SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.19.01.06
Vol.19 (32), No.1 (2023), 79–88

BLOW UP AND GROWTH OF SOLUTIONS FOR A KIRCHHOFF-TYPE PLATE EQUATION WITH DEGENERATE DAMPING

FATMA EKINCI AND ERHAN PIŞKIN

ABSTRACT. We investigate a Kirchhoff-type plate equation with a degenerate damping term. We prove blow up and exponential growth of the solution under some sufficient conditons on the initial data.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we focus on the blow up and exponential growth of solutions under sufficent conditions for the following problem

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u-\left(\int\limits_\Omega |\nabla u|^2\,dx\right)^\gamma \Delta u+|u|^\rho\, j'(u_t)=|u|^{q-1}u & \text{in } \Omega\times(0,+\infty), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) & \text{on } x\in\Omega, \\ u(x,t)=\frac{\partial}{\partial n}u(x,t)=0 & \text{on } x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $j(s)$ is a continuous, convex real-valued function defined on R and j' denotes the derivative of $j(\alpha)$ [1], $1<q<\infty$, $\rho\geq 0$, n is the outer normal and Ω is an open bounded domain in R^n with a smooth boundary $\partial\Omega$. Also, here

$$\Delta u-\left(\int\limits_\Omega |\nabla u|^2\,dx\right)^\gamma \Delta u \quad \text{and} \quad |u|^\rho\, j'(u_t),$$

represent the Kirchhoff-type term and the degenerate damping term, respectively.

1.1. **Kirchhoff-type plate problems**

To motivation for this problem comes from the following equation the so called Beam equation model

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2010 *Mathematics Subject Classification.* 35B40, 35B44, 35G61, 35L75.

Key words and phrases. Blow up, Exponential growth, Kirchhoff-type equation, Degenerate damping.

$$u_{tt}+\Delta^2 u-\left(\alpha+\beta\int_0^L |\nabla u|^2\, d\tau\right)\Delta u=|u|^{q-2}u, \qquad (1.2)$$

without source term $\left(|u|^{q-2}u\right)$ which was firstly introduced by Woinowsky-Krieger [2] to describe the dynamic bucking of a hinged extensible beam under an axial force. It was extensively studied by several researchers in different contexs. In [3, 4] the authors showed the global attractor, convergence and unboundedness of solutions with $|u_t|^{p-2}u_t$ nonlinear damping term. Then, the model also was investigated in [5, 6] and the existence, decay estimates of solutions and blow up of solutions with both negative and positive initial energy with $|u_t|^{p-2}u_t$ nonlinear damping term was obtained. Also, in absence of Kirchhoff-type term and with Δu_t, the problem (1.2) has been considerd by Pişkin and Polat [7] and decay estimates of the solution were proved by using the Nakaos inequality.

Recently, Periara et al. [8] and Pişkin and Yüksekkaya [9] studied this model with u_t. Periara et al. studied existence of the global solutions through the Faedo-Galerkin approximations and obtained the asymptotic behavior by using the Nakao method. Pişkin and Yüksekkaya proved the blow up of solutions with positive and negative initial energy.

1.2. **Problems with degenerate damping**

Hyperbolic models with degenerate damping are of much interest in material science and physics. They particularly appear in physics when the friction is modulated by the strains. There is wide literature about degenerate damping terms, namely $\delta(u)h(u_t)$ where $\delta(u)$ is a positive function and h is nonlinear.

This kind of degenerate damping effects were first investigated by Levine and Serrin [10] who considered the following equation

$$\left(|u_t|^{l-2}u_t\right)_t-\alpha\nabla\left(|\nabla u|^{\gamma-2}\nabla u\right)+\beta|u|^{\rho}|u_t|^{p-2}u_t=\lambda|u|^{q-2}u,$$

here $\alpha,\beta\geq 0$, $\lambda>0$. The authors studied blow up of solutions with negative initial energy. But Levine and Serrin obtain only a blow up solution with negative initial energy without any guarantees that the solution has a local solution.

In the absence of a Kirchhoff-type term and $\Delta^2 u$, the problem (1.1) takes the form

$$u_{tt}-\Delta u+|u|^{\rho}j'(u_t)=|u|^{q-1}u. \qquad (1.3)$$

The problem (1.3) has been widely investigated. For example, Barbu, Lasiecka and Rammaha [1, 11] studied the existence and uniqueness of varied kinds of solutions like weak solutions, generalized solutions and strong solutions. In addition, when initial energy is negative the blow up result of generalized and weak solution was proved in [11, 12].

For the case $j'(u_t) = |u_t|^p \, sgn(u_t)$ for the problem (1.3), Pitts and Rammaha [13] proved global and local existence for $\rho + p \geq q$ and for the case $\rho < 1$ established uniqueness. Also, the authors obtained blow up solutions with negative initial energy for $\rho + p < q$.

Rammaha and Strei [14] studied global existence and nonexistence of solutions with negative initial datum in case $j'(u_t) = u_t$ for problem (1.3). Then, Hu and Zhang [15] considered problem (1.3) and proved asymptotically stability and blow up with positive initial energy. Moreover, [16] investigated problem (1.3) and proved blow up results for arbitrary positive initial energy.

Recently, Ekinci and Pişkin [17–19] investigated the problem (1.1) and first showed nonexistence of weak solutions with arbitrary positive initial datum by constructing a energy perturbation function. Then, in [18], they showed the asymptotic stability of energy in the presence of a degenerate damping by potential well theory. Also, in [19], they studied blow up and exponential growth of solutions with negative initial energy for problem (1.1) with viscoelastic term. Furthermore, recently, blowing up and exponential growth of solutions for evolution systems with time degenerate damping have been studied by many authors [20–23].

For the last several decades, the mathematical analysis of Kirchhoff-type plate equations has attracted a lot of attention. Also, hyperbolic type equations with degenerate damping terms have attracted a lot of attention. Thus, we decided to combine Kirchhoff-type plate and degenerate damping in a equation.

Motivated by previous results, we prove several results concerning the blow up and exponential growth of solutions with positive initial datum for the problem (1.1). It should be noted here that we can say that the study is both quite difficult and interesting and the analysis more subtle because of the degenerate damping.

The remaining part of this paper is organized as follows: In the next section, we study the blow up of solutions. The exponential growth result is presented in Section 3.

Now, we present some preliminary material which will be helpful in the proof of our results. Throughout this paper, $W^{m,p}$ is denoted the Sobolev space and

$$\begin{cases} W^{0,p}(\Omega) = L^p(\Omega) \text{ if } m = 0, \\ W^{m,2}(\Omega) = H^m(\Omega) \text{ if } p = 2. \end{cases}$$

Also, we denote the standard $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm by $\|.\|_p = \|.\|_{L^p(\Omega)}$ for details see ([24,25]).

(A1) $\rho, p \geq 0$, $q > 1$; $\rho \leq \frac{n}{n-2}$, $q + 1 \leq \frac{2n}{n-2}$ if $n \geq 3$. There exist positive constants c, c_0, c_1 such that for all $\alpha, \beta \in R$, $j(\alpha) : R \to R$ is a C^1 convex real function satisfying

- $j(\alpha) \geq c|\alpha|^{p+1}$,
- $j'(\alpha)$ is single valued and $|j'(\alpha)| \leq c_0 |\alpha|^p$,
- $(j'(\alpha) - j'(\beta))(\alpha - \beta) \geq c_1 |\alpha - \beta|^{p+1}$.

(A2) $u_0(x) \in H_0^2(\Omega)$, $u_1(x) \in L^2(\Omega)$.

The mentioned solution of (1.1) satisfies the energy identity

$$E(t) + \int_0^t \int_\Omega |u(\tau)|^\rho j(u_t)(\tau)\, dx d\tau = E(0), \tag{1.4}$$

where

$$E(t) = \frac{1}{2}\left[\|u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right] - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \tag{1.5}$$

and

$$E(0) = \frac{1}{2}\left[\|u_1\|^2 + \|\Delta u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right] - \frac{1}{q+1}\|u_0\|_{q+1}^{q+1}. \tag{1.6}$$

Moreover, by computation, we get $E(t)$ is a non-increasing function, and then

$$E(t) \le E(0). \tag{1.7}$$

Now, we define

$$\alpha_1 = \lambda_1^{-\frac{2}{q-1}}, \quad E_1 = \left(\frac{1}{2(\gamma+1)} - \frac{1}{q+1}\right)\alpha_1^{q+1},$$

$$\alpha_2 = \left(\frac{1}{(q+1)\lambda_1^2}\right)^{\frac{1}{q-1}}, \quad E_2 = \frac{q+1}{2}\left(\frac{1}{2} - \frac{1}{q+1}\right)\alpha_2^{q+1},$$

$$W_0 = \left\{(\alpha, E) \in R^2, 0 \le \alpha < \alpha_2, 0 < E < E_2\right\},$$

$$V = \left\{(\alpha, E) \in R^2, \alpha > \alpha_1, 0 < E < E_1\right\}$$

where λ_1 is the embedding constant (where $H_0^2(\Omega)$ is embedded into $L^{q+1}(\Omega)$). We name W_0 the stable set, V the unstable set.

2. Blow up

In this section, we deal with the blow-up properties for the solutions of (1.1).

Lemma 2.1. *[26]. If $\left(\|u_0\|_{q+1}, E(0)\right) \in V$, then the solution of (1.1), satisfies $E(t) \le E(0)$ for all $t \in [0,T]$. Also, there exists $\alpha_0 > \alpha_1$ such that $\|u(t)\|_{q+1} \ge \alpha_0 > \alpha_1$ for all $t \in [0,T]$.*

Theorem 2.1. *Assume that (A1) and (A2) hold. Let $\left(\|u_0\|_{q+1}, E(0)\right) \in V$, $q > \max\{2\gamma+1, \rho+p\}$ and u be a local solution to (1.1) on the interval $[0,T]$. Then T is necessarily finite, i.e. u can't be continued for all $t > 0$.*

Proof. We assume that the solution exists for all time and we arrive to a contradiction. Set

$$G(t) = \|u(t)\|^2, \qquad H(t) = E_1 - E(t). \tag{2.1}$$

From (1.4), we have

$$H'(t) = -E'(t) = \int_\Omega |u(t)|^\rho \, j(u_t)(t)\, dx \geq 0. \tag{2.2}$$

Hence, $H(t)$ is an increasing function, we have

$$H(t) \geq H(0) = E_1 - E(0) > 0, \ \ t \geq 0. \tag{2.3}$$

Then, by using Lemma 1 and the definition of $E(t)$,

$$\begin{aligned} H(t) &\leq E_1 - \frac{1}{2}\left[\|\Delta u\|^2 + \|\nabla u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right] + \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\ &\leq E_1 - \frac{1}{2(\gamma+1)}\lambda_1^{-2}\alpha_1^2 + \frac{1}{q+1}\|u\|_{q+1}^{q+1}, \quad t \geq 0. \end{aligned} \tag{2.4}$$

Hence, since $E_1 - \frac{1}{2(\gamma+1)}\lambda_1^{-2}\alpha_1^2 = -\frac{1}{q+1}\alpha_1^{q+1} < 0$, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{q+1}\|u\|_{q+1}^{q+1}, \quad t \geq 0. \tag{2.5}$$

For simplicity, we denote

$$\chi(t) = \int_\Omega |u(t)|^\rho \, u(t)\, j'(u_t)(t)\, dx.$$

By Eq.(1.1) and the definition of $H(t)$,

$$\begin{aligned} \frac{1}{2}G''(t) &= \frac{d}{dt}\int_\Omega u(t)\, u_t(t)\, dx \\ &= \|u_t(t)\|^2 - \|\Delta u(t)\|^2 - \|\nabla u(t)\|^2 - \|\nabla u(t)\|^{2(\gamma+1)} + \|u(t)\|_{q+1}^{q+1} - \chi(t) \\ &= (\gamma+2)\|u_t(t)\|^2 + \gamma\|\nabla u\|^2 + \gamma\|\Delta u\|^2 + \left(1 - \frac{2(\gamma+1)}{q+1}\right)\|u\|_{q+1}^{q+1} \\ &\quad + 2(\gamma+1)H(t) - 2(\gamma+1)E_1 - \chi(t). \end{aligned}$$

By Lemma 1 (i.e $\|u\|_{q+1}^{q+1}\alpha_0^{-(q+1)} > 1$, or $E_1\|u\|_{q+1}^{q+1}\alpha_0^{-(q+1)} > E_1$),

$$\begin{aligned} \frac{1}{2}G''(t) &\geq (\gamma+2)\|u_t(t)\|^2 + \gamma\|\nabla u\|^2 + \gamma\|\Delta u\|^2 - \chi(t) \\ &\quad + \left(1 - \frac{2(\gamma+1)}{q+1} - 2(\gamma+1)E_1\alpha_0^{-(q+1)}\right)\|u\|_{q+1}^{q+1} + 2(\gamma+1)H(t) \\ &= (\gamma+2)\|u_t(t)\|^2 + \gamma\|\nabla u\|^2 + \gamma\|\Delta u\|^2 + C_1\|u\|_{q+1}^{q+1} + 2(\gamma+1)H(t) - \chi(t), \end{aligned} \tag{2.6}$$

where $C_1 = 1 - \frac{2(\gamma+1)}{q+1} - 2(\gamma+1)E_1\alpha_0^{-(q+1)} > 0$, because $\alpha_0 > \alpha_1$ by Lemma 1.

In order to estimate $\chi(t)$ in (2.6), since $q > \rho + p$, from assumption $(A1)$ and thanks to Holder's inequality and Young's inequality, we have

$$
\begin{aligned}
|\chi(t)| &\leq \int_\Omega |u(t)|^{\rho+1-\frac{\rho+p+1}{p+1}} |u(t)|^{\frac{\rho+p+1}{p+1}} |u_t(t)|^p\,dx \\
&\leq C_0 \left(\int_\Omega |u(t)|^\rho |u_t(t)|^{p+1}\,dx \right)^{\frac{p}{p+1}} \left(\int_\Omega |u(t)|^{\rho+p+1}\,dx \right)^{\frac{1}{p+1}} \\
&\leq C_0\lambda_0 \left(H'(t)\right)^{\frac{p}{p+1}} \|u(t)\|_{q+1}^{\frac{\rho+p+1}{p+1}} \\
&\leq C_0\lambda_0 \left(\delta^{-1} H'(t) + \delta^p \|u(t)\|_{q+1}^{\rho+p+1} \right), \qquad (2.7)
\end{aligned}
$$

where the constant $\delta > 0$ is specified later and λ_0 is the best embedding constant from $L^{\rho+p+1}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ since $\rho + p < q$.

We then define

$$\Gamma(t) = H^{1-\upsilon}(t) + \varepsilon G'(t),$$

where ε is to be chosen later, and $0 < \upsilon < \min\left\{ \frac{q-\rho-p}{p(q+1)}, \frac{q-1}{2(q+1)} \right\}$. Obviously, $0 < \upsilon < \frac{1}{2}$. Therefore, (2.6) and (2.7) yield

$$
\begin{aligned}
\Gamma'(t) &= (1-\upsilon) H^{-\upsilon}(t) H'(t) + \varepsilon G''(t) \\
&\geq (1-\upsilon) H^{-\upsilon}(t) H'(t) + 2\varepsilon(\gamma+2) \|u_t(t)\|^2 + 2\varepsilon\gamma \|\nabla u\|^2 \\
&\quad + 2\varepsilon\gamma \|\Delta u\|^2 + 2\varepsilon C_1 \|u\|_{q+1}^{q+1} + 2\varepsilon(\gamma+1) H(t) - 2\varepsilon\chi(t) \\
&\geq \left[(1-\upsilon) H^{-\upsilon}(t) - 2\varepsilon C_0\lambda_0\delta^{-1} \right] H'(t) + 2\varepsilon(\gamma+2) \|u_t(t)\|^2 + 2\varepsilon\gamma \|\nabla u\|^2 \\
&\quad + 2\varepsilon\gamma \|\Delta u\|^2 + 2\varepsilon C_1 \|u\|_{q+1}^{q+1} + 4\varepsilon(\gamma+1) H(t) - 2\varepsilon C_0\lambda_0\delta^p \|u\|_{q+1}^{\rho+p+1}. \qquad (2.8)
\end{aligned}
$$

Choosing $\delta = \left(\frac{C_1}{2C_0\lambda_0} \|u\|_{q+1}^{q-\rho-p} \right)^{\frac{1}{p}}$ we get

$$C_1 \|u\|_{q+1}^{q+1} - 2C_0\lambda_0\delta^p \|u\|_{q+1}^{\rho+p+1} = 0.$$

Hence, (2.8) becomes

$$
\begin{aligned}
\Gamma'(t) &\geq \left[(1-\upsilon) H^{-\upsilon}(t) - 2\varepsilon C_0\lambda_0\delta^{-1} \right] H'(t) + 2\varepsilon(\gamma+2) \|u_t(t)\|^2 + 2\varepsilon\gamma \|\nabla u\|^2 \\
&\quad + 2\varepsilon\gamma \|\Delta u\|^2 + \varepsilon C_1 \|u\|_{q+1}^{q+1} + 4\varepsilon(\gamma+1) H(t). \qquad (2.9)
\end{aligned}
$$

Since $H(t) \leq \frac{1}{q+1} \|u\|_{q+1}^{q+1}$ and the choice of δ $\left(\text{i.e. } \delta = 2^{-\frac{1}{p}} (C_0\lambda_0)^{-\frac{1}{p}} C_1^{\frac{1}{p}} \|u\|_{q+1}^{-\frac{\rho+p-q}{p}} \right)$, then

$$
\begin{aligned}
&(1-\upsilon) H^{-\upsilon}(t) - 2\varepsilon C_0\lambda_0\delta^{-1} \\
&\quad = H^{-\upsilon}(t) \left[1 - \upsilon - 2\varepsilon C_0\lambda_0\delta^{-1} H^{\upsilon}(t) \right] \\
&\quad \geq H^{-\upsilon}(t) \left[1 - \upsilon - 2^{1+\frac{1}{p}} \varepsilon (C_0\lambda_0)^{1+\frac{1}{p}} C_1^{-\frac{1}{p}} (q+1)^{-\upsilon} \|u\|_{q+1}^{\frac{\rho+p-q+\upsilon p(q+1)}{p}} \right]. \qquad (2.10)
\end{aligned}
$$

Furthermore, since $\|u\|_{q+1} \geq [(q+1)H(0)]^{\frac{1}{q+1}}$ by (2.5) and since υ was chosen so that $\rho + p - q + \upsilon p(q+1) \leq 0$, then

$$
\begin{aligned}
&(1-\upsilon) H^{-\upsilon}(t) - 2\varepsilon C_0\lambda_0\delta^{-1} \\
&\geq H^{-\upsilon}(t) \left[1 - \upsilon - 2^{1+\frac{1}{p}} \varepsilon (C_0\lambda_0)^{1+\frac{1}{p}} C_1^{-\frac{1}{p}} (q+1)^{-\frac{q-(\rho+p)}{p(q+1)}} H^{\upsilon+\frac{\rho+p-q}{p(q+1)}}(0) \right]. \qquad (2.11)
\end{aligned}
$$

Now, we choose ε to be sufficiently small so that

$$1-\upsilon-2^{1+\frac{1}{p}}\varepsilon\left(C_0\lambda_0\right)^{1+\frac{1}{p}}C_1^{-\frac{1}{p}}\left(q+1\right)^{-\frac{q-(\rho+p)}{p(q+1)}}H^{\upsilon+\frac{\rho+p-q}{(q+1)p}}\left(0\right)\geq 0,$$

then (2.11) and (2.12) yield

$$\Gamma'\left(t\right)\geq\varepsilon C_2\left[H\left(t\right)+\left\|u_t\left(t\right)\right\|^2+\left\|u\right\|_{q+1}^{q+1}+\left\|\nabla u\right\|^2+\left\|\Delta u\right\|^2\right], \tag{2.12}$$

where $C_2>0$ is a constant that does not depend on ε. Especially, (2.12) means that $\Gamma\left(t\right)$ is increasing on $\left(0,T\right)$, with

$$\Gamma\left(t\right)=H^{1-\upsilon}\left(t\right)+\varepsilon G'\left(t\right)\geq H^{1-\upsilon}\left(0\right)+\varepsilon G'\left(0\right).$$

We further choose ε to be sufficiently small so that $\Gamma\left(0\right)>0$, so $\Gamma\left(t\right)\geq\Gamma\left(0\right)>0$ for $t\geq 0$.

Now, let $\mu=\frac{1}{1-\upsilon}$. Since $0<\upsilon<\frac{1}{2}$, it is evident that $\mu>1$. By using the following inequality

$$\left|x+y\right|^{\mu}\leq 2^{\mu-1}\left(\left|x\right|^{\mu}+\left|y\right|^{\mu}\right)\ \text{ for }\ \mu\geq 1,$$

and applying Young's inequality, we have

$$\begin{aligned}\Gamma^{\mu}\left(t\right)&\leq 2^{\mu-1}\left(H\left(t\right)+\varepsilon\left\|u\left(t\right)\right\|^{\mu}\left\|u_t\left(t\right)\right\|^{\mu}\right)\\&\leq C_3\left(H\left(t\right)+\left\|u_t\left(t\right)\right\|^2+\left\|u\left(t\right)\right\|_{q+1}^{\frac{1}{\frac{1}{2}-\upsilon}}\right).\end{aligned} \tag{2.13}$$

By the choice of υ, we have $\frac{1}{2}-\upsilon>\frac{1}{q+1}$. Now applying the inequality

$$a^{\eta}\leq\left(1+\frac{1}{b}\right)\left(b+a\right),\ \ a\geq 0,\ \ 0\leq\eta\leq 1,\ \ b>0,$$

and taking $a=\left\|u\left(t\right)\right\|^{q+1}$, $\eta=\frac{1}{\left(\frac{1}{2}-\upsilon\right)(q+1)}<1$, and $b=H\left(0\right)$, we obtain

$$\begin{aligned}\left\|u\left(t\right)\right\|^{\frac{1}{\frac{1}{2}-\upsilon}}&\leq\left(1+\frac{1}{H\left(0\right)}\right)\left(H\left(0\right)+\left\|u\left(t\right)\right\|_{q+1}^{q+1}\right)\\&\leq C_4\left(H\left(t\right)+\left\|u\left(t\right)\right\|_{q+1}^{q+1}\right).\end{aligned} \tag{2.14}$$

Therefore, by combining of (2.13) and (2.14), we obatin

$$\begin{aligned}\Gamma^{\mu}\left(t\right)&\leq C_5\left(H\left(t\right)+\left\|u_t\left(t\right)\right\|^2+\left\|u\left(t\right)\right\|_{q+1}^{q+1}\right)\\&\leq C_6\left(H\left(t\right)+\left\|u_t\left(t\right)\right\|^2+\left\|u\left(t\right)\right\|_{q+1}^{q+1}+\left\|\nabla u\right\|^2+\left\|\Delta u\right\|^2\right).\end{aligned} \tag{2.15}$$

Thus, (2.12) and (2.15) imply that

$$\Gamma'\left(t\right)\geq C_7\Gamma^{\mu}\left(t\right),\ \ t\in\left[0,T\right]. \tag{2.16}$$

In the end, from (2.16) and $\mu=\frac{1}{1-\upsilon}>1$, we see that $\Gamma\left(t\right)=H^{1-\upsilon}\left(t\right)+\varepsilon G'\left(t\right)$ blow up in finite time. This completes the proof. $\square$

3. Growth

In this section, we deal with the exponential growth properties for the solutions of (1.1)

Theorem 3.1. *Assume that* $(A1)$ *and* $(A2)$ *hold. Furthermore, we assume that* $\left(\|u_0\|_{q+1}, E(0)\right) \in V$ *and* $q > \max\{2\gamma+1, \rho+p\}$. *Then the solution of the system (1.1) exponentially grows.*

Proof. We define

$$F(t) = H(t) + \varepsilon G'(t), \tag{3.1}$$

where $H(t)$ and $G(t)$ are specified in (2.1). By using (2.6) and (2.7), we have

$$\begin{aligned}
F'(t) &= H'(t) + \varepsilon G''(t) \\
&\geq H'(t) + 2\varepsilon(\gamma+2)\|u_t(t)\|^2 + 2\varepsilon\gamma\|\nabla u\|^2 \\
&\quad + 2\varepsilon\gamma\|\Delta u\|^2 + 2\varepsilon C_1\|u\|_{q+1}^{q+1} + 2\varepsilon(\gamma+1)H(t) - 2\varepsilon\chi(t) \\
&\geq \left[1 - 2\varepsilon C_0\lambda_0\delta^{-1}\right]H'(t) + 2\varepsilon(\gamma+2)\|u_t(t)\|^2 + 2\varepsilon\gamma\|\nabla u\|^2 \\
&\quad + 2\varepsilon\gamma\|\Delta u\|^2 + 2\varepsilon C_1\|u\|_{q+1}^{q+1} + 4\varepsilon(\gamma+1)H(t) - 2\varepsilon C_0\lambda_0\delta^p\|u\|_{q+1}^{\rho+p+1}.
\end{aligned} \tag{3.2}$$

Choosing $\delta = \left(\frac{C_1}{2C_0\lambda_0}\|u\|_{q+1}^{q-\rho-p}\right)^{\frac{1}{p}}$ then

$$C_1\|u\|_{q+1}^{q+1} - 2C_0\lambda_0\delta^p\|u\|_{q+1}^{\rho+p+1} = 0.$$

Hence, (3.2) becomes

$$\begin{aligned}
F'(t) \geq{}& \left[1 - 2\varepsilon C_0\lambda_0\delta^{-1}\right]H'(t) + 2\varepsilon(\gamma+2)\|u_t(t)\|^2 + 2\varepsilon\gamma\|\nabla u\|^2 \\
&+ 2\varepsilon\gamma\|\Delta u\|^2 + \varepsilon C_1\|u\|_{q+1}^{q+1} + 4\varepsilon(\gamma+1)H(t).
\end{aligned} \tag{3.3}$$

Since $H(t) \leq \frac{1}{q+1}\|u\|_{q+1}^{q+1}$ and the choice of δ $\left(\text{i.e. } \delta = 2^{-\frac{1}{p}}(C_0\lambda_0)^{-\frac{1}{p}}C_1^{\frac{1}{p}}\|u\|_{q+1}^{-\frac{\rho+p-q}{p}}\right)$, then

$$1 - 2\varepsilon C_0\lambda_0\delta^{-1} \geq 1 - 2^{1+\frac{1}{p}}\varepsilon(C_0\lambda_0)^{1+\frac{1}{p}}C_1^{-\frac{1}{p}}\|u\|_{q+1}^{\frac{\rho+p-q}{p}}. \tag{3.4}$$

Furthermore, since $\|u\|_{q+1} \geq [(q+1)H(0)]^{\frac{1}{q+1}}$ by (2.5) and $\rho + p - q \leq 0$, then

$$1 - 2\varepsilon C_0\lambda_0\delta^{-1} \geq 1 - 2^{1+\frac{1}{p}}\varepsilon(C_0\lambda_0)^{1+\frac{1}{p}}C_1^{-\frac{1}{p}}(q+1)^{\frac{\rho+p-q}{p(q+1)}}H^{\frac{\rho+p-q}{p(q+1)}}(0). \tag{3.5}$$

Now, we choose ε to be sufficiently small so that

$$1 - 2^{1+\frac{1}{p}}\varepsilon(C_0\lambda_0)^{1+\frac{1}{p}}C_1^{-\frac{1}{p}}(q+1)^{\frac{\rho+p-q}{p(q+1)}}H^{\frac{\rho+p-q}{p(q+1)}}(0) \geq 0,$$

and then (3.3) and (3.5) yield

$$F'(t) \geq \varepsilon\beta\left[H(t) + \|u_t(t)\|^2 + \|u\|_{q+1}^{q+1} + \|\nabla u\|^2 + \|\Delta u\|^2\right] \tag{3.6}$$

where $\beta > 0$ is a constant that does not depend on ε.

Now, applying Young's inequality and Sobolev Poincare inequality we have

$$\begin{aligned} F(t) &\leq H(t) + \varepsilon \|u\| \|u_t\| \\ &\leq C\left(H(t) + \|u_t\|^2 + \|u\|^2\right). \end{aligned}$$

Then, in order to estimate the $\|u\|^2$ term, we apply the inequality $a^l \leq (a+1) \leq (1+\frac{1}{b})(a+b)$ for $a = \|u\|_{q+1}^{q+1}$, $l = 2/q+1 < 1$, $b = H(0)$, and we have

$$\begin{aligned} \|u\|^2 &\leq C\|u\|_{q+1}^2 \\ &= C\left(\|u\|_{q+1}^{q+1}\right)^{\frac{2}{q+1}} \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|u\|_{q+1}^{q+1} + H(0)\right) \\ &\leq C\left(\|u\|_{q+1}^{q+1} + H(t)\right). \end{aligned} \tag{3.7}$$

Thus,

$$F(t) \leq C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{q+1}^{q+1} + \|\nabla u\|^2 + \|\Delta u\|^2\right]. \tag{3.8}$$

Therefore, (3.6) and (3.8) imply that

$$F'(t) \geq \xi F(t), \quad t \geq 0.$$

This completes the proof. □

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(Received: September 10, 2021)
(Revised: December 12, 2021)

Fatma Ekinci
Dicle University
Department of Mathematics
21280 Diyarbakır, Turkey
e-mail: *ekincifatma2017@gmail.com*
and
Erhan Pişkin
Dicle University
Department of Mathematics
21280 Diyarbakır, Turkey
e-mail: *ekincifatma2017@gmail.com*