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APPROXIMATION OF A CONTINUOUS FUNCTION BY A SEQUENCE OF CONVOLUTION OPERATORS VIA A MATRIX SUMMABILITY METHOD USING IDEALS

RIMA GHOSH AND SUDIPTA DUTTA

ABSTRACT. In this paper, in the line of Duman [7], we deal with Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a,b]$, the Banach space of all real valued continuous functions on $[a,b]$ endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a,b]$, based on the notion of A^I-summability. We construct an example to exhibit that the main result is more generalized than its statistical A-summable version. We also study the rate of A^I-summability.

1. INTRODUCTION AND BACKGROUND

The study of the Korovkin type approximation theory has a long history and is a well-established area of research (see [5, 9, 10, 12]). For a sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [17] first established the necessary and sufficient conditions for the uniform convergence of $\{L_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1,\ e_2 = x,\ e_3 = x^2$ [1].

We are interested in obtaining a general Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a,b]$ via a generalized matrix summability method, namely, the A^I-summability method. We study the rate of convergence via the A^I-summability method.

The concept of statistical convergence of a sequence of real numbers was first introduced by Fast [14]. This is a generalization of usual convergence. Further investigations started in this area after the works of Šalát [22] and Fridy [15]. Consequently, the notion of I-convergence of real sequences was introduced by Kostyrko et. al. [20]. Later a lot of works have been done on matrix summability (see [2,4,11,18,19,21,23,26]). A general Korovkin type approximation theory and the rate of convergence were studied using the notion of I-convergence in [6, 8]. In particular, in [24, 25] a very general notion of A^I-summability was studied.

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*Corresponding author (Email address: drsudipta.prof@gmail.com) .

Recall that a family $I \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if $(i) A, B \in I$ implies $A \cup B \in I$; $(ii) A \in I, B \subset A$ implies $B \in I$, while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. If I is a non-trivial proper ideal in Y (i.e. $Y \notin I, I \neq \{\emptyset\}$) then the family of sets $F(I) = \{M \subset Y :$ there exists $A \in I : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal I. The real number sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be I-convergent to L provided that for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I$ [20].
Throughout the paper I will denote a non-trivial admissible ideal on $\mathbb{N}$.

If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{jn})$ is an infinite matrix, then Ax is the sequence whose j-th term is given by

$$A_j(x) = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided the series converges for each j. We say that x is A-summable to L if $\lim_{j\to\infty} A_j(x) = L$ [18]. A matrix A is called regular if $A \in (c, c)$ and $\lim_{j\to\infty} A_j(x) = \lim_{n\to\infty} x_n$ for all $x = \{x_n\}_{n\in\mathbb{N}} \in c$ when c, as usual, stands for the set of all convergent sequences. The well-known characterization of regularity for two dimensional matrices is known as Silverman-Toeplitz conditions [16]. Connor and Leonetti recently introduced in [3] the larger class of matrices, namely $(I, \mathcal{J})$-regular matrices. We are concerned to extend the results in Korovkin type approximation theory using this new class of matrices in future.

2. Approximation for a sequence of convolution operators

For a non-negative regular matrix $A = (a_{jn})$ following [18], a set K is said to have A-density if $\delta_A(K) = \lim_j \sum_{n\in K} a_{jn}$ exists.

We first recall the definition

Definition 2.1 ([24]). *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then a real sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be A^I-summable to a number L if for every $\varepsilon > 0$, $\left\{j \in \mathbb{N} : |A_j(x) - L| \geq \varepsilon\right\} \in I$ where $A_j(x) = \sum_{n=1}^{\infty} a_{jn} x_n$.*

Thus $x = \{x_n\}_{n\in\mathbb{N}}$ is A^I-summable to a number L if and only if $\{A_j(x)\}_{j\in\mathbb{N}}$ is I-convergent to L. In this case, we write $I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} x_n = L$.

It should be noted that for $I = I_d$, the set of all subsets of $\mathbb{N}$ with natural density zero, A^I-summability reduces to statistical A-summability [13].

We consider the Banach space $C[a, b]$ endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a, b]$. Let L be a positive linear operator. Then $L(f) \geq 0$ for any positive function f. Also we denote the value of $L(f)$ at a point $x \in [a,b]$ by $L(f;x)$.

Theorem 2.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ and $A = (a_{jn})$ be a non-negative regular matrix. If*

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_i) - f_i \right\| = 0 \textit{ with } f_i(y) = y^i,\ i = 0,1,2$$

then for all $f \in C[a,b]$ we have

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\| = 0.$$

Proof. We start by observing for each $x \in [a,b]$, the function $0 \le \Psi \in C[a,b]$ defined by $\Psi(y) = (y-x)^2$. Since each L_n is positive, $L_n(\Psi;x)$ is a positive function. In particular, we have for each $x \in [a,b]$

$$\begin{aligned}
0 &\le \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x)\\
&= \sum_{n=1}^{\infty} a_{jn} L_n(y^2;x) - 2x \sum_{n=1}^{\infty} a_{jn} L_n(y;x) + x^2 \sum_{n=1}^{\infty} a_{jn} L_n(1;x)\\
&= \Big(\sum_{n=1}^{\infty} a_{jn} L_n(y^2;x) - y^2(x)\Big) - 2x\Big(\sum_{n=1}^{\infty} a_{jn} L_n(y;x) - y(x)\Big) + x^2\Big(\sum_{n=1}^{\infty} a_{jn} L_n(1;x) - 1(x)\Big)\\
&\le \left\| \sum_{n=1}^{\infty} a_{jn} L_n(y^2) - y^2 \right\| + 2b \left\| \sum_{n=1}^{\infty} a_{jn} L_n(y) - y \right\| + b^2 \left\| \sum_{n=1}^{\infty} a_{jn} L_n(1) - 1 \right\|.
\end{aligned}$$

Fix $f \in C[a,b]$. Let $M = \|f\|$. Then we can write $\big|f(y) - f(x)\big| < 2M$ for all $y,x \in [a,b]$. Also, since f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. Hence for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\big|f(y) - f(x)\big| < \varepsilon$ for all y,x satisfying $|\, y - x \,| < \delta$. On the other hand, if $|\, y - x \,| \ge \delta$, then it follows that,

$$-\frac{2M}{\delta^2}(y-x)^2 \le -2M \le f(y) - f(x) \le 2M \le \frac{2M}{\delta^2}(y-x)^2.$$

Therefore for all $y,x \in [a,b]$ we get,

$$|\, f(y) - f(x) \,| < \varepsilon + \frac{2M}{\delta^2}(y-x)^2$$

where δ is a fixed real number. Since each L_n is positive, we have

$$\begin{aligned}
-\varepsilon \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - \frac{2M}{\delta^2} \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) &\le \sum_{n=1}^{\infty} a_{jn} L_n(f(y);x) - f(x) \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x)\\
&\le \varepsilon \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) + \frac{2M}{\delta^2} \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x).
\end{aligned}$$

Next, let $K = \frac{2M}{\delta^2}$ and we get

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_{jn}L_n(f(y);x) - f(x)\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x)\right| &\leq \varepsilon \sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) + \frac{2M}{\delta^2}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x)\\
&= \varepsilon + \varepsilon\left[\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right]\\
&\quad + K\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x)\\
&\leq \varepsilon + \varepsilon\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|\\
&\quad + K\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x).
\end{aligned}$$

In particular,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_{jn}L_n(f(y);x) - f(x)\right| &\leq \left|\sum_{n=1}^{\infty} a_{jn}L_n(f(y);x) - f(x)\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x)\right|\\
&\quad + \left|f(x)\right|\left|\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right|\\
&\leq \varepsilon + K\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x) + (M+\varepsilon)\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right\|
\end{aligned}$$

which implies

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\| &\leq \varepsilon + C_2\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_2) - f_2\right\| + C_1\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_1) - f_1\right\|\\
&\quad + C_0\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|
\end{aligned}$$

where, $C_2 = K$, $C_1 = 2bK$ and $C_0 = (\varepsilon + b^2K + M)$ i.e. ,

$$\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\| \leq \varepsilon + C\sum_{i=0}^{2}\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_i) - f_i\right\|,\ i = 0,1,2$$

where $C = \max\{C_0,\ C_1,\ C_2\}$. For a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$ and let us define the following sets

$$D = \left\{ j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\| \geq \varepsilon' \right\};$$

$$D_1 = \left\{ j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\| \geq \frac{\varepsilon' - \varepsilon}{3C} \right\};$$

$$D_2 = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_1) - f_1 \right\| \geq \frac{\varepsilon' - \varepsilon}{3C} \right\};$$

$$D_3 = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_2) - f_2 \right\| \geq \frac{\varepsilon' - \varepsilon}{3C} \right\}.$$

It follows that $D \subseteq D_1 \cup D_2 \cup D_3$ and from the hypotheses we have D_1, D_2, D_3 belong to I. Therefore $D \in I$. Hence the proof is completed. □

We now consider the following convolution operators defined on $C[a,b]$ by

$$L_n(f;x) = \int_a^b f(y) K_n(y-x) dy, \ n \in \mathbb{N}, \ x \in [a,b] \text{ and } f \in C[a,b] \tag{2.1}$$

where a and b are two real numbers such that $a < b$.
Throughout the paper we assume that K_n is a continuous function on $[a-b, b-a]$ and also that $K_n(u) \geq 0$ for all $n \in \mathbb{N}$ and for every $u \in [a-b, b-a]$. Consider the function Ψ on $[a,b]$ defined by $\Psi(y) = (y-x)^2$ for each $x \in [a,b]$.
In [26], the authors investigated the classical versions of the following results in two variables and for sequences of infinite matrices. In particular, for the Frechet ideal I, the following results give the classical versions for a single variable.

Theorem 2.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of convolution operators from $C[a,b]$ into $C[a,b]$. If*

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\| = 0 \text{ with } f_0(y) = 1$$

and

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\| = 0$$

then for all $f \in C[a,b]$ we have

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\| = 0.$$

Proof. Fix $f \in C[a,b]$ and $x \in [a,b]$. Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of $f \in C[a,b]$ and $x \in [a,b]$, there exists a $\delta > 0$ such that $| f(y) - f(x) | < \varepsilon$ whenever $| y - x | \leq \delta$. Let $I_\delta = [x-\delta, x+\delta] \cap [a,b]$. So

$$\begin{aligned} \left| f(y) - f(x) \right| &= | f(y) - f(x) | \Psi_{I_\delta}(y) + | f(y) - f(x) | \Psi_{[a,b]-I_\delta}(y) \\ &\leq \varepsilon + 2M\delta^{-2}(y-x)^2. \end{aligned}$$

Since L_n's are positive and linear we have,

$$
\begin{aligned}
\left|\sum_{n=1}^{\infty} a_{jn} L_n(f;x) - f(x)\right| &= \left|\sum_{n=1}^{\infty} a_{jn} \int_a^b f(y) K_n(y-x)dy - f(x)\right| \\
&= \mid \sum_{n=1}^{\infty} a_{jn} \int_a^b (f(y)-f(x)) K_n(y-x)dy \\
&\quad + f(x) \sum_{n=1}^{\infty} a_{jn} \int_a^b K_n(y-x)dy - f(x) \mid \\
&\leq \left|\sum_{n=1}^{\infty} a_{jn} \int_a^b (f(y)-f(x)) K_n(y-x)dy\right| \\
&\quad + \left|f(x)\right| \left|\sum_{n=1}^{\infty} a_{jn} \int_a^b K_n(y-x)dy - 1\right| \\
&\leq \sum_{n=1}^{\infty} a_{jn} \int_a^b \left|f(y)-f(x)\right| \left|K_n(y-x)\right| dy \\
&\quad + \left|f(x)\right| \left|\sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x)\right| \\
&\leq \sum_{n=1}^{\infty} a_{jn} \int_a^b (\varepsilon + 2M\delta^{-2}(y-x)^2) K_n(y-x)dy \\
&\quad + M \left|\sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x)\right| \\
&= \varepsilon + (\varepsilon + M) \left|\sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x)\right| \\
&\quad + 2M\delta^{-2} \left|\sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x)\right| \\
&\leq \varepsilon + \alpha \left|\sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x)\right| + \alpha \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x)
\end{aligned}
$$

where $\alpha = \max\{\varepsilon + M, \frac{2M}{\delta^2}\}$.

Therefore

$$
\left\|\sum_{n=1}^{\infty} a_{jn} L_n(f) - f\right\| \leq \varepsilon + \alpha \left\{ \left\|\sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0\right\| + \left\|\sum_{n=1}^{\infty} a_{jn} L_n(\Psi)\right\| \right\}.
$$

For given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$ and define the following sets

$$
\begin{aligned}
D &= \left\{ j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn} L_n(f) - f\right\| \geq r \right\}; \\
D_1 &= \left\{ j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0\right\| \geq \frac{r-\varepsilon}{2\alpha} \right\}; \\
D_2 &= \left\{ j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn} L_n(\Psi)\right\| \geq \frac{r-\varepsilon}{2\alpha} \right\}.
\end{aligned}
$$

It follows that $D \subseteq D_1 \cup D_2$ and since D_1, D_2 belong to I then $D \in I$. Hence this completes the proof. $\square$

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$ and let $\|f\|_\delta = \sup\limits_{a+\delta \le x \le b-\delta} | f(x) |$, $f \in C[a,b]$.

We now study the main theorem of this paper.

Theorem 2.3. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by (1). If conditions*

$$I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y)dy = 1 \tag{2.2}$$

$$I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} (\sup_{|y|\ge\delta} K_n(y)) = 0 \tag{2.3}$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_\delta = 0.$$

In order to prove our main result we need the following lemma.

Lemma 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that δ is a fixed positive number such that $\delta < \dfrac{b-a}{2}$. If the conditions (2) and (3) hold, then for the operators L_n where $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$, $f \in C[a,b]$ and a,b are real numbers $a < b$, we have*

$$(i) \quad I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta = 0 \text{ with } f_0(y) = 1$$

and

$$(ii) \quad I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_\delta = 0 \ \text{ with } \Psi(y) = (y-x)^2.$$

Proof. (i) Let $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. Then $\delta \le x-a \le b-a \Rightarrow -(b-a) \le a-x \le -\delta$ and $\delta \le b-x \le b-a$. Now $L_n(f_0;x) = \int_a^b K_n(y-x)dy = \int_{a-x}^{b-x} K_n(y)dy$. Then we have,

$$\int_{-\delta}^{\delta} K_n(y)dy \le L_n(f_0;x) \le \int_{-(b-a)}^{b-a} K_n(y)dy.$$

Therefore

$$\left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta \le u_j$$

where $u_j := \max\left\{\left|\sum_{n=1}^{\infty} a_{jn}\int_{-\delta}^{\delta} K_n(y)dy - 1\right|, \left|\sum_{n=1}^{\infty} a_{jn}\int_{-(b-a)}^{b-a} K_n(y)dy - 1\right|\right\}$.

Therefore from condition (2), $I\text{-}\lim_j u_j = 0$ for all $\delta > 0$ such that $\delta < \dfrac{b-a}{2}$. Now for given a $\varepsilon > 0$

$$\text{(say) } D := \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_{\delta} \geq \varepsilon \right\} \subseteq \left\{ j \in \mathbb{N} : u_j \geq \varepsilon \right\}.$$

Since $\left\{ j \in \mathbb{N} : u_j \geq \varepsilon \right\} \in I$, $D \in I$. Hence this completes the proof of (i).

(ii) For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. Since $\Psi(y) = y^2 - 2xy + x^2$ then $\Psi \in C[a,b]$ for all $x \in [a+\delta, b-\delta]$. Now $L_n(\Psi;x) = L_n(f_2;x) - 2xL_n(f_1;x) + x^2 L_n(f_0;x)$ with $f_i(y) = y^i$, $i = 0,1,2$. Then for all $n \in \mathbb{N}$

$$\begin{aligned} L_n(\Psi;x) &= \int_a^b (y-x)^2 K_n(y-x)dy \\ &= \int_{a-x}^{b-x} y^2 K_n(y)dy \\ &\leq \int_{-(b-a)}^{b-a} y^2 K_n(y)dy \end{aligned}$$

Since the function f_2 is continuous at $y = 0$, given $\varepsilon > 0$ there exists $\eta > 0$ such that for every y satisfying $|\, y \,| \leq \eta$, $y^2 < \varepsilon$ holds. We have two cases $\eta \geq b - a$ *or* $\eta < b - a$.

Case 1

Let $\eta \geq b-a$. Therefore $0 \leq L_n(\Psi;x) \leq \varepsilon \int_{-(b-a)}^{b-a} K_n(y)dy$. By condition (2), $0 \leq \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) \leq \varepsilon$ and $I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_{\delta} = 0$ for $\eta \geq b-a$.

Case 2

Now let $\eta < b-a$. Therefore $L_n(\Psi;x) \leq \int_{|y|\geq\eta} y^2 K_n(y)dy + \int_{|y|\leq\eta} y^2 K_n(y)dy$ and hence we obtain for all $j \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_{\delta} &\leq \sum_{n=1}^{\infty} a_{jn} p_n \int_{\eta}^{b-a} y^2 dy + \varepsilon \sum_{n=1}^{\infty} a_{jn} \int_{|y|\leq\eta} K_n(y)dy \\ &= \frac{(b-a)^3 - \eta^3}{3} \sum_{n=1}^{\infty} a_{jn} p_n + \varepsilon \sum_{n=1}^{\infty} a_{jn} q_n \end{aligned}$$

where $p_n = \sup_{|y|\geq\eta} K_n(y)$ and $q_n = \int_{|y|\leq\eta} K_n(y)dy$.

Also we have from conditions (2) and (3),

$$I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} p_n = 0$$

and

$$I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} q_n = 1.$$

Taking $M = \max\{\frac{(b-a)^3-\eta^3}{3}, \varepsilon\}$ we have for all $j \in \mathbb{N}$ that

$$\left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_{\delta} \leq \varepsilon + M \left(\sum_{n=1}^{\infty} a_{jn} p_n + \left| \sum_{n=1}^{\infty} a_{jn} q_n - 1 \right| \right).$$

For given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$.
Let

$$D = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_{\delta} \geq r \right\};$$

$$D_1 = \left\{ j \in \mathbb{N} : \sum_{n=1}^{\infty} a_{jn} p_n \geq \frac{r-\varepsilon}{2M} \right\};$$

$$D_2 = \left\{ j \in \mathbb{N} : \left| \sum_{n=1}^{\infty} a_{jn} q_n - 1 \right| \geq \frac{r-\varepsilon}{2M} \right\}.$$

Therefore $D \subseteq D_1 \cup D_2$. Since from the hypotheses, D_1 and D_2 belong to I, $D \in I$. Hence this completes the proof. □

Proof. Proof of Theorem 2.3
The main result (Theorem 2.3) follows from Theorem 2.2, Lemma 2.1. □

If we take $I = I_d$, the ideal of all subsets of $\mathbb{N}$ with natural density zero, we get the following

Corollary 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by*

$$L_n(f;x) = \int_a^b f(y) K_n(y-x)dy$$

$n \in \mathbb{N}, x \in [a,b]$ and $f \in C[a,b]$ where a and b are two real numbers such that $a < b$. If conditions

$$st\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y)dy = 1$$

and

$$st\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} (\sup_{|y|\geq\delta} K_n(y)) = 0$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have

$$st\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_{\delta} = 0.$$

The above corollary can be proved independently in a straightforward way and it is the statistical A-summable version of Theorem 2.4. in [7].

Remark 2.1. We now exhibit a sequence of positive convolution operators for which Corollary 2.1 does not apply but Theorem 2.3 does. Let

$$u_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Let I be a non-trivial admissible ideal of $\mathbb{N}$ such that $I \neq I_{fin}$(Frechet ideal) and $I \neq I_d$. Choose an infinite subset $C = \{p_1 < p_2 < p_3 ...\}$ from $I \setminus I_d$ where I_d denotes the set of all subsets of $\mathbb{N}$ with natural density zero.

Let $A = (a_{jn})$ be given by

$$a_{jn} = \begin{cases} 1 & \text{if } j = p_i, n = 2p_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } j \neq p_i, \text{ for any } i,\ n = 2j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$y_j = \sum_{n=1}^{\infty} a_{jn} u_n = \begin{cases} 1 & \text{if } j = p_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } j \neq p_i, \text{ for any } i \in \mathbb{N}. \end{cases}$$

Let $\varepsilon > 0$ be given and $\{j \in \mathbb{N} : |y_j - 0| \geq \varepsilon\} = C \in I \setminus I_d$. Thus $\{u_n\}_{n \in \mathbb{N}}$ is A^I-summable to 0 but not statistically A-summable.

Now let the operators L_n on $C[a,b]$ be defined by

$$L_n(f;x) = \frac{n(1+y_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2} dy.$$

If we choose $K_n(y) = \frac{n(1+y_n)}{\sqrt{\pi}} e^{-n^2y^2}$ then

$$L_n(f;x) = \frac{n(1+y_n)}{\sqrt{\pi}} \int_a^b f(y) K_n(y-x) dy.$$

Now for every $\delta > 0$ such that $\delta < \frac{b-a}{2}$, we have

$$\begin{aligned} \int_{-\delta}^{\delta} K_n(y) dy &= \frac{n(1+y_n)}{\sqrt{\pi}} \Big(\int_{-\infty}^{\infty} e^{-n^2y^2} dy - \int_{|y| \geq \delta} e^{-n^2y^2} dy \Big) \\ &= \frac{2(1+y_n)}{\sqrt{\pi}} \Big(\int_0^{\infty} e^{-y^2} dy - \int_{\delta.n}^{\infty} e^{-y^2} dy \Big). \end{aligned}$$

Since $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} < \infty$, it is clear that $\lim_n \int_{\delta.n}^{\infty} e^{-y^2} dy = 0$.

Also since $I\text{-}\lim_j ||1 + y_j|| = 1$, we immediately get

$$I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y) dy = 1.$$

On the other hand, we have

$$\sup_{|y|\geq\delta} K_n(y) = \frac{n(1+y_n)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-n^2y^2}$$
$$\leq \frac{n(1+u_n)}{e^{n^2\delta^2}}.$$

Since $\lim_n \frac{n}{e^{n^2\delta^2}} = 0$ we conclude that

$$I\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn}(\sup_{|y|\geq\delta} K_n(y)) = 0.$$

Therefore from Theorem 2.3

$$I\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_{\delta} = 0 \text{ for all } f \in C[a,b].$$

However note that, as $\{u_n\}_{n\in\mathbb{N}}$ is not statistical A-summable to zero so Corollary 2.1. does not work for the operator defined above.

We now recall the following note from [3] and make a remark in support of the existence of the set C in the above remark.

Remark 2.2. The simple density ideal $\mathcal{Z}_g$ for which $\frac{n}{g(n)}$ is bounded does not necessarily coincide with $\mathcal{Z}$, where in particular, $\mathcal{Z}$ is the simple density ideal generated by $g(n) = n$ and in fact $\mathcal{Z} = I_d$. Consider the set $S_k = [(2k)!, (2k+1)!]$ for all $k \in \mathbb{N}$ and $S := \cup_k S_{2k}$. If we consider the simple density ideal $\mathcal{Z}_g$ where $g : \mathbb{N} \to [0,\infty)$ is defined by

$$g(n) = \begin{cases} n^2 & \text{if } n \in S \\ n & \text{if } n \notin S \end{cases}$$

then $S \in \mathcal{Z}_g \setminus \mathcal{Z}$ [3].

3. Rate of A^I-Summability

In this section we study the rates of A^I-summability in Theorem 2.3 using the modulus of continuity. Let $f \in C[a,b]$. The modulus of continuity denoted by $\omega(f,\alpha)$, is defined to be

$$\omega(f,\alpha) = \sup_{|y-x|\leq\alpha} |\, f(y) - f(x) \,| \,.$$

The modulus of continuity of the function f in $C[a,b]$ gives the maximum oscillation of f in any interval of length not exceeding $\alpha > 0$. It is well-known that if $f \in C[a,b]$, then

$$\lim_{\alpha\to 0} \omega(f,\alpha) = \omega(f,0) = 0,$$

and that for any constants $c > 0$, $\alpha > 0$,

$$\omega(f,c\alpha) \leq (1+[c])\omega(f,\alpha),$$

where $[c]$ is the greatest integer less than or equal to c.

Next we introduce the following definition

Definition 3.1. *Let $A=(a_{jn})$ be a non-negative regular summability matrix and let $\{c_n\}_{n\in\mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then a sequence $x=\{x_n\}_{n\in\mathbb{N}}$ is said to be A^I-summable to a number L with the rate of $o(c_n)$ if for every $\varepsilon>0$*

$$\left\{j\in\mathbb{N}:\left\|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}x_n-L\right\|\geq\varepsilon\right\}\in I.$$

In this case we write A^I-sum-$o(c_n)$-$\lim\limits_{n} x_n=L$.

In particular, for the non-increasing sequence $\{c_n\}_{n\in\mathbb{N}}$, where $c_n=1$ for all $n\in\mathbb{N}$, Definition 3.1 implies A^I-summability to a number L.

We establish the following Theorem

Theorem 3.1. *Let $A=(a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators given by (1). Assume further that $\{c_n\}_{n\in\mathbb{N}}$is a positive non-increasing sequence. If for a fixed $\delta>0$ such that $\delta<\frac{b-a}{2}$,*

$$A^I\text{-sum-}o(c_n)\text{-}\lim_n L_n(f_0)=f_0$$

and

$$I\text{-}\lim_j\omega(f,\alpha_j)=0$$

where $\alpha_j:=\sqrt{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(\Psi)\right\|_{\delta}}$, then for all $f\in C[a,b]$ we have

$$I\text{-}\lim_j\left\|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(f)-f\right\|_{\delta}=0.$$

Proof. Let $0<\delta<\frac{b-a}{2}$, $f\in C[a,b]$ and $x\in[a+\delta,b-\delta]$. By positivity and linearity of the operators L_n and using the inequalities for any $\alpha>0$ we get

$$\begin{aligned}\left|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(f;x)-f(x)\right| &\leq \frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(\mid f(y)-f(x)\mid;x)\\ &\quad+\mid f(x)\mid\left|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(f_0;x)-f_0(x)\right|\\ &\leq\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n\left(\omega(f,\alpha\frac{\mid y-x\mid}{\alpha}\right);x)\\ &\quad+\mid f(x)\mid\left|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(f_0;x)-f_0(x)\right|\\ &\leq\omega(f,\alpha)\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n\left(1+\left[\frac{\mid y-x\mid}{\alpha}\right];x\right)\\ &\quad+\mid f(x)\mid\left|\frac{1}{c_j}\sum_{n=1}^{\infty}a_{jn}L_n(f_0;x)-f_0(x)\right|\end{aligned}$$

$$\begin{aligned}&\leq \omega(f,\alpha)\left\{\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x)+\frac{1}{\alpha^2}\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x)\right\}\\ &+\mid f(x)\mid\left|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x)-f_0(x)\right|.\end{aligned}$$

Therefore for all $n\in\mathbb{N}$

$$\begin{aligned}\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f)-f\right\|_{\delta} &\leq \omega(f,\alpha)\left\{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)\right\|_{\delta}+\frac{1}{\alpha^2}\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi)\right\|_{\delta}\right\}\\ &+M_1\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}\end{aligned}$$

where $M_1:=\|f\|_{\delta}$. Now let $\alpha:=\alpha_j=\sqrt{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi)\right\|_{\delta}}$, then we have

$$\begin{aligned}\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f)-f\right\|_{\delta} &\leq \omega(f,\alpha_j)\left\{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)\right\|_{\delta}+1\right\}\\ &+M_1\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}\\ &\leq 2\omega(f,\alpha_j)+\omega(f,\alpha_j)\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}\\ &+M_1\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}.\end{aligned}$$

Let $M=\max\{2,\ M_1\}$. Then we can write for all $n\in\mathbb{N}$ that

$$\begin{aligned}\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f)-f\right\|_{\delta} &\leq M\left\{\omega(f,\alpha_j)+\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}\right\}\\ &+\omega(f,\alpha_j)\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}.\end{aligned}$$

Given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D&:=\left\{j\in\mathbb{N}:\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f)-f\right\|_{\delta}\geq\varepsilon\right\};\\ D_1&:=\left\{j\in\mathbb{N}:\omega(f,\alpha_j)\geq\frac{\varepsilon}{3M}\right\};\\ D_2&:=\left\{j\in\mathbb{N}:\omega(f,\alpha_j)\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}\geq\frac{\varepsilon}{3}\right\};\\ D_3&:=\left\{j\in\mathbb{N}:\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)-f_0\right\|_{\delta}\geq\frac{\varepsilon}{3M}\right\}.\end{aligned}$$

Then $D \subseteq D_1 \cup D_2 \cup D_3$. Also, we define

$$D_2^{'} = \left\{ j \in \mathbb{N} : \omega(f, \alpha_j) \geq \sqrt{\frac{\varepsilon}{3}} \right\};$$

$$D_2^{''} = \left\{ j \in \mathbb{N} : \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_{\delta} \geq \sqrt{\frac{\varepsilon}{3}} \right\}.$$

Therefore $D_2 \subseteq D_2^{'} \cup D_2^{''}$. Hence we get $D \subseteq D_1 \cup D_2^{'} \cup D_2^{''} \cup D_3$. Since $D_1, D_2^{'}, D_2^{''}, D_3$ belong to I then $D \in I$. This completes the proof. □

4. CONCLUSIONS

We generalize Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a,b]$ in some sense with a generalized matrix summability method, namely, A^I-summability method for real sequences. We construct an example in support of this generalization. We are very much interested whether the results of this paper are valid for the function f with two variables. Again we are interested whether the results are relevant on an infinite interval.

We now leave an open problem that the results of this paper may be extended to a larger class of matrices, namely $(I, \mathcal{J})$-regular matrices, the one which maps I-convergent sequences into $\mathcal{J}$-convergent sequences and preserves the ideal limits, for some choice of ideals I and $\mathcal{J}$ [3].

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Rima Ghosh
Department of Mathematics
Jadavpur University
Jadavpur, Kolkata-700032
West Bengal
India
e-mail: *rimag944@gmail.com*
and
Sudipta Dutta
Department of Mathematics
Govt. General Degree College
Manbazar-II, Purulia
Pin-723131
West Bengal
India
e-mail: *drsudipta.prof@gmail.com*