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ON A NEW CLASS RELATED TO THE SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. This paper is concerned with a generalized class of analytic functions which is related to the subclass of close-to-convex functions in the open unit disc $E = \{z : |z| < 1\}$. The coefficient estimates, distortion theorem, growth theorem, argument theorem, radius of convexity, Fekete-Szegö inequality and inclusion relation for the functions belonging to this class have been established. The results so obtained will provide a new direction in the study of certain new subclasses of analytic functions.

1. INTRODUCTION

Let $\mathcal{U}$ denote the class of Schwarzian functions of the form $w(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and with the conditions $w(0) = 0, |w(z)| < 1$. Also $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$. For two analytic functions f and g in E, we say that f is subordinate to g, if a Schwarzian function $w(z) \in \mathcal{U}$ exists, such that $f(z) = g(w(z))$ and this is denoted by $f \prec g$. If g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. Littlewood [5] and Reade [10] introduced the concept of subordination.

The class of functions f which are analytic in E and normalized by the condition $f(0) = f'(0) - 1 = 0$ is denoted by $\mathcal{A}$ and has the Taylor series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The well known classes of univalent, starlike and convex functions are denoted by $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{K}$ respectively.

A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a starlike function g such that $Re\left(\frac{zf'(z)}{g(z)}\right) > 0$. The class of close-to-convex functions is denoted by $\mathcal{C}$ and was introduced by Kaplan [3]. For $-1 \leq D < C \leq 1$, Mehrok [8] introduced and studied the subclass of close-to-convex functions $\mathcal{C}(C,D)$ which consists of

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the functions $f \in \mathcal{A}$ with the condition that $\frac{zf'(z)}{g(z)} \prec \frac{1+Cz}{1+Dz}$, where the condition holds for a starlike function g. Obviously $\mathcal{C}(1,-1) \equiv \mathcal{C}$.

Further Abdel Gawad and Thomas [1] studied the class $\mathcal{C}_1$ of functions $f \in \mathcal{A}$ satisfying the condition $Re\left(\frac{zf'(z)}{h(z)}\right) > 0$, where h is a convex function. Clearly $\mathcal{C}_1$ is a subclass of close-to-convex functions. Following this, Mehrok and Singh [9] studied the class $\mathcal{C}_1(C,D)$ consisting of the functions $f \in \mathcal{A}$ along with the condition that $\frac{zf'(z)}{h(z)} \prec \frac{1+Cz}{1+Dz}, h \in \mathcal{K}$. Particularly $\mathcal{C}_1(1,-1) \equiv \mathcal{C}_1$. Various properties related to other subclasses of analytic functions were studied recently by Mateljevic et al. [7]. Stelin and Selvaraj [13] studied the class $\mathcal{K}'_{\mathcal{C}}(\alpha)(\alpha \geq 0)$ consisting of the functions $f \in \mathcal{A}$ satisfying the following condition:

$$Re\left(\frac{f'(z)}{h'(z)}\right) > \alpha, h \in \mathcal{C}_1.$$

As a generalization, for $-1 \leq D < C \leq 1$, Singh and Singh [11] introduced the class $\mathcal{K}'_{\mathcal{C}}(C,D)$ containing the functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{f'(z)}{h'(z)} \prec \frac{1+Cz}{1+Dz}, h \in \mathcal{C}_1.$$

For $C = 1-2\alpha, D = -1$, the class $\mathcal{K}'_{\mathcal{C}}(C,D)$ agrees with the class $\mathcal{K}'_{\mathcal{C}}(\alpha)$.

Further, for $-1 \leq D \leq B < A \leq C \leq 1$, Singh and Singh [12] studied the class $\mathcal{K}'_{\mathcal{C}}(A,B;C,D)$ consisting of the functions $f \in \mathcal{A}$ satisfying the condition

$$\frac{f'(z)}{h'(z)} \prec \frac{1+Cz}{1+Dz}, h \in \mathcal{C}_1(A,B),$$

In particular, $\mathcal{K}'_{\mathcal{C}}(1,-1;C,D) \equiv \mathcal{K}'_{\mathcal{C}}(C,D)$.

Getting motivation from the above work, now we define the following class which is the subject of study in this paper;

Definition 1.1. *For $-1 \leq D \leq B < A \leq C \leq 1$, $\mathcal{K}^*_{\mathcal{C}}(A,B;C,D)$ denotes the class of functions $f \in \mathcal{A}$ satisfying the condition*

$$\frac{f'(z)}{g'(z)} \prec \frac{1+Cz}{1+Dz},$$

where

$$g(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{C}(A,B).$$

The following observations are obvious:

(i) $\mathcal{K}^*_{\mathcal{C}}(1,-1;C,D) \equiv \mathcal{K}^*_{\mathcal{C}}(C,D)$.
(ii) $\mathcal{K}^*_{\mathcal{C}}(1,-1;C,D) \equiv \mathcal{K}^*_{\mathcal{C}}(C,D)$.
(iii) $\mathcal{K}^*_{\mathcal{C}}(1,-1;C,D) \equiv \mathcal{K}^*_{\mathcal{C}}(C,D)$.

The present investigation deals with the study of the class $\mathcal{K}_{\mathcal{C}}^{*}(A,B;C,D)$. We establish the coefficient estimates, distortion theorem, growth theorem, argument theorem, radius of convexity, Fekete-Szegö inequality and inclusion relation for the functions in this class. This paper will motivate the other researchers for the further study in this direction.

2. Preliminary Results

Lemma 2.1. [2] *If* $P(z)=\dfrac{1+Cw(z)}{1+Dw(z)}=1+\sum_{k=1}^{\infty}p_kz^k$, *then*

$$|p_n|\leq (C-D), n\geq 1.$$

The bound is sharp for the function $P_n(z)=\dfrac{1+C\delta z^n}{1+D\delta z^n}, |\delta|=1.$

Lemma 2.2. [8] *If* $g(z)=z+\sum_{k=2}^{\infty}d_kz^k\in\mathcal{C}(A,B)$, *then,*

$$|d_n|\leq 1+\frac{(n-1)(A-B)}{2}.$$

Equality is attained for $g'(z)=\dfrac{1}{(1-\delta_1 z)^2}\left(\dfrac{1+A\delta_2 z^{n-1}}{1+B\delta_2 z^{n-1}}\right), |\delta_1|=1, |\delta_2|=1.$

Lemma 2.3. [8] *If* $g(z)=z+\sum_{k=2}^{\infty}d_kz^k\in\mathcal{C}(A,B)$, *then for* $|z|=r, 0<r<1$, *we have*

$$\frac{1-Ar}{(1-Br)(1+r)^2}\leq |g'(z)|\leq\frac{1+Ar}{(1+Br)(1-r)^2}.$$

Lemma 2.4. [8] *If* $g(z)=z+\sum_{k=2}^{\infty}d_kz^k\in\mathcal{C}(A,B)$, *then for* $|z|=r, 0<r<1$, *we have*

$$|arg(g'(z))|\leq 2sin^{-1}r+sin^{-1}\frac{(A-B)r}{1-ABr^2}.$$

Lemma 2.5. [8] *If* $g(z)=z+\sum_{k=2}^{\infty}d_kz^k\in\mathcal{C}(A,B)$, *then*

$$|d_2|\leq 1+\frac{(A-B)}{2},$$

$$|d_3|\leq 1+\frac{(A-B)}{3}(2+|B|)$$

and

$$|d_3-\mu d_2^2|\leq\frac{1}{3}max\{1,|3\mu-3|\}+\frac{(A-B)}{3}\left[|2+3\mu|+max\left\{1,B+\frac{3(A-B)\mu}{4}\right\}\right].$$

Lemma 2.6. [4] *If* $w(z)=\sum_{k=1}^{\infty}c_kz^k\in\mathcal{U}$, *then for* μ *complex,*

$$|c_2-\mu c_1^2|\leq max\{1,|\mu|\}.$$

Lemma 2.7. [6] *Let* $-1 \leq D_2 \leq D_1 < C_1 \leq C_2 \leq 1$. *Then*

$$\frac{1+C_1 z}{1+D_1 z} \prec \frac{1+C_2 z}{1+D_2 z}.$$

3. Main Results

Theorem 3.1. *If* $f(z) \in \mathcal{K}_{\mathcal{C}}^*(A,B;C,D)$*, then*

$$|a_n| \leq 1 + \frac{(n-1)}{2}\left[(A-B)+(C-D)+\frac{(A-B)(C-D)(n-2)}{3}\right]. \tag{3.1}$$

The bound is sharp.

Proof. By using the Principle of subordination in Definition 1.1, we have

$$f'(z) = g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), \tag{3.2}$$

where $w \in \mathcal{U}$ is a Schwarzian function.

After expanding (3.2), it yields

$$1+2a_2 z+3a_3 z^2+\ldots+na_n z^{n-1}+\ldots$$
$$=(1+2d_2 z+3d_3 z^2+\ldots+nd_n z^{n-1}+\ldots)(1+p_1 z+p_2 z^2+\ldots+p_{n-1}z^{n-1}+\ldots). \tag{3.3}$$

On equating the coefficients of z^{n-1} on both sides of (3.3), we obtain

$$na_n = nd_n + p_1(n-1)d_{n-1} + p_2(n-2)d_{n-2}\ldots+2p_{n-2}d_2+p_{n-1}. \tag{3.4}$$

Applying the triangle inequality in (3.4), it yields

$$n|a_n| \leq n|d_n|+(n-1)|p_1||d_{n-1}|+(n-2)|p_2||d_{n-2}|+\ldots+2|p_{n-2}||d_2|+|p_{n-1}|.$$

Again using Lemma 2.1, the above inequality reduces to

$$n|a_n| \leq n|d_n|+(C-D)\left[(n-1)|d_{n-1}|+(n-2)|d_{n-2}|\ldots+2|d_2|+1\right]. \tag{3.5}$$

Making use of Lemma 2.2 in (3.5), the result (3.1) can be easily obtained.

For $n \geq 2$, equality in (3.1) holds for the function $f_n(z)$ defined as

$$f_n'(z) = \frac{1}{(1-\delta_1 z)^2}\left(\frac{1+A\delta_1 z^{n-1}}{1+B\delta_2 z^{n-1}}\right)\left(\frac{1+C\delta_2 z^{n-1}}{1+D\delta_2 z^{n-1}}\right), |\delta_1|=1, |\delta_2|=1. \tag{3.6}$$

□

For $A=1, B=-1$, Theorem 3.1 gives the following result:

Corollary 3.1. *If* $f(z) \in \mathcal{K}_{\mathcal{C}}^*(C,D)$*, then,*

$$|a_n| \leq n + \frac{(n-1)(2n-1)(C-D)}{6}.$$

Substituting $A=1, B=-1, C=1-2\alpha, D=-1$, Theorem 3.1 agrees with the result given below:

Corollary 3.2. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(\alpha)$, then,*

$$|a_n| \leq n + \frac{(1-\alpha)(n-1)(2n-1)}{3}.$$

Substituting $A = 1, B = -1, C = 1, D = -1$ in Theorem 3.1, it yields the following result:

Corollary 3.3. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}$, then,*

$$|a_n| \leq \frac{2n^2+1}{3}.$$

Theorem 3.2. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(A,B;C,D)$, then for $|z| = r, 0 < r < 1$, we have*

$$\frac{(1-Cr)(1-Ar)}{(1-Dr)(1-Br)(1+r)^2} \leq |f'(z)| \leq \frac{(1+Cr)(1+Ar)}{(1+Dr)(1+Br)(1-r)^2}; \tag{3.7}$$

$$\int_0^r \frac{(1-Ct)(1-At)}{(1-Dt)(1-Bt)(1+t)^2}dt \leq |f(z)| \leq \int_0^r \frac{(1+Ct)(1+At)}{(1+Dt)(1+Bt)(1-t)^2}dt. \tag{3.8}$$

These estimates are sharp.

Proof. From (3.2), we have

$$|f'(z)| = |g'(z)| \left|\frac{1+Cw(z)}{1+Dw(z)}\right|. \tag{3.9}$$

It can be easily proved that the transformation

$$\frac{f'(z)}{g'(z)} = \frac{1+Cw(z)}{1+Dw(z)}$$

maps $|w(z)| \leq r$ onto the circle

$$\left|\frac{f'(z)}{g'(z)} - \frac{1-CDr^2}{1-D^2r^2}\right| \leq \frac{(C-D)r}{(1-D^2r^2)}, |z| = r.$$

This implies that

$$\frac{1-Cr}{1-Dr} \leq \left|\frac{1+Cw(z)}{1+Dw(z)}\right| \leq \frac{1+Cr}{1+Dr}. \tag{3.10}$$

Using Lemma 2.3 and (3.10) in (3.9), the result (3.7) is obvious. Again, on integrating (3.7) with limits from 0 to r, the result (3.8) can be easily obtained.

Sharpness follows for the function defined in (3.6).

For $A = 1, B = -1$, Theorem 3.2 gives the following result: **Corollary 3.4** If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(C,D)$, then

$$\frac{(1-Cr)(1-r)}{(1-Dr)(1+r)^3} \leq |f'(z)| \leq \frac{(1+Cr)(1+r)}{(1+Dr)(1-r)^3};$$

$$\int_0^r \frac{(1-Ct)(1-t)}{(1-Dt)(1+t)^3}dt \leq |f(z)| \leq \int_0^r \frac{(1+Ct)(1+t)}{(1+Dt)(1-t)^3}dt. \qquad \square$$

Substituting $A=1, B=-1, C=1-2\alpha, D=-1$, Theorem 3.2 agrees with the result given below:

Corollary 3.5. *If $f(z)\in\mathcal{K}_{\mathcal{C}}^{*}(\alpha)$, then,*

$$\frac{(1-(1-2\alpha)r)(1-r)}{(1+r)^4}\leq|f'(z)|\leq\frac{(1+(1-2\alpha)r)(1+r)}{(1-r)^4};$$

$$\int_0^r\frac{(1-(1-2\alpha)t)(1-t)}{(1+t)^4}dt\leq|f(z)|\leq\int_0^r\frac{(1+(1-2\alpha)t)(1+t)}{(1-t)^4}dt.$$

On Substituting $A=1, B=-1, C=1, D=-1$, Theorem 3.2 gives the following result:

Corollary 3.6. *If $f(z)\in\mathcal{K}_{\mathcal{C}}^{*}$, then,*

$$\frac{(1-r)^2}{(1+r)^4}\leq|f'(z)|\leq\frac{(1+r)^2}{(1-r)^4};$$

$$\int_0^r\frac{(1-t)^2}{(1+t)^4}dt\leq|f(z)|\leq\int_0^r\frac{(1+t)^2}{(1-t)^4}dt.$$

Theorem 3.3. *If $f(z)\in\mathcal{K}_{\mathcal{C}}^{*}(A,B;C,D)$, then*

$$\left|arg(f'(z))\right|\leq 2sin^{-1}r+sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right). \tag{3.11}$$

The estimate is sharp.

Proof. (3.2) can be expressed as

$$f'(z)=g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right).$$

Therefore, we have

$$\left|arg\,f'(z)\right|\leq\left|arg\left(\frac{1+Cw(z)}{1+Dw(z)}\right)\right|+\left|arg\,g'(z)\right|. \tag{3.12}$$

As in Theorem 2, it is clear that

$$\left|\frac{f'(z)}{g'(z)}-\frac{1-CDr^2}{1-D^2r^2}\right|\leq\frac{(C-D)r}{(1-D^2r^2)}.$$

So, it yields

$$\left|arg\left(\frac{1+Cw(z)}{1+Dw(z)}\right)\right|\leq sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right). \tag{3.13}$$

By using Lemma 2.4 and inequality (3.13) in (3.12), the result (3.11) is obvious. □

The result is sharp for the function defined in (3.6), where

$$\delta_1 = \frac{r}{z}\left[\frac{-(C+D)r+i((1-C^2r^2)(1-D^2r^2))^{\frac{1}{2}}}{(1+CDr^2)}\right], \delta_2 = \frac{r}{z}\left[-Dr+i(1-D^2r^2)^{\frac{1}{2}}\right].$$

For $A=1, B=-1$, Theorem 3.3 gives the following result:

Corollary 3.7. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(C,D)$, then*

$$\left|arg f'(z)\right| \leq 2sin^{-1}r + sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + sin^{-1}\left(\frac{2r}{1+r^2}\right).$$

Substituting $A=1, B=-1, C=1-2\alpha, D=-1$, Theorem 3.3 agrees with the result given below:

Corollary 3.8. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(\alpha)$, then,*

$$\left|arg f'(z)\right| \leq 2sin^{-1}r + sin^{-1}\left(\frac{(2-2\alpha)r}{1+(1-2\alpha)r^2}\right) + sin^{-1}\left(\frac{2r}{1+r^2}\right).$$

For $A=1, B=-1, C=1, D=-1$, Theorem 3.3 agrees with the result given below:

Corollary 3.9. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}$, then,*

$$\left|arg f'(z)\right| \leq 2sin^{-1}r + 2sin^{-1}\left(\frac{2r}{1+r^2}\right).$$

Theorem 3.4. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(A,B;C,D)$, then $f(z)$ is convex in $|z| < r_0$ where r_0 is the smallest positive root of*

$$\begin{aligned} &1+[2D-2A-1]r+[2B-2C+AB-AC+BC-3AD+CD-BD]r^2 \\ &\quad +(-AB+3BC+BD+2ABD-CD-2ACD+AC-AD)r^3 \\ &\quad +(-2ABC+2BCD+ABCD)r^4-ABCDr^5=0 \end{aligned} \tag{3.14}$$

in the interval $(0,1)$.

Proof. As $f(z) \in \mathcal{K}_{\mathcal{C}}^{*}(A,B;C,D)$, we have

$$f'(z) = g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right) = g'(z)P(z).$$

After differentiating it logarithmically, we get

$$1+\frac{zf''(z)}{f'(z)} = 1+\frac{zg''(z)}{g'(z)}+\frac{zP'(z)}{P(z)}. \tag{3.15}$$

Also from (3.10), we have

$$\left|\frac{1+Cw(z)}{1+Dw(z)}\right| = |P(z)| \leq \frac{1+Cr}{1+Dr},$$

which implies

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{r(C-D)}{(1+Cr)(1+Dr)}. \tag{3.16}$$

$f \in \mathcal{C}(A,B)$, so as proved by Mehrok [8], we have

$$Re\left(1+\frac{zg''(z)}{g'(z)}\right) \geq \frac{1-(1+2A)r+B(2+A)r^2-ABr^3}{(1+r)(1-Ar)(1-Br)}. \tag{3.17}$$

(3.15) yields,

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq Re\left(1+\frac{zg''(z)}{g'(z)}\right) - \left|\frac{zP'(z)}{P(z)}\right|. \tag{3.18}$$

Therefore using inequalities (3.16) and (3.17), (3.18)gives

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \frac{1-(1+2A)r+B(2+A)r^2-ABr^3}{(1+r)(1-Ar)(1-Br)} - \frac{r(C-D)}{(1+Cr)(1+Dr)}.$$

After simplification, the above inequality can be expressed as

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \frac{1+[2D-A]r+[CD-AC-AD+BC-BD]r^2-ACDr^3}{(1-Br)(1+Cr)(1+Dr)}.$$

Hence $f(z)$ is convex in $|z|<r_0$ where r_0 is the smallest positive root of

$$\begin{aligned}&1+[2D-2A-1]r+[2B-2C+AB-AC+BC-3AD+CD-BD]r^2\\&\quad+(-AB+3BC+BD+2ABD-CD-2ACD+AC-AD)r^3\\&\quad+(-2ABC+2BCD+ABCD)r^4-ABCDr^5=0 \text{ in the interval } (0,1).\end{aligned}$$

Sharpness follows for the function $f_n(z)$ defined in (3.6). □

For $A=1, B=-1$, Theorem 3.4 gives the following result:

Corollary 3.10. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^*(C,D)$, then $f(z)$ is convex in $|z|<r_1$ where r_1 is the smallest positive root of* $1+[2D-3]r+[-4C-2D+CD-3]r^2+(-2C-4D-3CD+1)r^3+(2C-3CD)r^4+CDr^5=0$ *in the interval* $(0,1)$.

Substituting $A=1, B=-1, C=1-2\alpha, D=-1$, Theorem 3.4 agrees with the result given below:

Corollary 3.11. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^*(\alpha)$, then $f(z)$ is convex in $|z|<r_2$ where r_2 is the smallest positive root of*
$1-5r+2(-3+5\alpha)r^2+2(3-\alpha)r^3+5(1-2\alpha)r^4-(1-2\alpha)r^5=0$
in the interval $(0,1)$.

Substituting $A=1, B=-1, C=1, D=-1$, Theorem 3.4 agrees with the result given below:

Corollary 3.12. *If $f(z) \in \mathcal{K}_{\mathcal{C}}^*$, then $f(z)$ is convex in $|z|<r_3$ where $r_3=3-2\sqrt{2}$.*

Theorem 3.5. *For $f \in \mathcal{K}_{\mathcal{C}}^{*}(A,B;C,D)$,*

$$|a_2| \leq 1 + \frac{1}{2}\left[(A-B)+(C-D)\right], \tag{3.19}$$

$$|a_3| \leq 1 + \frac{(A-B)}{3}\left[2+|B|+(C-D)\right]+(C-D) \tag{3.20}$$

and for μ complex,

$$|a_3-\mu a_2^2| \leq \tfrac{1}{3}max\{1,|3\mu-3|\}+\tfrac{(A-B)}{3}\left[|2+3\mu|+max\left\{1,B+\tfrac{3(A-B)\mu}{4}\right\}\right]$$
$$+\frac{(C-D)}{3}\left[\left(1+\frac{(A-B)}{2}\right)|2-3\mu|+max\left\{1,D+\frac{3(C-D)\mu}{4}\right\}\right]. \tag{3.21}$$

Proof. Expanding (3.2), gives
$1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1}+\ldots$

$$=(1+2d_2z+3d_3z^2+\ldots+nd_nz^{n-1}+\ldots)(1+(C-D)c_1z+(C-D)[c_2-Dc_1^2]z^2+\ldots). \tag{3.22}$$

Equating coefficients of z and z^2 in (3.22), it yields

$$a_2 = b_2 + \frac{(C-D)}{2}c_1, \tag{3.23}$$

and

$$a_3 = b_3 + \frac{2}{3}(C-D)b_2c_1 + \frac{(C-D)}{3}\left[c_2-Dc_1^2\right]. \tag{3.24}$$

After applying the triangle inequality, (3.23) and (3.24) reduce respectively to

$$|a_2| \leq |b_2| + \frac{(C-D)}{2}|c_1|, \tag{3.25}$$

and

$$|a_3| \leq |b_3| + \frac{2}{3}(C-D)|b_2||c_1| + \frac{(C-D)}{3}\left|c_2-Dc_1^2\right|. \tag{3.26}$$

Using $|c_1| \leq 1$ and Lemma 2.5, the result (3.19) can be easily obtained from (3.25).

Again applying Lemma 2.5, Lemma 2.6 and the inequality $|c_1| \leq 1$, the result (3.20) can be derived from (3.25).

From (3.23) and (3.24), we obtain $|a_3-\mu a_2^2| \leq |b_3-\mu b_2^2|+(C-D)|b_2||c_1||\frac{2}{3}-\mu|$

$$+\frac{(C-D)}{3}\left|c_2-\left\{D+\frac{3(C-D)\mu}{4}\right\}c_1^2\right|. \tag{3.27}$$

Using the inequality $|c_1| \leq 1$, and applying Lemma 2.5, Lemma 2.6, the result (3.21) can be easily obtained from (3.27). □

Theorem 3.6. *If $-1 \leq D_2 \leq D_1 < C_1 \leq C_2 \leq 1$, then*

$$\mathcal{K}_{\mathcal{C}}^{*}(A,B;C_1,D_1) \subset \mathcal{K}_{\mathcal{C}}^{*}(A,B;C_2,D_2).$$

Proof. As $\mathcal{K}_{\mathcal{C}}^{*}(A,B;C_1,D_1)$,

$$\frac{f'(z)}{g'(z)} \prec \frac{1+C_1z}{1+D_1z}.$$

As $-1 \leq D_2 \leq D_1 < C_1 \leq C_2 \leq 1$, by Lemma 2.7, we have

$$\frac{f'(z)}{g'(z)} \prec \frac{1+C_1z}{1+D_1z} \prec \frac{1+C_2z}{1+D_2z},$$

which proves the desired result. □

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