BRAUER-CLIFFORD GROUP OF POISSON (S,H)-HOPF ALGEBRAS

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ABSTRACT. In this paper, we extend the notion of the Brauer-Clifford group to the case of an Azumaya-Poisson (S,H)-Hopf algebra, when H is a commutative Hopf algebra and S is an H-comodule Poisson algebra. This is the situation that arises in applications with connections to algebraic geometry. We give three useful examples : an affine algebraic group acting rationally on a Poisson algebra, a Hopf algebra coacting on the localization of a Poisson algebra and a direct product of Hopf algebras coacting on a direct product of Poisson algebras.

1. INTRODUCTION

Let H be a cocommutative Hopf algebra over a field $\Bbbk$, and let S be an H-module Poisson algebra over $\Bbbk$. In [14], we studied the Brauer-Clifford group of (S,H)-Azumaya-Poisson algebras. In the present article, we will deal with the dual situation. More precisely, we will study the Brauer-Clifford group of (S,H)-Hopf Azumaya-Poisson algebras when H is a commutative Hopf algebra and S is an H-comodule Poisson algebra. Our study is also motivated by [12], where with A. Herman, we have studied the Brauer-Clifford group of (S,H)-Hopf Azumaya algebras when H is a commutative Hopf algebra and S is a commutative H-comodule algebra. In [11], K.R. Goodearl obtained interesting results when an algebraic torus G acts on a Poisson algebra. The coordinate ring $k[G]$ of G is a commutative Hopf algebra, and a rational G-action is equivalent to a $k[G]$-coaction. This result of Goodearl also suggests that we can study commutative Hopf algebra coactions on Poisson algebras. Poisson algebras appear naturally in Hamiltonian mechanics, and play an important role in the study of Poisson geometry and quantum groups. Let H be a commutative Hopf algebra over a field $\Bbbk$, and let S be an H-comodule Poisson algebra over $\Bbbk$. An (S,H)-Hopf Azumaya-Poisson algebra is an algebra in the category of Poisson (S,H)-Hopf modules which is also an Azumaya algebra over S - a precise definition is given in §2. After developing the background necessary for an understanding of (S,H)-Hopf Azumaya-Poisson algebras in §2 and §3, we define the Brauer-Clifford group $B^{\mathcal{P}co}(S,H)$ for a commutative Hopf

algebra H and an H-comodule Poisson algebra S. In many applications, H could be the coordinate ring of an algebraic group acting rationally on a Poisson algebra via Poisson algebras automorphisms. When H is finite-dimensional, we show that this Brauer-Clifford group is isomorphic to the Brauer-Clifford group $B^{\mathcal{P}}(S, H^{\star})$ of Azumaya-Poisson $(S, H^{\star})$-algebras studied in [14]. We present generalizations of the Rosenberg-Zelinsky exact sequence for automorphisms of Azumaya-Poisson (S, H)-Hopf algebras, and we discuss the central twist group actions on $B^{\mathcal{P}co}(S, H)$. If we consider the base field $\Bbbk$ as a trivial Hopf algebra, then $B^{\mathcal{P}co}(S, k)$ is the Brauer group $B^{\mathcal{P}}(S)$ of Azumaya-Poisson S-algebras studied in [13]. In Section 4, we give some examples of Brauer-Clifford groups. We refer to [1] and [16] for further information on Brauer groups of commutative rings and of H-comodule algebras.

2. THE CATEGORY OF POISSON (S, H)-HOPF MODULES

Let $\Bbbk$ be a field. A Hopf algebra over $\Bbbk$ is a $\Bbbk$-algebra H that possesses a multiplication $m_H : H \otimes H \to H$, a comultiplication $\Delta_H : H \to H \otimes H$, an antipode $S_H : H \to H$ and a counit $\varepsilon_H : H \to k$, satisfying the defining relations

$$(\Delta_H \otimes id_H) \otimes \Delta_H = (id_H \otimes \Delta_H) \otimes \Delta_H, \quad \text{(coassociativity)}$$

$$m_H \circ [(S_H \otimes id_H) \otimes \Delta_H] = m_H \circ [(id_H \otimes S_H) \otimes \Delta_H] = \varepsilon_H, \text{and}$$

$$(\varepsilon_H \otimes id_H) \otimes \Delta_H = (id_H \otimes \varepsilon_H) \otimes \Delta_H] = id_H.$$

For background on Hopf algebras and coactions of Hopf algebras on rings, we refer the reader to [24] and [20]. We will use Sweedler-Heyneman notation, and we write :

$$\Delta_H(h) = h_1 \otimes h_2, \text{ for all } h \in H.$$

By an H-comodule, we will mean a right H-comodule. When M and N are H-comodules, $M \otimes_k N$ is an H-comodule under the diagonal coaction, that is,

$$(m \otimes n)_0 \otimes (m \otimes n)_1 = m_0 \otimes n_0 \otimes m_1 n_1; \quad \forall \quad m \in M, n \in N.$$

A $\Bbbk$-algebra S is an *H-comodule algebra* if S is an H-comodule satisfying

$$(ss')_0 \otimes (ss')_1 = s_0 s'_0 \otimes s_1 s'_1, (1_S)_0 \otimes (1_S)_1 = 1_S \otimes 1_H, \forall s, s' \in S \qquad (1).$$

In this case, we will say that the H-coaction on S is compatible with the multiplication in S. A homomorphism of H-comodule algebras is a homomorphism of H-comodules which is also a homomorphism of $\Bbbk$-algebras.

Definition 2.1. *Let S be an H-comodule algebra. A vector space M is an (S, H)-Hopf module if M is an S-module and an H-comodule such that*

$$(sm)_0 \otimes (sm)_1 = s_0 m_0 \otimes s_1 m_1 \forall s, \in S, m \in M \qquad (2).$$

It is easy to see that S is an (S,H)-Hopf module whenever S is an H-comodule algebra.

If S is an H-comodule algebra and T is a sub-H-comodule algebra of S, it is also easy to see that S is a (T,H)-Hopf module. We denote by $_S\mathcal{M}^H$ the category of (S,H)-Hopf modules : its morphisms are the S-linear maps which are also H-colinear.

A Poisson algebra is a commutative associative unitary $\Bbbk$-algebra S endowed with a bilinear map $S \times S \to S$ denoted by $\{.,.\}$, called a Poisson bracket, providing S with a Lie algebra structure and satisfying the relation

$$\{s,s's''\} = s'\{s,s''\} + s''\{s,s'\} \quad \text{for} \quad s, \quad s', \quad s'' \in S \tag{3}.$$

The base field $\Bbbk$ is a Poisson algebra with a trivial bracket. A Poisson subalgebra of a Poisson algebra S is a subalgebra of S which is also a Lie subalgebra of S. We denote by $U(S)$ the enveloping algebra of S. Let S be a Poisson algebra. The Poisson center of S

$$Z_{\mathcal{P}}(S) = \{s \in S, \{s,S\} = 0\}$$

is a Poisson subalgebra of S with trivial Poisson bracket : $\{s,s'\} = 0$ for all $s,s' \in Z_{\mathcal{P}}(S)$.

Let S and T be two Poisson algebras. A homomorphism of Poisson algebras from S to T is a homomorphism of algebras f from S to T which preserves the brackets; i.e., $f(\{s,s'\}) = \{f(s), f(s')\}$ for all s, s' in S. Let us give some examples of Poisson algebras.

Let S be a Poisson algebra. A vector space M is a Poisson S-module if M is an S-module: $(s,m) \mapsto sm$, and a Lie S-module: $(s,m) \mapsto s \diamond m$ satisfying the following compatibility conditions:

$$s \diamond (s'm) = \{s,s'\}m + s'(s \diamond m) \tag{4}$$

and

$$(ss') \diamond m = s(s' \diamond m) + s'(s \diamond m) \tag{5}.$$

Recall that M is a Lie S- module means that the Lie action $\diamond$ satisfies

$$\{s,s'\} \diamond m = s \diamond (s' \diamond m) - s' \diamond (s \diamond m) \quad \forall \quad s, \quad s' \quad \in S, \quad m \in M \tag{6}.$$

We have $1_S \diamond m = 0$ for all $m \in M$. Recall also that M is a Lie S-module if and only if M is a $U(S)$-module in the natural way. The Poisson algebra S itself is a Poisson S-module with $s \diamond s' = \{s,s'\}$. A Poisson $\Bbbk$-module is just a vector space over $\Bbbk$. It follows from (3) that if S is a Poisson algebra, then S is a $U(S)$-module algebra, and we can form the smash product $S\#U(S)$. We deduce from (3) and (4) that M is a Poisson S-module if and only if M is an $S\#U(S)$-module and the relation (5) is satisfied. Given two Poisson S-modules M and N, a homomorphism of Poisson S-modules f from M to N is an S-linear map f which is also a Lie S-linear map from M to N, that is, an $S\#U(S)$-linear map f from M to N; i.e., $f(sm) = sf(m)$ and $f(s \diamond m) = s \diamond f(m)$.

Let S, T be two Poisson algebras and $f : T \to S$ a homomorphism of Poisson algebras. If M is a Poisson S-module, then M is a Poisson T-module: $tm = f(t)m$ and $t \diamond m = f(t) \diamond m$ for all $t \in T$, $m \in M$.

Definition 2.2. *Let S be a Poisson algebra. We say that S is an H-comodule Poisson algebra if S is an H-comodule algebra such that the H-coaction is compatible with the Poisson bracket; that is, S is an H-comodule algebra satisfying the relation*

$$\{s,s'\}_0 \otimes \{s,s'\}_1 = \{s_0,s'_0\} \otimes s_1 s'_1 \quad \forall s,s' \in S \tag{7}.$$

The base field $\Bbbk$ is an H-comodule Poisson algebra with a trivial Poisson bracket and a trivial H-coaction. An H-sub-comodule Poisson algebra of an H-comodule Poisson algebra S is an H-sub-comodule algebra of S which is also a Lie S-subalgebra of S.

Lemma 2.1. *Let S be an H-comodule Poisson algebra. Then the Poisson-center $Z_{\mathcal{P}}(S)$ of S is an H-sub-comodule Poisson algebra of S.*

Proof. We will adapt the proof of [2, Corollary 3.6]. We know that $\{1_S, S\} = 0$. So $1_S \in Z_{\mathcal{P}}(S)$. Using the relation (3), we can show that $tt' \in Z_{\mathcal{P}}(S)$ for all t,t' in $Z_{\mathcal{P}}(S)$. So $Z_{\mathcal{P}}(S)$ is a subalgebra of S. Let $t \in Z_{\mathcal{P}}(S)$, $s \in S$, $\rho(t) = t_0 \otimes t_1$ and $\rho(s) = s_0 \otimes s_1$. Using the relation (6), we have

$$\{t_0,s\} \otimes t_1 = [\{t,s_0\}_0 \otimes \{t,s_0\}_1] \otimes (1 \otimes S_H(s_1)).$$

Each $\{t,s_0\}$ is equal to 0 since $t \in Z_{\mathcal{P}}(S)$. It follows that the right term is equal to 0. We deduce that $\{t_0,s\} \otimes t_1 = 0$. Now taking the summands $\{t_1\}$ to be linearly independant, we have $\{t_0,s\} = 0$ for each summand t_0. So $Z_{\mathcal{P}}(S)$ is a subcomodule of S. It is well known that $Z_{\mathcal{P}}(S)$ is a Lie subalgebra of S. $\qquad\square$

Definition 2.3. *Let S be an H-comodule Poisson algebra. A vector space M is a Poisson (S,H)-Hopf module if M is a Poisson S-module and an (S,H)-Hopf module such that*

$$(s \diamond m)_0 \otimes (s \diamond m)_1 = (s_0 \diamond m_0) \otimes s_1 m_1 \quad \forall s \in S, m \in M \tag{8};$$

or equivalently, M is an $S\#U(S)$-module, an (S,H)-Hopf module and the relations (5) and (8) are satisfied.

The H-comodule Poisson algebra S itself is a Poisson (S,H)-Hopf module. When $\Bbbk$ is considered as a trivial Hopf algebra, a Poisson (S,k)-Hopf module is just a Poisson S-module. If we consider $\Bbbk$ as an H-comodule Poisson algebra with a trivial Poisson bracket and a trivial H-coaction, a Poisson (k,H)-Hopf module is just an H-comodule. A Poisson (S,H)-Hopf submodule of a Poisson (S,H)-Hopf module M is a Poisson S-submodule of M which is also an H-subcomodule of M, or equivalently, an (S,H)-Hopf submodule of M which is also a Lie S-submodule of M.

A Poisson (S,H)-Hopf module homomorphism between two Poisson (S,H)-Hopf modules M and N is an S-linear map from M to N which is also a Lie S-linear

map and an H-colinear map, that is, a Poisson S-linear map which is also an H-colinear map, that is, a Lie S-linear map which is also an (S,H)-Hopf module map, that is, an $S\#U(S)$-linear map which is also an H-colinear map.

We denote by $_{\mathcal{P}S}\mathcal{M}^H$ the category of Poisson (S,H)-Hopf modules with Poisson (S,H)-Hopf module homomorphisms. The category $_{\mathcal{P}S}\mathcal{M}^H$ is a subcategory of $_S\mathcal{M}^H$; it is also a subcategory of $_{S\#U(S)}\mathcal{M}$. This remark will enable us to use some well known results of modules over smash products and (S,H)-Hopf modules. For the remainder of the section, H is a commutative Hopf algebra, and S is an H-comodule Poisson algebra.

Lemma 2.2. *Let M and N be Poisson (S,H)-Hopf modules. Then*

(i) *$M \otimes_S N$ is a Poisson (S,H)-Hopf module : the actions and the coaction are given by*

$$s(m \otimes_S n) = (sm) \otimes_S n \quad : this\ is\ the\ natural\ S\text{-}action,$$

$$s \diamond (m \otimes_S n) = (s \diamond m) \otimes_S n + m \otimes_S (s \diamond n) : this\ is\ the\ diagonal\ U(S)\text{-}action,$$

$$(m \otimes_S n)_0 \otimes (m \otimes_S n)_1 = m_0 \otimes_S n_0 \otimes (m_1 n_1) : this\ is\ the\ diagonal\ H\text{-}coaction.$$

(ii) *the canonical S-isomorphism*

$$M \otimes_S N \to N \otimes_S M; m \otimes_S n \mapsto n \otimes_S m$$

is an isomorphism of Poisson (S,H)-Hopf modules.

Proof. (i) $M \otimes_S N$ is an $S\#U(S)$-module and an (S,H)-Hopf module, since S is commutative, $U(S)$ is cocommutative and H is commutative [12, Lemma 2.1]. By [14], the relation (5) is satisfied. We also have

$$(s \diamond (m \otimes_S n))_0 \otimes (s \diamond (m \otimes_S n))_1$$
$$= [(s \diamond m) \otimes_S n + m \otimes_S (s \diamond n)]_0 \otimes [(s \diamond m) \otimes_S n + m \otimes_S (s \diamond n)]_1$$
$$= ((s \diamond m) \otimes_S n)_0 \otimes ((s \diamond m) \otimes_S n)_1 + (m \otimes_S (s \diamond n))_0 \otimes (m \otimes_S (s \diamond n))_1$$
$$= ((s_0 \diamond m_0) \otimes_S n_0) \otimes ((s_1 m_1)n_1) + (m_0 \otimes_S (s_0 \diamond n_0)) \otimes (m_1(s_1 n_1))]$$
$$= [(s_0 \diamond m_0) \otimes_S n_0 + m_0 \otimes_S (s_0 \diamond n_0)] \otimes (s_1 m_1 n_1)$$
$$= s_0 \diamond (m_0 \otimes_S n_0) \otimes (s_1 m_1 n_1)$$
$$= s_0 \diamond (m \otimes_S n)_0 \otimes s_1 (m \otimes n)_1.$$

Thus we have shown that

$$(s \diamond (m \otimes_S n))_0 \otimes (s \diamond (m \otimes_S n))_1 = s_0 \diamond (m \otimes_S n)_0 \otimes s_1 (m \otimes n)_1,$$

and the relation (8) is satisfied.

(ii) By [20, Lemma 10.1.2], this natural S-isomorphism is $S\#U(S)$-linear and H-colinear, since S is commutative, $U(S)$ is cocommutative, and H is commutative
. $\qquad\square$

A *monoidal category* $(\mathcal{C}, \otimes, I)$ is a category $\mathcal{C}$ with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and object $I \in \mathcal{C}$ for which $\otimes$ satisfies natural coherence conditions, is associative up to natural transformation, and I is an identity object for $\otimes$ up to natural transformation. For background on monoidal categories, see [17]. A monoidal category $\mathcal{C}$ is *symmetric* if there are natural isomorphisms $\gamma_{M,N} : M \otimes N \cong N \otimes M$ in $\mathcal{C}$ for all $M, N \in \mathcal{C}$, such that $\gamma_{N,M} \circ \gamma_{M,N} = id_{M \otimes N}$ and certain hexagonal conditions are satisfied [17, p. 180]. Note that an H-comodule algebra is an algebra in the monoidal category $\mathcal{M}^H$ of H-comodules. One can use the previous lemma to show that $({}_{\mathcal{P}S}\mathcal{M}^H, \otimes_S, S)$ is a symmetric monoidal category.

Theorem 2.1. $({}_{\mathcal{P}S}\mathcal{M}^H, \otimes_S, S)$ *is a symmetric monoidal category.*

Proof. Let M, N, P be three Poisson (S, H)-Hopf modules. By Lemma 1.5(i), $(M \otimes_S N) \otimes_S P$ and $M \otimes_S (N \otimes_S P)$ are Poisson (S, H)-Hopf modules. It is well known that the natural S-isomorphism

$$(M \otimes_S N) \otimes_S P \to M \otimes_S (N \otimes_S P); (m \otimes_S n) \otimes_S p \mapsto m \otimes_S (n \otimes_S p)$$

is a homomorphism of $S\#U(S)$-modules and of (S, H)-Hopf modules since S is commutative, $U(S)$ is cocommutative and H is commutative. By [12], $({}_S\mathcal{M}^H, \otimes_S, S)$ and $({}_{S\#U(S)}\mathcal{M}, \otimes_S, S)$ are symmetric monoidal categories, since S is commutative, $U(S)$ is cocommutative and H is commutative. By Lemma 2.2(ii), the switch map $M \otimes_S N \to N \otimes_S M$ is an isomorphism of Poisson (S, H)-Hopf modules. The result follows. $\square$

Lemma 2.3. *Let M and N be Poisson (S, H)-Hopf modules with M finitely generated as an S-module. Then $Hom_S(M, N)$ is a Poisson (S, H)-Hopf module : the actions are given by*

$$(sf)(m) = sf(m) \quad : \textit{the natural S-action},$$

$$(s \diamond f)(m) = s \diamond f(m) - f(s \diamond m) : \textit{the diagonal U(S)-action},$$

and the coaction is the diagonal coaction, that is,

$$\rho(f) = f_0 \otimes f_1 \Leftrightarrow f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S_H(m_1)$$

for all $m \in M$, $s \in S$ and $f \in Hom_S(M, N)$.

Proof. It is well known that $Hom_S(M, N)$ is an $S\#U(S)$-module and an (S, H)-Hopf module, since S is commutative, $U(S)$ is cocommutative and H is commutative. By [13], we have

$$(ss') \diamond f = s(s' \diamond f) + s'(s \diamond f),$$

and the relation (5) is satisfied. For $s \in S$ and $m \in M$, we have

$$
\begin{aligned}
(s_0 \diamond f_0)(m) \otimes s_1 f_1 &= [(s_0 \diamond f_0(m) - f_0(s_0 \diamond m)] \otimes (s_1 f_1) \\
&= [(s_0 \diamond f_0(m)) \otimes (s_1 f_1)] - [f_0(s_0 \diamond m) \otimes (s_1 f_1)] \\
&= [s_0 \diamond (f(m_0)_0) \otimes (s_1 f(m_0)_1 S_H(m_1)] \\
&\quad - f((s_0 \diamond m)_0)_0 \otimes (s_1 f((s_0 \diamond m)_0)_1) S_H((s_0 \diamond m)_1) \\
&= [s_0 \diamond (f(m_0)_0) \otimes (s_1 f(m_0)_1 S_H(m_1)] \\
&\quad - f(s_{00} \diamond m_0)_0 \otimes (s_1 f(s_{00} \diamond m_0)_1) S_H(s_{01} m_1)) \\
&= (s_0 \diamond f(m_0)_0) \otimes (s_1 f(m_0)_1 S_H(m_1)) \\
&\quad - f(s_{00} \diamond m_0)_0 \otimes (s_1 f(s_{00} \diamond m_0)_1) S_H(m_1) S_H(s_{01})) \\
&= (s_0 \diamond f(m_0)_0) \otimes (s_1 f(m_0)_1 S_H(m_1)) \\
&\quad - f(s \diamond m_0)_0 \otimes (f(s \diamond m_0)_1 S_H(m_1)) \\
&= [s \diamond f(m_0)]_0 \otimes [s \diamond f(m_0)]_1 S_H(m_1) \\
&\quad - [f(s \diamond m_0)_0 \otimes f(s \diamond m_0)_1 S_H(m_1) \\
&= [s \diamond f(m_0) - f(s \diamond m_0)]_0 \otimes [s \diamond f(m_0) - f(s \diamond m_0)]_1 S_H(m_1) \\
&= (s \diamond f)(m_0)_0 \otimes ((s \diamond f)(m_0)_1) S_H(m_1) \\
&= (s \diamond f)_0(m) \otimes (s \diamond f)_1.
\end{aligned}
$$

This means that

$$
(s_0 \diamond f_0)(m) \otimes s_1 f_1 = (s \diamond f)_0(m) \otimes (s \diamond f)_1 \quad \forall m \in M.
$$

So the relation (8) is satisfied. $\qquad\square$

3. THE MAIN RESULTS

3.1. **Algebras in the category $_{PS}\mathcal{M}^H$**

In this section, H is a commutative Hopf algebra and S is an H-comodule Poisson algebra. We refer to [23] for more information on monoids in a monoidal category.

Definition 3.1. *An algebra A in $_{PS}\mathcal{M}^H$ is an object A of $_{PS}\mathcal{M}^H$ with the additional conditions that there are two morphisms $m_A : A \otimes_S A \to A$ (the multiplication map) and $\mu_A : S \to A$ (the unit map) in $_{PS}\mathcal{M}^H$ satisfying*

$$
m_A \circ (id_A \otimes m_A) = m_A \circ (m_A \otimes id_A) \quad \text{and}
$$
$$
m_A \circ (\mu_A \otimes id_A) = id_A = m_A \circ (id_A \otimes \mu_A).
$$

It follows from this definition that A is an algebra in $_{PS}\mathcal{M}^H$ if and only if A is an algebra in $_S\mathcal{M}^H$, an algebra in $_{S\#U(S)}\mathcal{M}$ with the same product map and the same unit map and the relations (5) and (8) are satisfied. An algebra in $_S\mathcal{M}^H$ is called an (S,H)-Hopf algebra in [12]. Following this terminology, we will call an algebra in $_{PS}\mathcal{M}^H$ a Poisson (S,H)-Hopf algebra. The H-comodule Poisson algebra S is

an example of a Poisson (S,H)-Hopf algebra. If we consider $\Bbbk$ as an H-comodule Poisson algebra with a trivial Poisson bracket and a trivial H-coaction, a Poisson (k,H)-Hopf algebra is just an H-comodule algebra. When $\Bbbk$ is considered as a trivial Hopf algebra, a Poisson (S,k)-Hopf algebra is just a Poisson S-algebra in the terminology of [13]. A homomorphism of Poisson (S,H)-Hopf algebras is an S-algebra homomorphism which is also Lie S-linear and H-colinear. If A is an S-algebra, we denote by A^o the opposite algebra of A.

Lemma 3.1. *Let A and B be Poisson (S,H)-Hopf algebras. Then*

(i) *$A \otimes_S B$ is a Poisson (S,H)-Hopf algebra;*

(ii) *the canonical S-algebra isomorphism $A \otimes_S B \simeq B \otimes_S A$ is an isomorphism of Poisson (S,H)-Hopf algebras;*

(iii) *A^o is a Poisson (S,H)-Hopf algebra, where the co-action of H is given by $a^o{}_o \otimes a^o{}_1 = a_0{}^o \otimes a_1$, the Lie S-action is given by $s \diamond a^o = (s \diamond a)^o$ and $sa^o = (sa)^o$ for all $s \in S, a \in A$ with $a \mapsto a^o$ being the canonical antiautomorphism; and*

(iv) *$A^e := A \otimes_S A^o$ is a Poisson (S,H)-Hopf algebra.*

Proof. (i) A and B are Poisson (S,H)-Hopf modules. By Lemma 2.2(i), $A \otimes_S B$ is a Poisson (S,H)-Hopf module. By [12, Lemma 3.2], $A \otimes_S B$ is an (S,H)-Hopf algebra and an $(S,U(S))$-algebra with the same product $m_{A \otimes_S B}$ and the same unit map $\mu_{A \otimes_S B}$. This shows the first assertion.

(ii) By [12, Lemma 3.2], the natural S-algebra isomorphism $A \otimes_S B \to B \otimes_S A$ given by $a \otimes_S b \mapsto b \otimes_S a$, for all $a \in A$ and $b \in B$, is an isomorphism of (S,H)-Hopf modules and of $S\#U(S)$-modules. So it is an isomomorphism of Poisson (S,H)-Hopf modules.

(iii) By [12, Lemma 3.2], A^o is an (S,H)-Hopf algebra and an $(S,U(S))$-algebra. By [13, Lemma 2.3(iii)], the relation (5) is satisfied for A^0, since A^0 is a Poisson S-module. Clearly, the relation (8) is satisfied for A^0.

(iv) Follows from (i) and (iii).   □

Lemma 3.2. *Let M and N be Poisson (S,H)-Hopf modules with M finitely generated as an S-module.*

(i) *Then $End_S(M)$ is a Poisson (S,H)-Hopf algebra.*

(ii) *If M and N are finitely generated projective as S-modules, then*

$$End_S(M) \otimes_S End_S(N) \simeq End_S(M \otimes_S N)$$

as Poisson (S,H)-Hopf algebras.

Proof. (i) By Lemma 2.3, $End_S(M)$ is a Poisson (S,H)-Hopf module. By [12, Lemma 3.2], $End_S(M)$ is an (S,H)-Hopf algebra and an $(S,U(S))$-algebra with the same product and the same unit map.

(ii) By (i), $End_S(M \otimes_S N)$ is an algebra in ${}_{PS}\mathcal{M}^H$. By (i) and Lemma 2.2(i), $End_S(M) \otimes_S End_S(N)$ is an algebra in ${}_{PS}\mathcal{M}^H$. It is well known that the canonical

map from $End_S(M) \otimes_S End_S(N)$ to $End_S(M \otimes_S N)$ is an S-algebra isomorphism when M and N are finitely generated projective as S-modules. It is obviously H-colinear and $U(S)$-linear. $\square$

An S-module is an S-progenerator if it is finitely generated projective and faithful (i.e. a generator) in the category of S-modules. If A is an S-algebra, we denote by $End_S(A)$ the algebra of S-endomorphisms of A, and by $A \otimes_S A^o$ the enveloping S-algebra of A. Note that A is a left $A \otimes_S A^o$-module with the action defined by $(a \otimes b^0).x = axb$, for all $a, x \in A$ and $b \in A^o$. We say that A is an S-*Azumaya algebra* if it is an S-progenerator as an S-module and the canonical map $A \otimes_S A^o \to End_S(A)$ is an S-algebra isomorphism. We will freely use well-known properties of tensor products, S-progenerators, and Azumaya algebras, all of which can be found in [6] or [15].

Definition 3.2. *An Azumaya algebra A in $_{PS}\mathcal{M}^H$ is an algebra A in $_{PS}\mathcal{M}^H$ which is an S-progenerator as an S-module and the canonical map $A \otimes_S A^o \to End_S(A)$ is an isomorphism of Poisson (S,H)-Hopf algebras.*

It follows from [12] that A is an Azumaya algebra in $_{PS}\mathcal{M}^H$ if and only if A is an Azumaya algebra in $_S\mathcal{M}^H$, an Azumaya algebra in $_{S\#U(S)}\mathcal{M}$ and the relations (5) and (8) are satisfied. In [12], an Azumaya algebra in $_S\mathcal{M}^H$ is called an (S,H)-Hopf Azumaya algebra. We will call an Azumaya algebra in $_{PS}\mathcal{M}^H$ an Azumaya-Poisson (S,H)-Hopf algebra. An Azumaya-Poisson (S,H)-Hopf algebra is just a Poisson (S,H)-Hopf algebra that is also an S-Azumaya algebra. A homomorphism of Azumaya-Poisson (S,H)-Hopf algebras is a homomorphism of Poisson (S,H)-Hopf algebras.

If P is an S-module, we set $P^* := Hom_S(P,S)$. A **Poisson (S,H)-Hopf lattice** is a Poisson (S,H)-Hopf module that is also an S-progenerator.

Lemma 3.3.   (i) *If P is a Poisson (S,H)-Hopf lattice, then $End_S(P)$ is an Azumaya-Poisson (S,H)-Hopf algebra.*
 (ii) *If P is a Poisson (S,H)-Hopf lattice, then P^* is a Poisson (S,H)-Hopf lattice.*
(iii) *If A is an Azumaya-Poisson (S,H)-Hopf algebra, then A^o is an Azumaya-Poisson (S,H)-Hopf algebra.*
(iv) *If P is a Poisson (S,H)-Hopf lattice, then $End_S(P)^o$ and $End_S(P^*)$ are isomorphic as Poisson (S,H)-Hopf algebras.*
 (v) *If A and B are Azumaya-Poisson (S,H)-Hopf algebras, then $A \otimes_S B$ is an Azumaya-Poisson (S,H)-Hopf algebra.*

Proof. (i) $End_S(P)$ is an S-Azumaya algebra since P is an S-progenerator. By Lemma 3.2(i), it is also a Poisson (S,H)-Hopf algebra. The result follows.

(ii) By Lemma 2.3, P^* is a Poisson (S,H)-Hopf module. By [6, Corollary I.3.4(f)], P^* is an S-progenerator.

(iii) By Lemma 3.1(iii), A^o is a Poisson (S,H)-Hopf algebra. By [6] A^o is an S-Azumaya algebra.

(iv) Since P is an S-progenerator, it follows from [6, Corollary I.3.4(d)] that the S-linear map $\theta : End_S(P)^o \to End_S(P^*)$ defined by

$$\theta(b)(y) = y \circ b, \text{ for all } b \in End_S(P)^o \text{ and } y \in P^*,$$

is a ring isomorphism. θ is S-linear since b and y are S-linear. By [12], θ is H-colinear and $U(S)$-linear.

(v) This is straightforward, since $A \otimes_S B$ is both a Poisson (S,H)-Hopf algebra (Lemma 3.1(i)) and an S-Azumaya algebra. $\hfill\square$

We are now ready to define our Brauer-Clifford group $B^{\mathcal{P}co}(S,H)$. We will follow the first approach of Pareigis [23] in his definition of the Brauer group in a symmetric monoidal category.

Definition 3.3. *We say that two Azumaya-Poisson (S,H)-Hopf algebras A and B are **Brauer equivalent** if there exists a pair of Poisson (S,H)-Hopf lattices P and Q such that $A \otimes_S End_S(P) \simeq B \otimes_S End_S(Q)$ as Azumaya-Poisson (S,H)-Hopf algebras.*

It is easy to see that the above relation is an equivalence relation on the set of Azumaya- Poisson (S,H)-Hopf algebras. Let $[A]$ be the equivalence class of an Azumaya-Poisson (S,H)-Hopf algebra A under this relation.

Theorem 3.1. *The set of equivalence classes of Azumaya-Poisson (S,H)-Hopf algebras under the operation $[A][B] = [A \otimes_S B]$ is an abelian group.*

Proof. It follows from Lemmas 3.1, 3.2 and 3.3 that this operation is a well-defined associative and commutative product on the collection of equivalence classes of Azumaya-Poisson (S,H)-Hopf algebras, and that $[S]$ will be the identity for this operation. When A is an Azumaya-Poisson (S,H)-Hopf algebra, it follows from Definition 3.2 that A is a Poisson (S,H)-Hopf lattice. So the fact that $A \otimes_S A^o \simeq End_S(A)$ as Azumaya-Poisson (S,H)-Hopf algebras implies that $[A^o]$ is the inverse of $[A]$ with respect to this operation. $\hfill\square$

Definition 3.4. *The abelian group of Theorem 3.1 is called the Brauer-Clifford group of the equivalence classes of Azumaya-Poisson (S,H)-Hopf algebras, and is denoted $B^{\mathcal{P}co}(S,H)$.*

Now we will consider the case where H is finite dimensional. Set $H^\star = Hom_k(H,k)$. If H is finite-dimensional, then H^* is a Hopf algebra, and by [4, Proposition 7.2.1], there is an equivalence between the categories of right H-comodules and left H^*-modules : if M is left H^*-module and if $\{h_i, h_i^*; i = 1,...,n\}$ is a dual basis of H, then the map $\rho : M \to M \otimes_k H$ given by $\rho(m) = \sum_i h_i^* m \otimes h_i$ is an H-coaction on M.

Using the same dual basis, if M is a right H-comodule, then M becomes a left H^*-module with $h_i^* m = \langle h_i^*, m_1 \rangle$, for all $m \in M$. These correspondences give an equivalence between the categories of (S,H)-Hopf modules and left $S\#H^*$-modules if H is commutative and S is a commutative H-comodule algebra [4, Proposition 8.1.2] : if M is an (S,H)-Hopf module, $(s\#h^*)m = s(h^*m)$ makes M a left $S\#H^*$-module. If M is a left $S\#H^*$-module, then $h^* m = (1\#h^*)m$, and then M is a right H-comodule, therefore an (S,H)-Hopf module. Let S be a Poisson H-comodule algebra. Then S is a left H^*-module algebra, a Poisson algebra, and we have

$$
\begin{aligned}
h^*.\{s,s'\} &= \langle h^*, \{s,s'\}_1 \rangle \{s,s'\}_0 \\
&= \langle h^*, s_1 s_1' \rangle \{s_0, s_0'\} \\
&= \langle h^*_1, s_1 \rangle \langle h^*_2, s_1' \rangle \{s_0, s_0'\} \\
&= \{ \langle h^*_1, s_1 \rangle s_0, \langle h^*_2 s_1' \rangle s_0' \} \\
&= \{ h^*_1.s, h^*_2.s' \}.
\end{aligned}
$$

It follows that S is a Poisson H^*-module algebra [14]. Then we can construct the Brauer-Clifford group $B^{\mathcal{P}}(S,H^*)$ of Azumaya-Poisson (S,H^*)-algebras [14].

Theorem 3.2. *Let H be a commutative finite-dimensional Hopf algebra, and let S be an H-comodule Poisson algebra. Then*

$$
B^{\mathcal{P}co}(S,H) = B^{\mathcal{P}}(S,H^\star).
$$

Proof. The category of Poisson (S,H^*) will be denoted $_{\mathcal{P}S,H^*}\mathcal{M}$. Let M be an object of $_{\mathcal{P}S}\mathcal{M}^H$. Then M is an (S,H)-Hopf module, a Poisson S-module, and

$$
(s \diamond m)_0 \otimes (s \diamond m)_1 = (s_0 \diamond m_0) \otimes s_1 m_1.
$$

We deduce that M is a left $S\#H^*$-module, a Poisson S-module, and we have

$$
\begin{aligned}
rclh^*(s \diamond m) &= \langle h^*, s_1 m_1 \rangle s_0 m_0 \\
&= \langle h^*_1, s_1 \rangle \langle h^*_2, m_1 \rangle s_0 m_0 \\
&= (\langle h^*_1, s_1 \rangle s_0) \diamond (\langle h^*_2, m_1 \rangle m_0) \\
&= (h^*_1.s) \diamond (h^*_2 m).
\end{aligned}
$$

Therefore, M is a Poisson (S,H^*)-module, that is, M is an object of $_{\mathcal{P}S,H^*}\mathcal{M}$. Let M be an object of $_{\mathcal{P}S,H^*}\mathcal{M}$. Then M is a left $S\#H^*$-module, a Poisson S-module, and we have

$$
h^*(s \diamond m) = (h^*_1.s) \diamond (h^*_2 m).
$$

We deduce that M is an (S,H)-Hopf module, a Poisson S-module, and we have

$$
\begin{aligned}
(s \diamond m)_0 \otimes (s \diamond m)_1 &= h_i^*(s \diamond m) \otimes h_i \\
&= (h_{i\,1}^*.s) \diamond (h_{i\,2}^* m) \otimes h_i
\end{aligned}
$$

$$= (\langle h^*_{i\,1}, s_1 \rangle s_0) \diamond (\langle h^*_{i\,2}, m_1 \rangle m_0) \otimes h_i$$
$$= (s_0 \diamond m_0) \otimes \langle h^*_{i\,1}, s_1 \rangle \langle h^*_{i\,2}, m_1 \rangle h_i$$
$$= (s_0 \diamond m_0) \otimes \langle h^*_i, s_1 m_1 \rangle h_i$$
$$= (s_0 \diamond m_0) \otimes s_1 m_1.$$

Therefore, M is a Poisson (S,H)-Hopf module, that is, M is an object of $_{\mathcal{P}S}\mathcal{M}^H$. It is clear that the morphisms of the two categories are the same. So we have an equivalence of categories between $_{\mathcal{P}S}\mathcal{M}^H$ and $_{\mathcal{P}S,H^*}\mathcal{M}$. It is routine to check that the functors involved preserve algebras, Azumaya algebras, algebra homomorphisms, and lattices in these categories, and hence preserve Brauer equivalence classes.   □

The Brauer-Clifford group $B^{co}(S,H)$ is the abelian group of the equivalence classes of (S,H)-Hopf Azumaya algebras under the operation $[A][B] = [A \otimes_S B]$, for all (S,H)-Hopf Azumaya algebras A and B, where $[A]$ denotes the equivalence class of an Azumaya (S,H)-Hopf algebra A [12, Definition 3.7].

The Brauer-Clifford group $B(S,U(S))$ is the abelian group of the equivalence classes of $(S,U(S))$-Azumaya algebras under the operation $[A][B] = [A \otimes_S B]$, for all $(S,U(S))$-Azumaya algebras A and B, where $[A]$ denotes the equivalence class of an $(S,U(S))$-Azumaya algebra A [12, Definition 3.7].

Lemma 3.4.   (i) *We have a map from $B^{\mathcal{P}co}(S,H) \to B^{co}(S,H)$, sending a class $[A]$ in $B^{\mathcal{P}co}(S,H)$ to the class of (S,H)-Hopf Azumaya algebra represented by A, viewed as an (S,H)-Hopf algebra, forgetting the Lie S-action. This map is probably not injective.*

(ii) *We have a map from $B^{\mathcal{P}co}(S,H) \to B(S,U(S))$, sending a class $[A]$ in $B^{\mathcal{P}co}(S,H)$ to the class of $(S,U(S))$-Azumaya algebra represented by A, viewed as an $(S,U(S))$-algebra, forgetting the H-coaction. This map is probably not injective.*

We now present two natural homomorphisms between Brauer-Clifford groups. The proof is obvious, and is therefore omitted.

Theorem 3.3. *Let H be a commutative Hopf algebra, and let S be an H-comodule Poisson algebra.*

(i) *(Corestriction) Let $H \hookrightarrow H'$ be a surjective Hopf algebra homomorphism. Then any Azumaya-Poisson (S,H)-Hopf algebra can be considered as an Azumaya-Poisson (S,H')-Hopf algebra. This induces a natural group homomorphism*

$$B^{\mathcal{P}co}(S,H) \to B^{\mathcal{P}co}(S,H').$$

(ii) *(Specialization of H-comodule Poisson algebras) Let T be an H-sub-comodule Poisson algebra of S (for example, T could be the Poisson center of S). If A*

is an Azumaya-Poisson (T,H)-Hopf algebra, then $A \otimes_T S$ is an Azumaya-Poisson (S,H)-Hopf algebra. This induces a group homomorphism

$$B^{\mathcal{P}co}(T,H) \to B^{\mathcal{P}co}(S,H), [A] \mapsto [A \otimes_T S].$$

3.2. **Generalizations of the Rosenberg-Zelinsky sequence**

In this subsection, $U(S)$ is the enveloping algebra of S. We will present a generalization of the Rosenberg-Zelinsky sequence for H-colinear $U(S) - linear$ S-algebra automorphisms of Azumaya-Poisson (S,H)-Hopf algebras. The approach is based on that of [15, Section 4.1] and [4, Section 13.6].

An equivalent form of the definition of an S-algebra A to be an S-Azumaya algebra is that there is a pair of inverse equivalent adjoint functors between the category of S-modules and the category of A^e-modules

$$F : N \to A \otimes_S N, G : M \to M^A,$$

where

$$M^A = \{m \in M, (a \otimes 1)m = (1 \otimes a^o)m; \quad \forall \quad a \in A\}.$$

We will see in this subsection that these same functors are an adjoint pair of functors between the symmetric monoidal category $_{\mathcal{P}S}\mathcal{M}^H$ of Poisson (S,H)-Hopf modules and the category $_{A^e,\mathcal{P}S}\mathcal{M}^H$ of A^e-modules in $_{\mathcal{P}S}\mathcal{M}^H$ for any Poisson (S,H)-Hopf algebra A, and they are a pair of inverse equivalences exactly when A is an Azumaya-Poisson (S,H)-Hopf algebra.

Let A be a Poisson (S,H)-Hopf algebra. By Lemma 3.1(iv), A^e is a Poisson (S,H)-Hopf algebra. Let us denote by $_{A^e,\mathcal{P}S}\mathcal{M}^H$ the category of left A^e-modules in the symmetric monoidal category $_{\mathcal{P}S}\mathcal{M}^H$: its morphisms are the homomorphisms of Poisson (S,H)-Hopf modules which are also A^e-linear maps, that is, the A^e-linear maps which are also Lie S-linear and H-colinear. The objects of $_{A^e,\mathcal{P}S}\mathcal{M}^H$ are in fact the (A^e,H)-Hopf modules which are also $A^e\#U(S)$-modules satisfying the relations (5) and (8). Clearly A is an object of $_{A^e,\mathcal{P}S}\mathcal{M}^H$ with the A^e-action on A being the natural one. If $N \in {}_{\mathcal{P}S}\mathcal{M}^H$, then $A \otimes_S N$ is an object of $_{A^e,\mathcal{P}S}\mathcal{M}^H$. So $N \mapsto A \otimes_S N$ defines a functor

$$F_H : {}_{\mathcal{P}S}\mathcal{M}^H \to {}_{A^e,\mathcal{P}S}\mathcal{M}^H.$$

On the other hand, if $M \in {}_{A^e,\mathcal{P}S}\mathcal{M}^H$, we know that M is an object of $_{A^e}\mathcal{M}^H$ and an object of $_{A^e\#U(S)}\mathcal{M}$ which satisfies the relations (5) and (8). Define

$$M^A := \{m \in M, (a \otimes 1)m = (1 \otimes a^o)m \quad \forall a \in A\}.$$

If we consider M with its natural structure of an (A,A)-bimodule, we have

$$M^A := \{m \in M, am = ma \quad \forall a \in A\}.$$

It is easy to see that

$$(1 \otimes S_H(x_1))\rho_M(mx_0) = m_0x \otimes m_1,$$

and

$$(1 \otimes S_H(x_1))\rho_M(x_0 m) = x m_0 \otimes m_1.$$

It follows that

$$(1 \otimes S_H(x_1))\rho_M(m x_0 - x_0 m) = (m_0 x - x m_0) \otimes m_1.$$

Now, let $m \in M^A$. Then each $m x_0 - x_0 m$ is equal to 0. We deduce that $(m_0 x - x m_0) \otimes m_1 = 0$. Now taking the summands $\{m_1\}$ to be linearly independant, we have $m_0 x - x m_0 = 0$ for each summand m_0, that is, each $m_0 \in M^A$. Thus M^A is a subcomodule of M. Clearly, M^A is an S-submodule of M. Therefore, M^A is an (S, H)-Hopf submodule of M. By [12], M^A is an $S \# U(S)$-module. Clearly, M^A satisfies the relations (5) and (8) as M does. Therefore, M^A is an object of $_{\mathcal{P}S}\mathcal{M}^H$, and we have that $M \mapsto M^A$ defines a functor

$$G_H : {}_{A^e, \mathcal{P}S}\mathcal{M}^H \to {}_{\mathcal{P}S}\mathcal{M}^H.$$

Proposition 3.1. *F_H and G_H are adjoint functors between $_{\mathcal{P}S}\mathcal{M}^H$ and $_{A^e, \mathcal{P}S,}\mathcal{M}^H$.*

Proof. Let N be a Poisson (S, H)-Hopf module and M an object of $_{A^e, \mathcal{P}S}\mathcal{M}^H$. It is well known that $F : N \to A \otimes_S N$ and $G : M \to M^A$ are adjoint functors between the category of S-modules and the category of A^e-modules. This means that we have a bijection

$$\phi : {}_{A^e}Hom(A \otimes_S N, M) \to {}_S Hom(N, M^A).$$

Recall that ϕ is defined by $\phi(f)(n) = f(1_A \otimes_S n)$ for all $f \in {}_{A^e}Hom(A \otimes_S N, M)$. Its inverse ψ is given by $\psi(g)(a \otimes_S n) = a g(n)$ for all $g \in {}_S Hom(N, M^A)$. The unit map of the adjunction is

$$\eta_N : N \to (A \otimes_S N)^A, \quad \eta_N(n) = 1 \otimes_S n \quad \text{and}$$

its counit is

$$\varepsilon_M : A \otimes_S M^A \to M, \quad \varepsilon_M(a \otimes_S m) = am.$$

Clearly, η_N is H-colinear and Lie S-linear. It is easy to show that $\phi(f)$ is H-colinear and Lie S-linear if f is H-colinear and Lie S-linear. Note that for every $m \in M$, we have $am = (a \otimes_S 1)m$. We can show that ε_M is H-colinear and Lie S-linear. Using this fact, we get that $\psi(g)$ is H-colinear and Lie S-linear if g is H-colinear and Lie S-linear. So ϕ is a homomorphism of (A^e, H)-Hopf modules and an $A^e \# U(S)$-linear map. Since $N, A \otimes_S N, M$ and M^A satisfy the relations (5) and (8), ϕ defines a bijection from $_{A^e, \mathcal{P}S}Hom^H(A \otimes_S N, M)$ to $_{\mathcal{P}S}Hom^H(N, M^A)$. We have seen that F_H and G_H are functors between $_{\mathcal{P}S}\mathcal{M}^H$ and $_{A^e, \mathcal{P}S}\mathcal{M}^H$, so the result follows. $\qquad \square$

Theorem 3.4. *Let A be a Poisson (S, H)-Hopf algebra. Then A is an Azumaya-Poisson (S, H)-Hopf algebra if and only if F_H and G_H are inverse equivalences of categories between $_{\mathcal{P}S}\mathcal{M}^H$ and $_{A^e, \mathcal{P}S}\mathcal{M}^H$.*

Proof. If A is an Azumaya-Poisson (S,H)-Hopf algebra, then A is S-Azumaya, so the corresponding adjoint functors F and G are inverse equivalences. This means that the unit maps η_N and ε_M are bijective for all S-modules N and left A^e-modules M, and therefore for all $N \in {}_{\mathcal{PS}}\mathcal{M}^H$ and $M \in {}_{A^e,\mathcal{PS}}\mathcal{M}^H$. Hence F_H and G_H are inverse equivalences.

Conversely, suppose F_H and G_H are inverse equivalences of categories. Let $M = End_S(A)$. A straightforward verification shows that the left A^e-action given by
$$(a\phi b)(x) = a\phi(bx)$$
with the diagonal action of $U(S)$ and the diagonal H-coaction makes M into an object of ${}_{A^e,\mathcal{PS}}\mathcal{M}^H$. Our assumption implies that the counit map $\varepsilon_M : A \otimes_S M^A \to M$ is an isomorphism in ${}_{A^e,\mathcal{PS}}\mathcal{M}^H$. Furthermore, $\phi \in M^A$ if and only if ϕ is left A-linear. So ϕ is of type ϕ_b, for some $b \in A$, with $\phi_b(a) = ab$. Observe that $M^A = C_{End_S(A)}(A)$ is an S-algebra. The map ϕ is an S-algebra isomorphism $\phi : A^o \to M^A$, sending b to ϕ_b. This ϕ is also H-colinear and $U(S)$-linear. We have a commutative diagram

$$
\begin{array}{ccc}
A \otimes_S M^A & \xrightarrow{\varepsilon_M} & M \\
{\scriptstyle 1 \otimes \phi} \uparrow & & \downarrow {\scriptstyle =} \\
A \otimes_S A^o & \xrightarrow{\alpha} & End_S(A)
\end{array}
$$

since
$$\varepsilon_M(a \otimes_S \phi_b)(x) = (a\phi_b)(x) = axb = \alpha(a \otimes_S b)(x).$$
Therefore, $\alpha : A^e \to End_S(A)$ is an isomorphism of Poisson (S,H)-Hopf algebras. Finally, that A is an S-progenerator follows now since the Morita context defined by the S-module A is strict. $\qquad\square$

Let A be a Poisson (S,H)-Hopf algebra. Let $(U(S),H)$-$Aut_S(A)$ be the group of all automorphisms of Poisson (S,H)-Hopf algebras of A, that is, the group of all H-colinear $U(S)$-linear S-algebra automorphisms of A. An element α of $(U(S),H)$-$Aut_S(A)$ is called $(U(S),H)$-inner if $\alpha(x) = uxu^{-1}$ for all $x \in A$, for some unit u of S. It is obvious that the set of all $(U(S),H)$-inner automorphisms of A is a subgroup of $(U(S),H)$-$Aut_S(A)$, denoted $(U(S),H)$-$Inn(A)$. An element α of $(U(S),H)$-$Aut_S(A)$ is $(U(S),H)$-INNER if there is an invertible H-*coinvariant and* $U(S)$-*invariant* element u of A such that $\alpha(x) = uxu^{-1}$ for all $x \in A$: (an element $u \in A$ is H-coinvariant if $u_0 \otimes u_1 = u \otimes 1_H$, and $U(S)$-invariant if $s \diamond u = 0$ for all $s \in S$.) The set of all $(U(S),H)$-INNER automorphisms of A is a subgroup of $(U(S),H)$-$Inn(A)$, denoted $(U(S),H)$-$INN(A)$.

For $\alpha, \beta \in (U(S),H)$-$Aut_S(A)$, we define an object ${}_\alpha A_\beta$ in ${}_{A^e,\mathcal{PS}}\mathcal{M}^H$ as follows: ${}_\alpha A_\beta$ is equal to A as a Poisson (S,H)-Hopf module, but with left A^e-action $(a \otimes b^o)x = \alpha(a)x\beta(b)$ for all $a,b,x \in A$.

Lemma 3.5. *If* $\alpha, \beta, \gamma \in (U(S),H)$-$Aut_S(A)$, *then*
  (i) ${}_\alpha A_\beta \simeq {}_{\gamma\alpha} A_{\gamma\beta}$ *in* ${}_{A^e,\mathcal{PS}}\mathcal{M}^H$;
  (ii) ${}_1 A_\alpha \otimes_A {}_1 A_\beta \simeq {}_1 A_{\alpha\beta}$ *in* ${}_{A^e,\mathcal{PS}}\mathcal{M}^H$;

(iii) $_1A_\alpha \simeq {}_1A_1$ *as A^e-modules if and only if α is in $(U(S),H)$-Inn(A); and*

(iv) $_1A_\alpha \simeq {}_1A_1$ *in $_{A^e,\mathcal{P}S}\mathcal{M}^H$ if and only if α is in $(U(S),H)$-INN(A).*

Proof. We can adapt the proof of [12, Lemma 3.3].   □

The Picard group $Pic(S)$ of the commutative algebra S is the group of isomorphism classes of invertible S-modules; i.e. those S-modules M for which there exists an S-module N with $M \otimes_S N \simeq S$. The isomorphism class in $Pic(S)$ represented by an invertible S-module M will be denoted by $[M]$. A Poisson (S,H)-Hopf module M is invertible if there exists a Poisson (S,H)-Hopf module N with $M \otimes_S N \simeq S$ as Poisson (S,H)-Hopf modules. The Picard group $Pic^{\mathcal{P},H}(S,H)$ is the group of isomorphism classes of invertible Poisson (S,H)-Hopf modules. The isomorphism class in $Pic^{\mathcal{P},H}(S,H)$ represented by an object $M \in {}_{\mathcal{P}S}\mathcal{M}^H$ will be denoted by $\{M\}$.

Theorem 3.5. *Let A be an Azumaya-Poisson (S,H)-Hopf algebra. Then there are exact sequences of groups*

$$1 \to (U(S),H)\text{-}Inn(A) \to (U(S),H) - \text{Aut}_S(A) \xrightarrow{\Psi} Pic(S)$$

and

$$1 \to (U(S),H)\text{-}INN(A) \to (U(S),H) - \text{Aut}_S(A) \xrightarrow{\Phi} Pic^{\mathcal{P},H}(S,H).$$

The homomorphisms Ψ and Φ are respectively defined by $\Psi(\alpha) = [I_\alpha]$ and $\Phi(\alpha) = \{I_\alpha\}$, for every α in $(U(S),H)$-Aut$_S(A)$, where $I_\alpha = ({}_1A_\alpha)^A$.

Proof. Ψ is simply the restriction of the map

$$\Psi : \text{Aut}_S(A) \to Pic(S)$$

used in the original Rosenberg-Zelinsky exact sequence

$$1 \to Inn(A) \to \text{Aut}_S(A) \xrightarrow{\Psi} Pic(S)$$

to the subgroup $(U(S),H)$-Aut$_S(A)$. So exactness of the first sequence is immediate. For the other sequence, the fact that Φ is a group homomorphism follows from Lemma 3.5(ii) and the fact that G_H is a category equivalence. Furthermore, if α is in $(U(S),H)$-Aut$_S(A)$, we will have $I_\alpha \simeq S$ if and only if $_1A_\alpha \simeq {}_1A_1$. Therefore, $\text{Ker}\Phi = H$-INN(A). Exactness of the second sequence follows.   □

3.3. **Central Twists**

In this subsection, $U(S)$ is the enveloping algebra of the Lie algebra S. Let $(U(S),H)$-Aut$_k(S)$ be the group of H-colinear $U(S)$-linear $\Bbbk$-algebra automorphisms of S. We will show that there is an action of $(U(S),H)$-Aut$_k(S)$ on the Brauer group $B^{\mathcal{P}co}(S,H)$.

For $M \in {}_{\mathcal{P}S,H}\mathcal{M}$ and $\tau \in (U(S),H)$-Aut$_k(S)$, let $_\tau M$ be equal to M as a right H-comodule and $U(S)$-module but S-module structure given by $s*m = \tau^{-1}(s)m$ for all $s \in S$, $m \in M$. By [13], $_\tau M$ is a Poisson S-module. By [12], $_\tau M$ is an object of $_S\mathcal{M}^H$. $_\tau M$ is equal to M as a Lie S-module and as an H-comodule, and the relation (8) is satisfied for M. Then $_\tau M \in {}_{\mathcal{P}S}\mathcal{M}^H$.

Lemma 3.6. *Let $M, N, P \in {}_{\mathcal{P}S}\mathcal{M}^H$ with M finitely generated as an S-module. If τ is in $(U(S), H)$-$\mathrm{Aut}_k(S)$, then*

(i) *${}_\tau Hom_S(M, N)$ and $Hom_S({}_\tau M, {}_\tau N)$ are isomorphic in ${}_{\mathcal{P}S}\mathcal{M}^H$;*
(ii) *${}_\tau End_S(M)$ and $End_S({}_\tau M)$ are isomorphic as algebras in ${}_{\mathcal{P}S}\mathcal{M}^H$;*
(iii) *P is an S-progenerator if and only if ${}_\tau P$ is an S-progenerator.*

Proof. (i) ${}_\tau Hom_S(M, N)$ and $Hom_S({}_\tau M, {}_\tau N)$ are objects of ${}_{\mathcal{P}S}\mathcal{M}^H$. By [13], they are isomorphic as Poisson S-modules. By [12], they are isomorphic as (S, H)-Hopf modules, the result follows.

(ii) By (i), ${}_\tau End_S(M)$ and $End_S({}_\tau M)$ are isomorphic in ${}_{\mathcal{P}S}\mathcal{M}^H$. By [13], they are isomorphic as algebras in the category of Poisson S-modules. By [12], they are isomorphic as algebras in the category of (S, H)-Hopf modules.

(iii) Since this statement only depends on the S-module structure of P, the result follows from [5, Lemma 1(c)]. $\qquad\square$

Definition 3.5. *Let A be an algebra in ${}_{\mathcal{P}S}\mathcal{M}^H$. For any $\tau \in (U(S), H)$-$\mathrm{Aut}_k(S)$, we define ${}_\tau A$ to be equal to A as an H-comodule algebra and $U(S)$-module algebra, but equal to ${}_\tau A$ as an S-module.*

Lemma 3.7. *Let $\tau \in H$-$\mathrm{Aut}_k(S)$.*

(i) *If A is an algebra in ${}_{\mathcal{P}S}\mathcal{M}^H$, then ${}_\tau A$ is an algebra in ${}_{\mathcal{P}S}\mathcal{M}^H$.*
(ii) *If A is an Azumaya algebra in ${}_{\mathcal{P}S}\mathcal{M}^H$, then ${}_\tau A$ is an Azumaya algebra in ${}_{\mathcal{P}S}\mathcal{M}^H$.*
(iii) *If A and B are algebras in ${}_{\mathcal{P}S}\mathcal{M}^H$, then ${}_\tau(A \otimes_S B)$ is an algebra in ${}_{\mathcal{P}S}\mathcal{M}^H$ and ${}_\tau(A \otimes_S B) \simeq {}_\tau A \otimes_S {}_\tau B$ as algebras in ${}_{\mathcal{P}S}\mathcal{M}^H$.*

Proof. (i) ${}_\tau A$ is an object of ${}_{\mathcal{P}S}\mathcal{M}^H$. By [12], ${}_\tau A$ is an algebra in ${}_S\mathcal{M}^H$ and an algebra in ${}_{S\#U(S)}\mathcal{M}$.

(ii). We can see from the definition of the action of S on ${}_\tau A$ that if A is S-Azumaya, then ${}_\tau A$ is S-Azumaya [5, Lemma 1(d)]. The result follows from (i).

(iii). ${}_\tau(A \otimes_S B)$ is an object of ${}_{\mathcal{P}S}\mathcal{M}^H$. By [12], ${}_\tau(A \otimes_S B)$ is an (S, H)-Hopf algebra and an $(S, U(S))$-algebra. Hence ${}_\tau(A \otimes_S B)$ is a Poisson (S, H)-Hopf algebra. By (i), ${}_\tau A$ and ${}_\tau B$ are Poisson (S, H)-Hopf algebras. So by Lemma 3.1(i), ${}_\tau A \otimes_S {}_\tau B$ is a Poisson (S, H)-Hopf algebra. By [12], ${}_\tau(A \otimes_S B)$ and ${}_\tau A \otimes_S {}_\tau B$ are isomorphic as (S, H)-Hopf algebras and as $(S, U(S))$-algebras. $\qquad\square$

Proposition 3.2. *$(U(S), H)$-$\mathrm{Aut}_k(S)$ acts by automorphisms on $B^{\mathcal{P}co}(S, H)$: the action is given by $\tau.[A] = [{}_\tau A]$, for any Azumaya algebra A in ${}_{\mathcal{P}S}\mathcal{M}^H$ and $\tau \in (U(S), H)$-$\mathrm{Aut}_k(S)$.*

Proof. Let A and B be two equivalent Azumaya algebras in ${}_{\mathcal{P}S}\mathcal{M}^H$. Then there exist lattices M and N in ${}_{\mathcal{P}S}\mathcal{M}^H$ such that

$$A \otimes_S End_S(M) \simeq B \otimes_S End_S(N)$$

as algebras in $_{\mathcal{P}S}\mathcal{M}^H$. By Lemma 3.6 and Lemma 3.7 (iii), we have the following isomorphisms of algebras in $_{\mathcal{P}S}\mathcal{M}^H$:

$$
\begin{aligned}
_\tau A \otimes_S End_S(_\tau M) &\simeq {}_\tau A \otimes_{S\tau} End_S(M) \\
&\simeq {}_\tau(A \otimes_S End_S(M)) \\
&\simeq {}_\tau(B \otimes_S End_S(N)) \\
&\simeq {}_\tau B \otimes_{S\tau} End_S(N) \\
&\simeq {}_\tau B \otimes_S End_S(_\tau N).
\end{aligned}
$$

Therefore, $_\tau A$ and $_\tau B$ are Brauer equivalent Azumaya algebras in $_{\mathcal{P}S}\mathcal{M}^H$.

Let A and B be Azumaya algebras in $_{\mathcal{P}S}\mathcal{M}^H$. Then by Lemma 3.6 and Lemma 3.7 (iii), $_\tau(A \otimes_S B) \simeq {}_\tau A \otimes_S {}_\tau B$ as algebras in $_{\mathcal{P}S}\mathcal{M}^H$. So we have $\tau.([A][B]) = (\tau.[A])(\tau.[B])$.

Let A be an Azumaya algebra in $_{\mathcal{P}S}\mathcal{M}^H$. If $\tau, \tau' \in (U(S), H)$-$\mathrm{Aut}_k(S)$, then by [12, 6.4], $_\tau(_{\tau'}A)$ and $_{(\tau\tau')}A$ are isomorphic as (S, H)-Hopf Azumaya algebras and as $(S, U(S))$-Azumaya algebras. Since $_\tau(_{\tau'}A)$ and $_{(\tau\tau')}A$ are objects of $_{\mathcal{P}S}\mathcal{M}^H$, they are isomorphic as Azumaya-Poisson (S, H)-Hopf algebras. This means that $\tau.(\tau'.A) = (\tau\tau').A$. It follows that the group $(U(S), H)$-$\mathrm{Aut}_k(S)$ acts on the group of these classes, as required. $\square$

4. SOME EXAMPLES OF BRAUER-CLIFFORD GROUPS

Example 4.1. *When $\Bbbk$ is considered as a trivial Hopf algebra, $B^{\mathcal{P}co}(S, k)$ is just the Brauer group $B^{\mathcal{P}}(S)$ of Azumaya-Poisson S-algebras studied in [13].*

Example 4.2. *Let H be a commutative Hopf algebra. We know that $\Bbbk$ is an H-comodule Poisson algebra when $\Bbbk$ is considered as a Poisson algebra with a trivial bracket and a trivial H-coaction. So we get the Brauer-Clifford group $B^{\mathcal{P}co}(k, H)$. This Brauer-Clifford group is exactly the Brauer group of Azumaya H-comodule algebras $BC(k, H)$ studied in [16].*

Example 4.3. *Let G be an affine algebraic group. By [19], a $\Bbbk$-vector space M is a rational G-module if it is a G-module, and for every $m \in M$, the translates of m span a finite dimensional subspace N of M and the induced map $G \to \mathrm{Aut}_\Bbbk(N)$ is a morphism of algebraic groups. A rational G-module algebra is an algebra S which is a rational G-module such that*

$$g.(ss') = (g.s)(g.s'), \quad g.1_S = 1_S \quad \forall s, s' \in S, g \in G.$$

If S is a rational G-module algebra, a $\Bbbk$-vector space M is a rational (S, G)-module if it is an S-module, a rational G-module, and

$$g(sm) = (g.s)(gm), \quad \forall s \in S, m \in M, g \in G.$$

Let $\Bbbk[G]$ be the affine coordinate ring of G (it is a commutative Hopf algebra). We recall that rational G-modules are $\Bbbk[G]$-comodules with (right) coaction $\rho : M \to M \otimes \Bbbk[G]$ characterized by the condition

$$\rho(m) = m_0 \otimes m_1 \in M \otimes \Bbbk[G] \Leftrightarrow g.m = m_0 m_1(g), \quad \forall g \in G.$$

Let S be a rational G-module Poisson algebra, that is, S is a Poisson algebra which is also a commutative rational G-module algebra such that

$$g.\{s,s'\} = \{g.s, g.s'\}, \quad \forall s,s' \in S, g \in G.$$

Then S is a right $\Bbbk[G]$-comodule algebra, and

$$\{s,s'\}_0 \otimes \{s,s'\}_1 = \{s_0, s'_0\} \otimes s_1 s'_1, \quad s,s' \in S;$$

that is, S is a right $\Bbbk[G]$-comodule Poisson algebra. Thus we have the Brauer-Clifford group $B^{\mathcal{P}co}(S, \Bbbk[G])$ of Azumaya-Poisson $(S, \Bbbk[G])$-Hopf algebras.

The category of rational $(S - G)$-modules and the category of $(S, \Bbbk[G])$ Hopf modules are equivalent. We refer to [18], [19], [7] and [8] for further information on rational actions of an algebraic group.

A rational Poisson (S,G)-module will be a rational left (S,G)-module M which is also a Poisson S-module such that

$$g(s \diamond m) = (g.s) \diamond (gm), \quad s \in S, g \in G, m \in M.$$

It follows that S is a rational Poisson (S,G)-module. We denote by $_{\mathcal{P}S,G}\mathcal{M}$ the category of rational Poisson (S,G)-modules: its morphisms are the Poisson S-linear maps which are also G-linear. It is easy to see that $(_{\mathcal{P}S,G}\mathcal{M}, \otimes_S, S)$ is a symmetric monoidal category. In this category, we can define algebras, lattices, Azumaya algebras and Brauer equivalent Azumaya algebras : we will call them respectively, rational Poisson (S,G)-algebras, rational Poisson (S,G)-lattices, rational Azumaya-Poisson (S,G)-algebras, and Brauer equivalent rational Azumaya-Poisson (S,G)-algebras. Then we can define the Brauer-Clifford group $B^{\mathcal{P}}(S,G)$ of rational Azumaya-Poisson (S,G)-algebras.

We can show that the category of rational Poisson $(S - G)$-modules and the category of Poisson $(S, k[G])$ Hopf modules are equivalent. It follows that the Brauer-Clifford groups $B^{\mathcal{P}co}(S, \Bbbk[G])$ and $B^{\mathcal{P}}(S,G)$ are isomorphic.

Example 4.4. *Let V be an affine Poisson variety over $\mathbb{C}$. Then the coordinate ring $\mathbb{C}[V]$ of V is a Poisson algebra. Let G be an algebraic group acting morphically on V via automorphisms of Poisson varieties. There is an induced action of G on $\mathbb{C}[V]$ by Poisson automorphisms. If $\mathbb{C}[G]$ is the coordinate ring of G, we have a $\mathbb{C}[G]$-coaction on $\mathbb{C}[V]$. So we get the Brauer-Clifford group $B^{\mathcal{P}co}(\mathbb{C}[V], \mathbb{C}[G])$.*

Example 4.5. *Let S be an H-comodule Poisson algebra, and $Z_{\mathcal{P}}(S)$ the Poisson-center of S. By Lemma 1.3, we get the Brauer-Clifford group $B^{\mathcal{P}co}(Z_{\mathcal{P}}(S), H)$.*

Example 4.6. *Let K and L be two commutative Hopf $\Bbbk$-algebras. Then $H = K \times L$ is a commutative Hopf algebra over $\Bbbk \times \Bbbk$ (with component-wise operations). Let M be a vector space over $\Bbbk$. Then M can be seen as a left $\Bbbk \times \Bbbk$-module via the first projection $\Bbbk \times \Bbbk \longrightarrow \Bbbk; (\lambda, \lambda') \mapsto \lambda$. If M is a $\Bbbk$-comodule (resp. an L-comodule, an H-comodule), we denote by $m \mapsto m_0 \otimes m_1$ (resp. $m \mapsto m_{(0)} \otimes m_{(1)}$,*

$m \mapsto m_{[0]} \otimes m_{[1]}$) *its comodule structure. Let S be a $\Bbbk$-comodule Poisson algebra and T be an L-comodule Poisson algebra. Then $S \times T$ is an H-comodule Poisson algebra :* $(\lambda,\lambda')(s,t) = (\lambda s,\lambda' t)$, $(k,l).(s,t) = (k.s,l.t)$ *and* $(s,t)_{[0]} \otimes_R (s,t)_{[1]} = (s_0,t_{(0)}) \otimes_R (s_1,t_{(1)})$, $([s,t],[s',t']) = ([s,s'],tt') + (ss',[t,t'])$ *for* $\lambda,\lambda' \in k$, $k \in K, l \in L, s, s' \in S$ *and* $t, t' \in T$. *Let S be a $\Bbbk$-comodule Poisson algebra and T an L-comodule Poisson algebra. Thus we can consider the categories ${}_{\mathcal{P}S}\mathcal{M}^K$, ${}_{\mathcal{P}T}\mathcal{M}^L$, ${}_{\mathcal{P}(S \times T)}\mathcal{M}^H$.*

We have three Brauer groups $B^{\mathcal{P}co}(S,K)$, $B^{\mathcal{P}co}(T,L)$ and $B^{\mathcal{P}co}(S \times T,H)$. We want to establish some relations between these Brauer groups. Every object M of the category ${}_{\mathcal{P}S}\mathcal{M}^K$ is an object of ${}_{\mathcal{P}(S \times T)}\mathcal{M}^H$: $(s,t)m = sm$, $(k,l).m = km$, $[s,t] \diamond m = s \diamond m$, *and* $m_{[0]} \otimes m_{[1]} = m_0 \otimes (m_1,0_L)$ *for $s \in S$, $t \in T$ and $m \in M$. Every object M of the category ${}_{\mathcal{P}T}\mathcal{M}^L$ is an object of ${}_{\mathcal{P}(S \times T)}\mathcal{M}^H$:* $(s,t)m = tm$, $(k,l).m = lm$, $[s,t] \diamond m = t \diamond m$ *and* $m_{[0]} \otimes m_{[1]} = m_{(0)} \otimes (0_K,m_{(1)})$.

For every M in ${}_{\mathcal{P}S}\mathcal{M}^K$, we have $End_S(M) = End_{S \times T}(M)$. For every M in ${}_{\mathcal{P}T}\mathcal{M}^L$, we have $End_T(M) = End_{S \times T}(M)$.

Every S-progenerator object in ${}_{\mathcal{P}S}\mathcal{M}^K$ is an $S \times T$-progenerator object in ${}_{\mathcal{P}(S \times T)}\mathcal{M}^H$. Likewise, every T-progenerator object in ${}_{\mathcal{P}T}\mathcal{M}^L$ is an $S \times T$-progenerator object in ${}_{\mathcal{P}(S \times T)}\mathcal{M}^H$. Let A be a Poisson (S,K)-Hopf algebra (resp. a Poisson (T,L)-Hopf algebra). Then A is a Poisson $(S \times T,H)$-Hopf algebra. If A^o is the opposite algebra of A as a Poisson (S,K)-Hopf algebra, then A^o is the opposite algebra of A as a Poisson $(S \times T,H)$-Hopf algebra. Likewise, if A^o is the opposite algebra of A as a Poisson (T,L)-Hopf algebra, then A^o is the opposite algebra of A as a Poisson $(S \times T,H)$-Hopf algebra. Azumaya-Poisson (S,K)-Hopf algebras and Azumaya-Poisson (T,L)-Hopf algebras are Azumaya-Poisson $(S \times T,H)$-Hopf algebras.

We have injective group homomorphisms $B^{\mathcal{P}co}(S,K) \hookrightarrow B^{\mathcal{P}co}(S \times T,H)$ and $B^{\mathcal{P}co}(T,L) \hookrightarrow B^{\mathcal{P}co}(S \times T,H)$.

If $T = \Bbbk$ as a trivial L-comodule Poisson algebra, we deduce an injective group homomorphism $B^{\mathcal{P}co}(S,K) \hookrightarrow B^{\mathcal{P}co}(S \times T,H)$.

If $S = \Bbbk$ as a trivial $\Bbbk$-comodule Poisson algebra, we deduce an injective group homomorphism $B^{\mathcal{P}co}(T,L) \hookrightarrow B^{\mathcal{P}co}(S \times T,H)$.

Let A be an (S,K)-comodule Poisson algebra and B a (T,L)-comodule Poisson algebra or let A be a (T,L)-comodule Poisson algebra and B an (S,K)-comodule Poisson algebra. Then the classes of A and B in $B^{\mathcal{P}co}(S \times T,H)$ commute. Furthermore if A and B are Azumaya, then $A \otimes_{S \times T} B$ is an Azumaya-Poisson $(S \times T,H)$-Hopf algebra. We have a well-defined injective group homomorphism

$$B^{\mathcal{P}co}(S,K) \times B^{\mathcal{P}co}(T,L) \to B^{\mathcal{P}co}(S \times T,H); ([A],[B]) \to [A \otimes_{S \times T} B].$$

We can show that the intersection of $B^{\mathcal{P}co}(S,K)$ and $B^{\mathcal{P}co}(T,L)$ in $B^{\mathcal{P}co}(S \times T,H)$ is trivial.

We can apply this example to the following situation : Let G and G' be two abelian finite groups and $\Bbbk$ a field. Set $H = Maps(G \uplus G', \Bbbk)$ the set of all maps

from the disjoint union $G \uplus G'$ of G and G' to $\Bbbk$. Then $H = K \times L$, where $K = Maps(G, \Bbbk)$ and $L = Maps(G', \Bbbk)$. We claim that K and L are commutative (co-commutative) Hopf algebras over $\Bbbk$ and that $H = K \times L$ is a commutative (cocom-mutative) Hopf algebra over $\Bbbk \times \Bbbk$ (with component-wise operations).

Example 4.7. *We refer to [21] for further information on localization of Poisson algebras. Let S be an H-comodule Poisson algebra, T a multiplicative subset of S stable under the H-coaction. Then the localization $T^{-1}S$ of S with respect to T is an H-comodule Poisson algebra:*

$$\{st^{-1}, s't'^{-1}\} = t^{-2}t'^{-2}\big(tt'\{s\ s'\} - ts'\{s\ t'\} - st'\{t\ s'\} + ss'\{t\ t'\}\big),$$

the H-coaction is defined by $(st^{-1})_0 \otimes (st^{-1})_1 = (s_0 t_0^{-1}) \otimes (s_1 t_1)$ for all $t, t' \in T$ and $s, s' \in S, m \in M$.

If M is a Poisson (S,H)-Hopf module, then the localization $T^{-1}M$ of M with respect to T is a Poisson $(T^{-1}S, H)$-Hopf module: the Lie $T^{-1}S$-action is given by

$$(st^{-1}) \diamond (mt'^{-1}) = t^{-2}t'^{-2}\big(tt'(s \diamond m) - tm\{s\ t'\} - (st')(t \diamond s') + sm\{t\ t'\}\big),$$

the coaction is defined by $(mt^{-1})_0 \otimes (mt^{-1})_1 = (m_0 t_0^{-1}) \otimes (m_1 t_1)$, for all $t, t' \in T$, $s \in S, m \in M$. Let A be an Azumaya-Poisson (S,H)-Hopf algebra. Since A is an S-progenerator, it is finitely generated projective as an S-module. So it is finitely presented as an S-module. Using [14, Remark 2.2], we can show that $T^{-1}A$ is a Poisson $(T^{-1}S, H)$-Hopf algebra. It is well known that if A is an S-progenerator as an S-module, then $T^{-1}A$ is a $T^{-1}S$-progenerator as a $T^{-1}S$-module. It follows that $T^{-1}A$ is an Azumaya-Poisson $(T^{-1}S, H)$-Hopf algebra. Furthermore, since A^o is an Azumaya-Poisson (S,H)-Hopf algebra, $T^{-1}A^o$ is an Azumaya-Poisson $(T^{-1}S, H)$-Hopf algebra. Clearly, we have $T^{-1}A^o = (T^{-1}A)^o$. Thus we have a well-defined group homomorphism

$$B^{\mathcal{P}co}(S,H) \to B^{\mathcal{P}co}(T^{-1}S, H), [A] \mapsto [T^{-1}A].$$

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