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IDEAL *e*-CONVERGENCE OF DOUBLE SEQUENCES AND A KOROVKIN-TYPE APPROXIMATION THEOREM

YURDAL SEVER AND ÖZER TALO

ABSTRACT. In the present paper, we introduce the concept of ideal e-convergence of double sequences and prove some fundamental properties. Next, we define the concepts of ideal e-limit superior and inferior for double sequences. After that, some properties of this type of convergence are examined. Finally, we give a Korovkin-type approximation theorem for double sequences of positive linear operators on the space of all continuous real-valued functions defined on any compact subset of the real two-dimensional space via ideal e-convergence.

1. INTRODUCTION AND PRELIMINARIES

Throughout this study, the symbols $\mathbb{N}, \mathbb{N}^2$ and $\mathbb{R}$ are used for the set of all positive integers, of all pairs of positive integers, and of all real numbers, respectively. Additionally, we are only interested in real double sequences.

The ordering of elements of the index set $\mathbb{N}^2$ can be defined in different forms, which leads to multiple types of convergence of double sequences. One of them is Pringsheim [22] convergence which is best known and well-studied.

The fundamental flaw of Pringsheim convergence is that it does not require a convergent sequence to be bounded. The concept of regular convergence introduced by Hardy [12] eliminates this disadvantage for double sequences. In regular convergence, both the row-index and the column-index of the double sequence need to be convergent besides convergence in Pringsheim's sense.

The notion of e-convergence of double sequences, which is substantially weaker than Pringsheim convergence and regular convergence, was introduced and examined by Boos et al. in [2]. While in Pringsheim convergence, the row-index k and the column-index l independently go to infinity, e-convergence states that the row-index k is linked to the column-index l whenever it goes to infinity. Some authors studied on e-convergence (see [23, 32–34]).

We say that a double sequence $x = (x_{kl})$ e-converges to a number a, written $e - \lim_{kl} x_{kl} = a$, if $\forall \varepsilon > 0\ \exists l_0 \in \mathbb{N}\ \forall l \geq l_0\ \exists k_l \in \mathbb{N}\ \forall k \geq k_l\ :\ |x_{kl} - a| < \varepsilon$.

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Recently, the notions of e-limit inferior and e-limit superior for a real double sequence have been studied in [26].

The notion of statistical convergence as a generalization of the ordinary convergence of real numbers was introduced independently by Fast [10] and Schoenberg [25]. Let K be a subset of the set $\mathbb{N}$ natural numbers. In this case, the natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$ if this limit exists, where the symbol $|A|$ denotes the number of elements in A. A sequence (x_k) is said to be statistically convergent to l, if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}) = 0$. Symbolically, this is denoted by $\text{st}-\lim_{k\to\infty} x_k = l$.

Since then, a number of authors have studied the statistical analogues of different types of convergence for double sequences in various spaces (see, for example, [5, 16, 18]).

The notion of ideal convergence, a common generalization of ordinary convergence and statistical convergence, is defined by Kostyrko et al. [14] using the concept of an ideal. This approach is much more general as most of the known convergence types become special cases. Therefore, it is one of the most active research areas in Topology and Analysis. Then, this convergence has been examined by many authors (see, for example, [3, 4, 15, 24]).

Let X be a nonempty set. A family of sets I of subsets of X is said to be an ideal on X if $I \neq \emptyset$, for each $A, B \in I$ implies $A \cup B \in I$, and for each $A \in I$ and each $B \subset A$ implies $B \in I$. An ideal I on X is called a nontrivial ideal if $X \notin I$. A nontrivial ideal I on X is called admissible if $\{x\} \in I$ for each $x \in X$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter on X if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $A \subset B$ implies $B \in \mathcal{F}$. It is obvious that if I is a nontrivial ideal on X, then the family $\mathcal{F}(I) = \{M \subset X : X \setminus M \in I\}$ is a filter on X, called the filter associated with the ideal I.

Lemma 1.1. *[24, Lemma 2.5] $K \in F(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$ then $M \cap K \notin I$.*

Tripathy generalized the notion of ideal convergence from single to double sequences in [31]. To avoid confusion of the ideals defined on $2^{\mathbb{N}}$ and $2^{\mathbb{N}\times\mathbb{N}}$ we shall symbolize the ideals on $2^{\mathbb{N}}$ by I and that on $2^{\mathbb{N}\times\mathbb{N}}$ by I_2, respectively. It is clear that there is no relationship between I and I_2 in general.

Let I_2 be an ideal on $2^{\mathbb{N}\times\mathbb{N}}$. Then a double sequence (x_{kl}) is said to be I_2-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, the set $\{(k,l) \in \mathbb{N}\times\mathbb{N} : |x_{kl} - L| \geq \varepsilon\}$ belongs to I_2. Symbolically, this is denoted as $I_2 - \lim x_{kl} = L$.

Let $C(X)$ be the space of all continuous functions with real-valued on a compact subset X of real numbers. Assume that $\{L_n\}$ be a sequence of positive linear operators from $C(X)$ in itself. Then, Korovkin [13] first gave necessary and sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test functions f_i defined by $f_i(x) = x^i$ $(i = 0, 1, 2)$. Since then many researchers studied these theorems for various concepts of convergence (see, for example, [1, 8, 9, 11, 20]). Also, in the last decades, by virtue of various types of convergence for double sequences, some Korovkin-type approximation theorems for functions of two variables have been investigated (see, for instance, [6, 7, 17, 19, 21, 27, 28]).

2. Ideal e-convergence of double sequences

Quite recently, Sever and Talo [27] rigorously examined the concept of e-convergence of double sequences to define the concept of e-statistical convergence of double sequences as follows:

$e-\lim x_{kl}=a$ if and only if $\forall\varepsilon>0\ \exists l_0\in\mathbb{N}\ \forall l\geq l_0\ \exists k_l\in\mathbb{N}\ \forall k\geq k_l\ :|x_{kl}-a|<\varepsilon$. Namely, $e-\lim x_{kl}=a$ if and only if $\forall\varepsilon>0\ \exists l_0\in\mathbb{N}\ \forall l\geq l_0$ the sets $H_l:=\{k:|x_{kl}-a|\geq\varepsilon\}$ are finite. Define

$$\mathcal{P}_{\triangle}(\mathbb{N})=\{A\in\mathcal{P}(\mathbb{N}):\mathbb{N}\backslash A \text{ has a finite number of elements}\}.$$

Therefore we can state that $e-\lim x_{kl}=a$ if and only if for every $\varepsilon>0$ the sets $\{l:\{k:|x_{kl}-a|\geq\varepsilon\}\text{ is finite }\}\in\mathcal{P}_{\triangle}(\mathbb{N})$.

We know that the natural density of a finite set is zero and the natural density of an element of $\mathcal{P}_{\triangle}(\mathbb{N})$ is 1. Keeping above approach in mind, the concept of statistical e-convergence was defined for double sequences [27] as follows:

A double sequence (x_{kl}) is said to be statistically e-convergent to the number a, written $st_e-\lim_{kl}x_{kl}=a$, if for every $\varepsilon>0$ the set $\{l:\delta(\{k:|x_{kl}-a|\geq\varepsilon\})=0\}$ has natural density 1.

In the present paper, using the same manner we extend the concept of e-convergence and statistical e-convergence of double sequence to ideal e-convergence of double sequence. Considering the definition of e-convergence we can make this expansion more comprehensive by taking two ideals due to the two indices.

Definition 2.1. *Let I and $\mathcal{J}$ denote non-trivial ideals of subsets of the natural numbers $\mathbb{N}$. A double sequence $x=(x_{kl})$ is said to be ideal e-convergent to the number a if for every $\varepsilon>0$ the complement of the set*

$$\left\{l:\{k:|x_{kl}-a|\geq\varepsilon\}\in\mathcal{J}\right\}$$

belongs to I, i.e,

$$\mathbb{N}\setminus\left\{l:\{k:|x_{kl}-a|\geq\varepsilon\}\in\mathcal{J}\right\}\in I.$$

We denote it with $(I,\mathcal{J})_e-\lim_{kl}x_{kl}=a$.

We describe the definition of ideal e-convergent as follows:

$$\begin{aligned}(I,\mathcal{J})_e-\lim_{kl}x_{kl}=a &\Longleftrightarrow \forall\varepsilon>0\ \exists N\in I\ \forall\ l\notin N\ :\ \{k:|x_{kl}-a|\geq\varepsilon\}\in\mathcal{J}\\ &\Longleftrightarrow \forall\varepsilon>0\ \exists N\in\mathcal{F}(I)\ \forall\ l\in N\ :\ \{k:|x_{kl}-a|\geq\varepsilon\}\in\mathcal{J}\end{aligned}$$

or equivalently

$$\forall\varepsilon>0\ \exists N\in I:\forall\ l\notin N\ \exists\ M_l\in\mathcal{J}\ \textit{such that}\ k\notin M_l\ :\ |x_{kl}-a|<\varepsilon.$$

In the following, we will give some examples of ideals to clarify the given definition, and corresponding $(I,\mathcal{J})_e$-convergence.

(I) Let I_f be the class of all finite subsets of natural numbers $\mathbb{N}$. Then, it is clear that I_f is an admissible ideal on $\mathbb{N}$. If we take both ideals I and $\mathcal{J}$ as I_f, then $(I,\mathcal{J})_e$-convergence corresponds to e-convergence of double sequences [2].

(II) Denote by I_δ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then I_δ is an admissible ideal. If we take both ideals I and $\mathcal{J}$ as I_δ, then $(I,\mathcal{J})_e$-convergence corresponds to the statistical e-convergence of double sequences [27].

(III) Depending on the choice of ideals I and $\mathcal{J}$, the types of convergence in the e-sense that has not been studied before can be obtained. For instance, taking I and $\mathcal{J}$, I_f and $\mathcal{J}_\delta$ respectively or vice versa. They are new kinds of convergence in the e-sense.

Example 2.1. *We decompose the natural numbers $\mathbb{N}$ into countably many disjoint sets*

$$N_p = \{2^{p-1}(2q-1) : q \in \mathbb{N}\},\ (p = 1,2,3,...).$$

It is clear that $\mathbb{N} = \bigcup_{p=1}^{\infty} N_p$ and $N_i \cap N_j = \emptyset$ for $i \neq j$, and $\delta(N_p) = \frac{1}{2^p}$. Let $\mathcal{J}$ be the class of all $A \subset \mathbb{N}$ such that A intersects only a finite number of N_p and $I = I_f$. It is easy to see that I and $\mathcal{J}$ are admissible ideals on $\mathbb{N}$. Given a sequence $x = (x_{kl})$ defined as follows: For each $l \in \mathbb{N}$ and $k \in N_p$ define $x_{kl} = \frac{1}{p}$, $p = 1,2,3,\dots$. We get $(I,\mathcal{J})_e - \lim_{kl} x_{kl} = 0$ but (x_{kl}) is not statistical e-convergent.

Lemma 2.1. *Let I and $\mathcal{J}$ be admissible ideals on the natural numbers $\mathbb{N}$ and let (x_{kl}) be a double sequence of real numbers. If $e - \lim_{kl} x_{kl} = a$ then $(I,\mathcal{J})_e - \lim_{kl} x_{kl} = a$.*

Proof. The statement is a consequence of the inclusion $I_f \subset I$. □

Theorem 2.1. *Let I and $\mathcal{J}$ be admissible ideals on the natural numbers $\mathbb{N}$. Let (x_{kl}) and (y_{kl}) be two double sequences of real numbers and $c \in \mathbb{R}$. If $(I,\mathcal{J})_e - \lim_{kl} x_{kl} = a$ and $(I,\mathcal{J})_e - \lim_{kl} y_{kl} = b$, then*

(i) $(I,\mathcal{J})_e - \lim_{kl} c.x_{kl} = c.a$,

(ii) $(I,\mathcal{J})_e - \lim_{kl}(x_{kl} + y_{kl}) = a + b$.

Proof. (i) If $c = 0$, then the equality is trivially true. Now, let $c \neq 0$. Hence for every $\varepsilon > 0$, we have

$$\Big\{k : |c.x_{kl} - c.a| \geq \varepsilon\} = \{k : |x_{kl} - a| \geq \frac{\varepsilon}{|c|}\Big\}.$$

Consequently the complement of the set

$$\Big\{l : \{k : |c.x_{kl} - c.a| \geq \varepsilon\} \in \mathcal{J}\Big\} = \Big\{l : \{k : |x_{kl} - a| \geq \frac{\varepsilon}{|c|}\} \in \mathcal{J}\Big\}$$

belongs to ideal I. This implies that $(I,\mathcal{J})_e - \lim_{kl} c.x_{kl} = c.a$.

(ii) Since the assumption of the theorem, we can write for given $\varepsilon > 0$

$$K_1 = \Big\{l : \{k : |x_{kl} - a| \geq \frac{\varepsilon}{2}\} \in \mathcal{J}\Big\}$$

and

$$K_2 = \left\{ l : \{k : |y_{kl} - b| \geq \frac{\varepsilon}{2}\} \in \mathcal{J} \right\}$$

with $K_1 \in \mathcal{F}(I)$ and $K_2 \in \mathcal{F}(I)$. If we take $K = K_1 \cap K_2$, then we have $\mathbb{N} \setminus K \in I \equiv K \in \mathcal{F}(I)$. Since

$$|x_{kl} + y_{kl} - (a+b)| \leq |x_{kl} - a| + |y_{kl} - b|$$

for each $l \in K$ we obtain

$$\{k : |x_{kl} + y_{kl} - (a+b)| \geq \varepsilon\} \subseteq \left\{ k : |x_{kl} - a| \geq \frac{\varepsilon}{2} \right\} \cup \left\{ k : |y_{kl} - a| \geq \frac{\varepsilon}{2} \right\}$$

and the set on the right side of the inclusion belongs to ideal $\mathcal{J}$. This implies that the set on the left side of the inclusion belongs to ideal $\mathcal{J}$. Then

$$\{k : |x_{kl} + y_{kl} - (a+b)| \geq \varepsilon\} \in \mathcal{J}$$

holds and we get

$$K \subseteq \{l : \{k : |x_{kl} + y_{kl} - (a+b)| \geq \varepsilon\} \in \mathcal{J}\}.$$

Therefore, we have

$$\mathbb{N} \setminus \{l : \{k : |x_{kl} + y_{kl} - (a+b)| \geq \varepsilon\} \in \mathcal{J}\} \in I. \qquad \square$$

The concepts of statistical e-limit superior and inferior for double sequences of real numbers were introduced by Sever and Talo in [27]. We now define the concepts of ideal e-limit superior and inferior for double sequences and prove some analogues. Firstly, we give the concept of ideal e-bounded double sequences.

Definition 2.2. *Let I and $\mathcal{J}$ be admissible ideals on the natural numbers $\mathbb{N}$ and let $x = (x_{kl})$ be a double sequences of real numbers. We say that the double sequence $x = (x_{kl})$ is $(I, \mathcal{J})_e$-bounded above if there exists $M \in \mathbb{R}$ such that the complement of the set $\{l : \{k : x_{kl} < M\} \in \mathcal{F}(\mathcal{J})\}$ belongs to I, and also we say that the double sequence $x = (x_{kl})$ is $(I, \mathcal{J})_e$-bounded below if there exists $N \in \mathbb{R}$ such that the complement of the set $\{l : \{k : x_{kl} > N\} \in \mathcal{F}(\mathcal{J})\}$ belongs to I. If the sequence $x = (x_{kl})$ is both $(I, \mathcal{J})_e$-bounded above and $(I, \mathcal{J})_e$-bounded below then it is called $(I, \mathcal{J})_e$-bounded.*

Now we can give the definitions of ideal e-limit superior and inferior.

Definition 2.3. *Let I and $\mathcal{J}$ be admissible ideal of subsets of $\mathbb{N}$ and let $x = (x_{kl})$ be a double sequences of real numbers. $(I, \mathcal{J})_e$-limit superior of $x = (x_{kl})$ is defined by*

$$(I, \mathcal{J})_e - \limsup x := \begin{cases} \inf B_x, & B_x \neq \emptyset, \\ \infty, & \textit{otherwise} \end{cases}$$

and $(I, \mathcal{J})_e$-limit inferior of $x = (x_{kl})$ is defined by

$$(I, \mathcal{J})_e - \liminf x := \begin{cases} \sup A_x, & A_x \neq \emptyset, \\ -\infty, & \textit{otherwise} \end{cases}$$

where

$$A_x := \left\{a \in \mathbb{R} : \{l : \{k : x_{kl} > a\} \in \mathcal{F}(\mathcal{J})\} \in \mathcal{F}(I)\right\},$$
$$B_x := \left\{b \in \mathbb{R} : \{l : \{k : x_{kl} < b\} \in \mathcal{F}(\mathcal{J})\} \in \mathcal{F}(I)\right\}.$$

Clearly, if $x = (x_{kl})$ is $(I,\mathcal{J})_e$-bounded, then $A_x \neq \emptyset$ and $B_x \neq \emptyset$. Hence, both of $(I,\mathcal{J})_e - \limsup x$ and $(I,\mathcal{J})_e - \liminf x$ are finite numbers. Moreover, if $(I,\mathcal{J})_e - \lim x$ exists, then $x = (x_{kl})$ is $(I,\mathcal{J})_e$-bounded.

Now we will give an example to clarify the concepts just defined.

Example 2.2. *Let us take $I = I_f$ and $\mathcal{J} = I_\delta$, and define*

$$x_{kl} := \begin{cases} k, & k \text{ is a square and } l \text{ is an even}, \\ 1, & k \text{ is a nonsquare and } l \text{ is an even}, \\ -k, & k \text{ is a square and } l \text{ is an odd}, \\ -1, & k \text{ is a nonsquare and } l \text{ is an odd}. \end{cases}$$

Then the sets $A_x = (-\infty, -1)$ and $B_x = (1, \infty)$ are obtained. Consequently, we have $(I,\mathcal{J})_e - \liminf x = -1$ and $(I,\mathcal{J})_e - \limsup x = 1$.

Theorem 2.2. *If $u = (I,\mathcal{J})_e - \limsup x$ is finite, then for every $\varepsilon > 0$*

$$\left\{l : \{k : x_{kl} < u+\varepsilon\} \in \mathcal{F}(\mathcal{J})\right\} \in \mathcal{F}(I), \ \left\{l : \{k : x_{kl} > u-\varepsilon\} \notin \mathcal{J}\right\} \notin I. \qquad (2.1)$$

Conversely, if for every $\varepsilon > 0$ (2.1) holds then $u = (I,\mathcal{J})_e - \limsup x$.

Proof. Assume that $(I,\mathcal{J})_e - \limsup x = u$. Hence $u = \inf B_x$. By using the definition of infimum, for given $\varepsilon > 0$, there exists $u_\varepsilon \in B_x$ such that $u_\varepsilon \leq u+\varepsilon$. Since $u_\varepsilon \in B_x$ and considering the definition of the set B_x, we have $\{l : \{k : x_{kl} < u_\varepsilon\} \in \mathcal{F}(\mathcal{J})\} \in \mathcal{F}(I)$. Since

$$\{l : \{k : x_{kl} < u_\varepsilon\} \in \mathcal{F}(\mathcal{J})\} \subseteq \{l : \{k : x_{kl} < u+\varepsilon\} \in \mathcal{F}(\mathcal{J})\},$$

we obtain that the set $\{l : \{k : x_{kl} < u+\varepsilon\} \in \mathcal{F}(I)\}$ belongs to $\mathcal{F}(I)$.

We now show the second formula of (2.1). Define $L = \{l : \{k : x_{kl} > u-\varepsilon\} \notin \mathcal{J}\}$ and suppose that the set $L \in I$. Therefore, for each $l \in L^c$ we obtain $\{k : x_{kl} > u-\varepsilon\} \in \mathcal{J}$, i.e., $\{k : x_{kl} \leq u-\varepsilon\} \in \mathcal{F}(\mathcal{J})$ for each $l \in L^c$. So,

$$L^c \subseteq \{l : \{k : x_{kl} \leq u-\varepsilon\}\}.$$

Hence, the set

$$\{l : \{k : x_{kl} \leq u-\varepsilon\}\}$$

belongs to $\mathcal{F}(I)$. This implies that $u-\varepsilon \in B_x$. Therefore $u-\varepsilon \geq \inf B_x = u$, which is a contradiction. This gives us $L \notin I$.

Conversely, take a real number u that provides the conditions (2.1). In this case, for given $\varepsilon > 0$ we get $u+\varepsilon \in B_x$. Hence we have

$$(I,\mathcal{J})_e - \limsup x = \inf B_x \leq u+\varepsilon. \qquad (2.2)$$

On the other hand for each $b \in B_x$ we obtain the set $K = \{l : \{k : x_{kl} < b\} \in \mathcal{F}(\mathcal{J})\}$ with $K \in \mathcal{F}(I)$. So $L \notin I$, there exists $l_1 \in K \cap L$ such that

$$\{k : x_{kl_1} < b\} \in \mathcal{F}(\mathcal{J}) \text{ and } \{k : x_{kl_1} > u - \varepsilon\} \notin \mathcal{J}.$$

Therefore there exists k_1 such that $u - \varepsilon < x_{k_1 l_1} < b$. Since this holds for each $b \in B_x$ we have

$$u - \varepsilon \leq \inf B_x = (I, \mathcal{J})_e - \limsup x. \tag{2.3}$$

Take (2.2) and (2.3) together, since ε is arbitrary we obtain $u = (I, \mathcal{J})_e - \limsup x$. □

The following theorem is the dual statement of Theorem 2.2 for $(I, \mathcal{J})_e - \liminf x$. Therefore we omit the proof of the theorem.

Theorem 2.3. *If* $v = (I, \mathcal{J})_e - \liminf x$ *is finite, then for every* $\varepsilon > 0$

$$\{l : \{k : x_{kl} < v + \varepsilon\} \notin \mathcal{J}\} \notin I, \ \{l : \{k : x_{kl} > v - \varepsilon\} \in \mathcal{F}(\mathcal{J})\} \in \mathcal{F}(I). \tag{2.4}$$

Conversely, if for every $\varepsilon > 0$ *(2.4) holds then* $v = (I, \mathcal{J})_e - \liminf x$.

We will now give a theorem that gives the condition for the existence of the e-limit.

Theorem 2.4. $(I, \mathcal{J})_e - \limsup x = (I, \mathcal{J})_e - \liminf x = a$ *if and only if* $(I, \mathcal{J})_e - \lim x = a$.

Proof. Let $(I, \mathcal{J})_e - \lim x = a$. In this case for any $\varepsilon > 0$ the set

$$K = \{l : \{k : |x_{kl} - a| \geq \varepsilon\} \in \mathcal{J}\}$$

belongs to $\mathcal{F}(I)$. So, we have for $l \in K$,

$$\{k : x_{kl} \geq a + \varepsilon\} \in \mathcal{J} \quad \text{and} \quad \{k : x_{kl} \leq a - \varepsilon\} \in \mathcal{J}$$

i.e.,

$$\{k : x_{kl} < a + \varepsilon\} \in \mathcal{F}(\mathcal{J}) \quad \text{and} \quad \{k : x_{kl} > a - \varepsilon\} \in \mathcal{F}(\mathcal{J}).$$

This implies that $a + \varepsilon \in B_x$ and $a - \varepsilon \in A_x$. Consequently we obtain

$$a - \varepsilon \leq (I, \mathcal{J})_e - \liminf x = \sup A_x \leq (I, \mathcal{J})_e - \limsup x = \inf B_x \leq a + \varepsilon.$$

Since ε is an arbitrary, we have $(I, \mathcal{J})_e - \limsup x = (I, \mathcal{J})_e - \liminf x = a$.

Conversely, let us take $(I, \mathcal{J})_e - \limsup x = (I, \mathcal{J})_e - \liminf x = a$. Hence, for given $\varepsilon > 0$ there exist two sets $K_1 := \{l : \{k : x_{kl} < a + \varepsilon\} \in \mathcal{F}(\mathcal{J})\}$, $K_2 := \{l : \{k : x_{kl} > a - \varepsilon\} \in \mathcal{F}(\mathcal{J})\}$ with $K_1 \in \mathcal{F}(I)$ and $K_2 \in \mathcal{F}(I)$. If we take $K = K_1 \cap K_2$, then we have $K \in \mathcal{F}(I)$. For $l \in K$ we have

$$\{k : |x_{kl} - a| < \varepsilon\} \in \mathcal{F}(\mathcal{J}) \qquad \text{or} \qquad \{k : |x_{kl} - a| \geq \varepsilon\} \in \mathcal{J}.$$

Since

$$K \subseteq \{l : \{k : |x_{kl} - a| \geq \varepsilon\} \in \mathcal{J}\},$$

we get the set $\{l : \{k : |x_{kl} - a| \geq \varepsilon\} \in \mathcal{J}\}$ belongs to $\mathcal{F}(I)$. Consequently we obtain $(I, \mathcal{J})_e - \lim x = a$. □

We now give a theorem that can be easily proved by a similar argument used for the $(\mathcal{I},\mathcal{J})_e$-convergence of double sequences (see [26]).

Theorem 2.5. *Let $x=(x_{kl})$ and $y=(y_{kl})$ be two real double sequences. Then the following statements are satisfied:*

(i) $(\mathcal{I},\mathcal{J})_e-\liminf x \leq (\mathcal{I},\mathcal{J})_e-\limsup x$,
(ii) $(\mathcal{I},\mathcal{J})_e-\limsup\,(-x) = -((\mathcal{I},\mathcal{J})_e-\liminf x)$,
(iii) $(\mathcal{I},\mathcal{J})_e-\liminf(x+y) \geq (\mathcal{I},\mathcal{J})_e-\liminf x + (\mathcal{I},\mathcal{J})_e-\liminf y$,
(iv) $(\mathcal{I},\mathcal{J})_e-\limsup(x+y) \leq (\mathcal{I},\mathcal{J})_e-\limsup x + (\mathcal{I},\mathcal{J})_e-\limsup y$.

Theorem 2.6. *Let $\mathcal{I}$ and $\mathcal{J}$ be admissible ideals on natural numbers $\mathbb{N}$. Then the following inequalities*

$$e-\liminf x \leq (\mathcal{I},\mathcal{J})_e-\liminf x \leq (\mathcal{I},\mathcal{J})_e-\limsup x \leq e-\limsup x \tag{2.5}$$

hold for every real double sequences $x=(x_{kl})$.

Proof. The proof of the theorem is a consequence of the inclusion $\mathcal{I}_f \subset \mathcal{I}$. □

Similar to the e-core [26] we will define the $(\mathcal{I},\mathcal{J})_e$-core of double sequences of real numbers.

Definition 2.4. *Let $x=(x_{kl})$ be a $(\mathcal{I},\mathcal{J})_e$-bounded real double sequence. The ideal $(\mathcal{I},\mathcal{J})_e$-core of the sequence x is defined as the closed interval*

$$[(\mathcal{I},\mathcal{J})_e-\liminf x, (\mathcal{I},\mathcal{J})_e-\limsup x].$$

If x is not $(\mathcal{I},\mathcal{J})_e$-bounded, then $(\mathcal{I},\mathcal{J})_e$-core of the sequence x is defined by either $(-\infty,(\mathcal{I},\mathcal{J})_e-\limsup x]$, $[(\mathcal{I},\mathcal{J})_e-\liminf x,\infty)$ or $(-\infty,\infty)$. The ideal e-core of the sequence $x=(x_{kl})$ will be denoted by $(\mathcal{I},\mathcal{J})_e-core(x)$.

We can easily obtain the following corollary from the inequality (2.5).

Corollary 2.1. *If $x=(x_{kl})$ is any real double sequence, then the following inclusion holds*

$$(\mathcal{I},\mathcal{J})_e-core(x) \subset e\text{-}core(x).$$

We will examine a Korovkin-type approximation theorem for double sequences of positive linear operators on the space of all continuous real-valued functions defined on any compact subset of the real two-dimensional space via ideal.

3. A Korovkin-type approximation theorem

Let $C(K)$ be the space of all continuous real-valued functions on any compact subset of the real two-dimensional space. Then $C(K)$ is a Banach space with the norm $\|\cdot\|_{C(K)}$ defined as

$$\|f\|_{C(K)} := \sup_{(x,y)\in K} |f(x,y)|, \quad (f \in C(K)).$$

Let L be a linear operator from $C(K)$ into $C(K)$. We denote the value of Lf at a point $(x,y) \in K$ by $L(f;x,y)$. Also we say that L is positive linear operator if $f \geq 0$ implies $Lf \geq 0$. The positive linear operator L is monotone and isotonic. In other words, if f and g are elements of $C(K)$ with $f \leq g$ then we have $Lf \leq Lg$ and $|Lf| \leq L|f|$.

We will give a Korovkin-type approximation theorem, where the classic version was given by Volkov [30] and the ideal version was given by Duman [8].

Theorem 3.1. *Let (L_{kl}) be a double sequence of positive linear operators acting from $C(K)$ into $C(K)$. Then, for all $f \in C(K)$,*

$$(I,\mathcal{J})_e - \lim_{k,l} \|L_{kl}(f) - f\|_{C(K)} = 0 \tag{3.1}$$

if and only if

$$(I,\mathcal{J})_e - \lim_{k,l} \left\|L_{kl}(f_r) - f_r\right\|_{C(K)} = 0, \quad (r = 0,1,2,3), \tag{3.2}$$

where $f_0(x,y) = 1,\ f_1(x,y) = x,\ f_2(x,y) = y,\ f_3(x,y) = x^2 + y^2$.

Proof. Since each of the functions $f_0(x,y) = 1, f_1(x,y) = x, f_2(x,y) = y, f_3(x,y) = x^2 + y^2$ belongs to $C(K)$, the condition (3.2) follows immediately from (3.1).

On the other hand, let $f \in C(K)$. We can write $f(x,y) \leq M$, where $M := \|f\|_{C(K)}$. Hence $|f(u,v) - f(x,y)| \leq 2M$, for all $u,v,x,y \in \mathbb{R}$. Also, since f is continuous on K, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ for all $(u,v) \in K$ satisfying $|u-x| < \delta$ and $|v-y| < \delta$. The other case; $|u-x| \geq \delta$ implies $\frac{|u-x|}{\delta} \geq 1$, which in turn is equivalent to $\frac{(u-x)^2}{\delta^2} \geq 1$,(also, $|v-y| \geq \delta$ implies $\frac{|v-y|}{\delta} \geq 1$, which in turn is equivalent to $\frac{(u-x)^2}{\delta^2} \geq 1$).

Therefore, we get

$$|f(u,v) - f(x,y)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\}. \tag{3.3}$$

Since (L_{kl}) is linear and positive, from (3.3) we obtain

$$\begin{aligned}
|L_{kl}(f;x,y) - f(x,y)| &= |L_{kl}(f(u,v) - f(x,y);x,y) - f(x,y)(L_{kl}(f_0;x,y) - f_0(x,y))| \\
&\leq L_{kl}(|f(u,v) - f(x,y)|;x,y) + M|L_{kl}(f_0;x,y) - f_0(x,y)| \\
&\leq \left|L_{kl}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\};x,y\right)\right| \\
&\quad + M|L_{kl}(f_0;x,y) - f_0(x,y)| \\
&\leq \left(\varepsilon + M + \frac{2M}{\delta^2}(E^2 + F^2)\right)|L_{kl}(f_0;x,y) - f_0(x,y)| \\
&\quad + \frac{4M}{\delta^2}E|L_{kl}(f_1;x,y) - f_1(x,y)| + \frac{4M}{\delta^2}F|L_{kl}(f_2;x,y) - f_2(x,y)| \\
&\quad + \frac{2M}{\delta^2}|L_{kl}(f_3;x,y) - f_3(x,y)| + \varepsilon,
\end{aligned}$$

where $E := \max|x|,\ F := \max|y|$. Taking the supremum over $(x,y) \in K$ we get

$$\|L_{kl}(f)-f\|_{C(K)} \leq P\Big\{\|L_{kl}(f_0)-f_0\|_{C(K)}+\|L_{kl}(f_1)-f_1\|_{C(K)} + \|L_{kl}(f_2)-f_2\|_{C(K)}+\|L_{kl}(f_3)-f_3\|_{C(K)}\Big\}+\varepsilon \qquad (3.4)$$

where $P = \max\left\{\varepsilon + M + \frac{2M}{\delta^2}(E^2+F^2),\ \frac{4M}{\delta^2}E,\ \frac{4M}{\delta^2}F,\ \frac{2M}{\delta^2}\right\}$.

Now, for a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$ and define

$$K = \left\{l : \{k : \|L_{kl}(f)-f\|_{C(K)} \geq \varepsilon'\} \in \mathcal{J}\right\},$$

$$K_r = \left\{l : \left\{k : \|L_{kl}(f_r)-f_r\|_{C(K)} \geq \frac{\varepsilon'-\varepsilon}{4P}\right\} \in \mathcal{J}\right\}.$$

Then, by the equality (3.2), we have $\mathbb{N}\setminus K_r \in \mathcal{I},\ r = 0,1,2,3$. If we take $K_4 = \bigcap_{r=0}^{3} K_r$, then we have $\mathbb{N}\setminus K_4 \in \mathcal{I}$. For each $l \in K_4$ we define

$$N_r^l = \left\{k : \|L_{kl}(f_r)-f_r\|_{C(K)} \geq \frac{\varepsilon'-\varepsilon}{4P}\right\}.$$

From the inequality (3.4) for each $l \in K_4$ we get

$$\{k : \|L_{kl}(f)-f\|_{C(K)} \geq \varepsilon'\} \subseteq \bigcup_{r=0}^{3} N_r^l.$$

Since, for $l \in K_4$, $N_r^l \in \mathcal{J}$ i.e., $\bigcup_{r=0}^{3} N_r^l \in \mathcal{J}$ we obtain that the set

$$\{k : \|L_{kl}(f)-f\|_{C(K)} \geq \varepsilon'\}$$

belongs to $\mathcal{J}$. This implies $K_4 \subseteq K$. So $\mathbb{N}\setminus K \in \mathcal{I}$. Namely, (3.1) holds, which completes the proof. □

Example 3.1. *Let the ideals $\mathcal{I}$, $\mathcal{J}$ and double sequence (x_{kl}) be as in Example 2.1. Then $(\mathcal{I},\mathcal{J})_e - \lim x = 0$. Now we define Bernstein polynomials of two variables [29] by*

$$B_{kl}(f;x,y) = \sum_{i=0}^{k}\sum_{j=0}^{l} f\left(\frac{i}{k},\frac{j}{l}\right)\binom{k}{i}x^i(1-x)^{k-i}\binom{l}{j}y^j(1-y)^{l-j}, \qquad (3.5)$$

where $(x,y) \in K = [0,1]\times[0,1]$ and $f \in C(K)$. We consider the following positive linear operators on $C(K)$ as follows:

$$L_{kl}(f;x,y) = (1+x_{kl})B_{kl}(f;x,y).$$

Then, the double sequence (L_{kl}) satisfies the conditions of Theorem 3.1. Therefore, we obtain

$$e - \lim_{k,l}\|L_{kl}(f)-f\|_{C(K)} = 0.$$

On the other hand, since (x_{kl}) is not statistical e-convergent, (L_{kl}) does not satisfy the conditions of Theorem 3.3 in [27].

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Yurdal SEVER
Department of Mathematics
Faculty of Art and Sciences
Afyon Kocatepe University
03200 Afyonkarahisar, Turkey
e-mail: *yurdalsever@hotmail.com, ysever@aku.edu.tr*
and
Özer TALO
Manisa Celal Bayar Üniversitesi Küme evleri
45140 Yunusemre, Manisa, Turkey
e-mail: *ozertalo@hotmail.com*