+m)} = (\alpha + i)\, \alpha^{k-m-1}$$

for all $k, m \in \mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Fibonacci function with Fibonacci function f. Note that from Theorem 1.2, the limit of quotient $\frac{f(x+1)}{f(x)}$ exists and $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \alpha$. Given $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y + n$. By using the formulas

$$f_G(x+n) = GF_n f(x+1) + GF_{n-1} f(x)$$

and

$$f(x+n) = F_n f(x+1) + F_{n-1} f(x)$$

we get

$$\begin{aligned}
\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{f_G(y+n+k)}{f(y+n+m)} = \lim_{x\to\infty}\lim_{n\to\infty}\frac{f_G(x+n+k)}{f(x+n+m)} \\
&= \lim_{x\to\infty}\lim_{n\to\infty}\frac{GF_{n+k}\dfrac{f(x+1)}{f(x)}+GF_{n+k-1}}{F_{n+m}\dfrac{f(x+1)}{f(x)}+F_{n+m-1}} \\
&= \lim_{x\to\infty}\lim_{n\to\infty}\frac{\dfrac{GF_{n+k}}{F_{n+m-1}}\dfrac{f(x+1)}{f(x)}+\dfrac{GF_{n+k-1}}{F_{n+m-1}}}{\dfrac{F_{n+m}}{F_{n+m-1}}\dfrac{f(x+1)}{f(x)}+1}.
\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{F_{n+p}}{F_{n+q}} = \alpha^{p-q}, \quad p,q\in\mathbb{Z},$$

$$\lim_{n\to\infty}\frac{GF_{n+p}}{F_{n+q}} = \lim_{n\to\infty}\frac{F_{n+p}}{F_{n+q}}+i\frac{F_{n+p-1}}{F_{n+q}} = \alpha^{p-q}+i\alpha^{p-1-q}, \quad p,q\in\mathbb{Z},$$

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \alpha,$$

it follows that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} = (\alpha+i)\,\alpha^{k-m-1}. \qquad \square$$

5. Matrix Formulation of $f(x)$ and $f_G(x+n)$

The matrix method is a very useful method in order to obtain some identities for special sequences. We define the square matrix M of order 2 as:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

such that $\det M = -1$. Note that for all $n\in\mathbb{Z}$, we have

$$M^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Matrix formulation of F_n can be given as

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}.$$

Consider the matrices N_F, E_F defined by as follows:

$$N_F = \begin{pmatrix} 1+i & 1 \\ 1 & i \end{pmatrix},$$

$$E_F = \begin{pmatrix} GF_{n+2} & GF_{n+1} \\ GF_{n+1} & GF_n \end{pmatrix}.$$

The following theorem presents the relations between M^n, N_F and E_F.

Theorem 5.1. *For all $n \in \mathbb{Z}$, we have*
$$M^n N_F = E_F.$$

Proof. It can be proved by mathematical induction. $\square$

Define
$$A_f = \begin{pmatrix} f(x+2) & f(x+1) \\ f(x+1) & f(x) \end{pmatrix},$$
$$B_f = \begin{pmatrix} f(x+n+2) & f(x+n+1) \\ f(x+n+1) & f(x+n) \end{pmatrix}.$$

Theorem 5.2. *For all integers $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*
$$M^n A_f = B_f. \tag{5.1}$$

Proof. By using
$$f(x+n) = F_n f(x+1) + F_{n-1} f(x)$$
and
$$f(x+2) = f(x+1) + f(x),$$
the case $n \geq 0$ can be proved by mathematical induction. Then for the case $n \leq 0$, we take $m = -n$ in (5.1) and then the case $m \geq 0$ can be proved by mathematical induction, as well. $\square$

Note that if we consider the Fibonacci function
$$f(x) = \alpha^x$$
then, we see that
$$A_f = \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} \\ \alpha^{x+1} & \alpha^x \end{pmatrix}, B_f = \begin{pmatrix} \alpha^{x+n+2} & \alpha^{x+n+1} \\ \alpha^{x+n+1} & \alpha^{x+n} \end{pmatrix}$$
and
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} \\ \alpha^{x+1} & \alpha^x \end{pmatrix} = \begin{pmatrix} \alpha^{x+n+2} & \alpha^{x+n+1} \\ \alpha^{x+n+1} & \alpha^{x+n} \end{pmatrix}$$
for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Define
$$D_{GF} = \begin{pmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{pmatrix}$$
and
$$C_{f_G} = \begin{pmatrix} f_G(x+n+2) & f_G(x+n) \\ f_G(x+n+1) & f_G(x+n-1) \end{pmatrix}.$$

Theorem 5.3. *For all integers $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*
$$D_{GF} A_f = C_{f_G} \tag{5.2}$$

Proof. By using
$$f_G(x+n) = GF_n f(x+1) + GF_{n-1} f(x).$$
and
$$f(x+2) = f(x+1) + f(x),$$
the case $n \geq 0$ can be proved by mathematical induction. Then for the case $n \leq 0$,

we take $m = -n$ in (5.2) and then the case $m \geq 0$ can be proved by mathematical induction, as well. $\square$

Note that if we consider the Fibonacci function

$$f(x) = \alpha^x$$

and the Gaussian Fibonacci function

$$f_G(x+n) = (1+i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$A_f = \begin{pmatrix} \alpha^{x+2} & \alpha^x \\ \alpha^{x+1} & \alpha^{x-1} \end{pmatrix},$$

$$D_{GF} = \begin{pmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{pmatrix}$$

and

$$C_{f_G} = \begin{pmatrix} (1+i\alpha^{-1})\alpha^{x+n+2} & (1+i\alpha^{-1})\alpha^{x+n} \\ (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n-1} \end{pmatrix},$$

and so

$$\begin{pmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{pmatrix}\begin{pmatrix} \alpha^{x+2} & \alpha^x \\ \alpha^{x+1} & \alpha^{x-1} \end{pmatrix} = \begin{pmatrix} (1+i\alpha^{-1})\alpha^{x+n+2} & (1+i\alpha^{-1})\alpha^{x+n} \\ (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n-1} \end{pmatrix}.$$

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