

## SOME REMARKS ASSOCIATED WITH THE SCHWARZ LEMMA ON THE BOUNDARY

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ABSTRACT. The aim of this paper is to introduce the class of the analytic functions called  $\mathcal{N}(\alpha, c)$  and to investigate the various properties of the functions belonging to this class. For the functions in this class, some inequalities related to the angular derivative have been obtained. That is, the modulus of the angular derivative of the function  $\frac{zf'(z)}{f(z)}$  under hypothesis  $\sum_{p=2}^{\infty} \{(p-1) + |c(1+\alpha) + \alpha(p-1)|\} |c_p| \leq |c(1+\alpha)|$  is estimated from below.

### 1. INTRODUCTION

Let  $g$  be an analytic function in the unit disc  $D = \{z : |z| < 1\}$ ,  $g(0) = 0$  and  $g : D \rightarrow D$ . In accordance with the classical Schwarz lemma, for any point  $z$  in the unit disc  $D$ , we have  $|g(z)| \leq |z|$  and  $|g'(0)| \leq 1$ . In addition, if the equality  $|g(z)| = |z|$  holds for any  $z \neq 0$ , or  $|g'(0)| = 1$ , then  $g$  is a rotation; that is  $g(z) = ze^{i\theta}$ ,  $\theta$  real ([5], p.329). Schwarz lemma has important applications in engineering [14, 15]. In this study, the Schwarz lemma will be obtained for the following class  $\mathcal{N}(\alpha, c)$  which will be given.

Let  $\mathcal{A}$  denote the class of functions  $f(z) = z + \sum_{p=2}^{\infty} c_p z^p$  that are analytic in  $D$ . Also, let  $\mathcal{N}(\alpha, c)$  be the subclass of  $\mathcal{A}$  consisting of all functions  $f(z)$  satisfying

$$\sum_{p=2}^{\infty} \{(p-1) + |c(1+\alpha) + \alpha(p-1)|\} |c_p| \leq |c(1+\alpha)|, \quad (1.1)$$

where  $\alpha = \frac{m-1}{m}$  ( $m > \frac{1}{2}$ ) and  $0 \neq c \in \mathbb{C}$ . In this paper, we study some of the properties of the classes  $\mathcal{N}(\alpha, c)$ . Namely, an upper bound will be obtained for the modulus of the coefficient  $c_2 = \frac{f''(0)}{2}$  for this class.

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Let  $f \in \mathcal{N}(\alpha, c)$  and consider the following function

$$w(z) = \frac{\frac{zf'(z)}{f(z)} - 1}{c(1 + \alpha) + \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right)}. \quad (1.2)$$

It is an analytic function in  $D$  and  $w(0) = 0$ . Now, let us show that  $|w(z)| < 1$  in  $D$ . From (1.2), we have

$$w(z) = \frac{zf'(z) - f(z)}{c(1 + \alpha)f(z) + \alpha(zf'(z) - f(z))}.$$

Now let us look at the difference in the modulus of the between the numerator and denominator of the function  $w(z)$ . Therefore, we take

$$\begin{aligned} & |zf'(z) - f(z)| - |c(1 + \alpha)f(z) + \alpha(zf'(z) - f(z))| \\ &= \left| \sum_{p=2}^{\infty} (p-1)c_p z^p \right| - \left| c(1 + \alpha) \left\{ z + \sum_{p=2}^{\infty} c_p z^p \right\} + \alpha \sum_{p=2}^{\infty} (p-1)c_p z^p \right| \\ &\leq \sum_{p=2}^{\infty} (p-1)|c_p||z|^p - \left\{ |c(1 + \alpha)||z| - \sum_{p=2}^{\infty} |c(1 + \alpha) + \alpha(p-1)||c_p||z|^p \right\} \\ &= \sum_{p=2}^{\infty} \{(p-1) + |c(1 + \alpha) + \alpha(p-1)|\} |c_p||z|^p - |c(1 + \alpha)||z|. \end{aligned}$$

If we pass to the limit in the last expression as  $|z| \rightarrow 1^-$ , we obtain

$$\begin{aligned} & |zf'(z) - f(z)| - |c(1 + \alpha)f(z) + \alpha(zf'(z) - f(z))| \\ &\leq \sum_{p=2}^{\infty} \{(p-1) + |c(1 + \alpha) + \alpha(p-1)|\} |c_p| - |c(1 + \alpha)|. \end{aligned}$$

From (1.1), we obtain

$$|zf'(z) - f(z)| - |c(1 + \alpha)f(z) + \alpha(zf'(z) - f(z))| \leq 0$$

and

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{c(1 + \alpha) + \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right)} \right| < 1, \quad |z| < 1.$$

Thus, the analytical function  $w(z)$  satisfies the conditions for the Schwarz lemma. So, if we apply the Schwarz lemma to the function  $w(z)$ , we obtain

$$\begin{aligned} w(z) &= \frac{\frac{zf'(z)}{f(z)} - 1}{c(1 + \alpha) + \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right)} = \frac{c_2 z + (2c_3 - c_2^2) z^2 + \dots}{c(1 + \alpha) + \alpha (c_2 z + (2c_3 - c_2^2) z^2 + \dots)} \\ \frac{w(z)}{z} &= \frac{c_2 + (2c_3 - c_2^2) z + \dots}{c(1 + \alpha) + \alpha (c_2 z + (2c_3 - c_2^2) z^2 + \dots)} \end{aligned}$$

$$|w'(0)| = \left| \frac{c_2}{c(1+\alpha)} \right| \leq 1$$

and

$$|c_2| \leq |c|(1+\alpha).$$

We thus obtain the following lemma.

**Lemma 1.1.** *If  $f \in \mathcal{N}(\alpha, c)$ , then we have the inequality*

$$|f''(0)| \leq 2(1+\alpha)|c|. \quad (1.3)$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz Lemma, are about estimation from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [12, 17]:

**Lemma 1.2.** *Let  $g(z)$  be an analytic function in  $D$ ,  $g(0) = 0$  and  $|g(z)| < 1$  for  $z \in D$ . If  $g(z)$  extends continuously at the boundary point  $1 \in \partial D = \{z : |z| = 1\}$ , and if  $|g(1)| = 1$  and  $g'(1)$  exists, then*

$$|g'(1)| \geq \frac{2}{1+|g'(0)|} \quad (1.4)$$

and

$$|g'(1)| \geq 1. \quad (1.5)$$

Moreover, the equality in (1.4) holds if and only if

$$g(z) = z \frac{z - \sigma}{1 - \sigma z}$$

for some  $\sigma \in (-1, 0]$ . Also, the equality in (1.5) holds if and only if  $g(z) = ze^{i\theta}$ .

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 6–13].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [16]).

**Lemma 1.3 (Julia-Wolff lemma).** *Let  $g$  be an analytic function in  $D$ ,  $g(0) = 0$  and  $g(D) \subset D$ . If, in addition, the function  $g$  has an angular limit  $g(1)$  at  $1 \in \partial D$ ,  $|g(1)| = 1$ , then the angular derivative  $g'(1)$  exists and  $1 \leq |g'(1)| \leq \infty$ .*

**Corollary 1.1.** *The analytic function  $g$  has a finite angular derivative  $g'(1)$  if and only if  $g'$  has the finite angular limit  $g'(1)$  at  $1 \in \partial D$ .*

## 2. MAIN RESULTS

In this section, we discuss different versions of the boundary Schwarz lemma for the  $\mathcal{N}(\alpha, c)$  class. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of the angular derivative have been obtained.

**Theorem 2.1.** *Let  $f \in \mathcal{N}(\alpha, c)$ . Assume that, for  $1 \in \partial D$ ,  $f$  has an angular limit  $f(1)$  at the point 1,  $f'(1) = (1 - c)f(1)$ . Then we have the inequality*

$$\left| \left( \frac{zf'(z)}{f(z)} \right)'_{z=1} \right| \geq \frac{|c|}{1 + \alpha}. \quad (2.1)$$

*Proof.* Consider the function

$$w(z) = \frac{p(z) - 1}{c(1 + \alpha) + \alpha(p(z) - 1)}, \quad p(z) = \frac{zf'(z)}{f(z)}.$$

Also, since  $f'(1) = (1 - c)f(1)$ , we have  $|w(1)| = 1$ . Therefore, from (1.5), we obtain

$$\begin{aligned} 1 &\leq |w'(1)| = \frac{|p'(1)||c|(1 + \alpha)}{|c(1 + \alpha) + \alpha(p(1) - 1)|^2} = \frac{|p'(1)||c|(1 + \alpha)}{|c|^2} \\ &= \frac{|p'(1)|(1 + \alpha)}{|c|} \end{aligned}$$

and

$$|p'(1)| \geq \frac{|c|}{1 + \alpha}. \quad \square$$

The inequality (2.1) can be strengthened from below by taking into account,  $c_2 = \frac{f''(0)}{2}$ , the first coefficient of the expansion of the function  $f(z) = z + c_2z^2 + c_3z^3 + \dots$

**Theorem 2.2.** *Under the same assumptions as in Theorem 2.1, we have*

$$\left| \left( \frac{zf'(z)}{f(z)} \right)'_{z=1} \right| \geq \frac{2|c|^2}{|c|(1 + \alpha) + |c_2|}. \quad (2.2)$$

*Proof.* Let  $w(z)$  function be the same as (1.2). So, from (1.4), we obtain

$$\frac{2}{1 + |w'(0)|} \leq |w'(1)| = \frac{|p'(1)|(1 + \alpha)}{|c|}.$$

Since

$$|w'(0)| = \left| \frac{c_2}{c(1 + \alpha)} \right|,$$

we take

$$\frac{2}{1 + \frac{|c_2|}{|c|(1 + \alpha)}} \leq \frac{|p'(1)|(1 + \alpha)}{|c|}$$

and

$$|p'(1)| \geq \frac{2|c|^2}{|c|(1+\alpha) + |c_2|}. \quad \square$$

The inequality (2.2) can be strengthened as below by taking into account  $c_3 = \frac{f'''(0)}{3!}$  which is the coefficient in the expansion of the function  $f(z) = z + c_2z^2 + c_3z^3 + \dots$

**Theorem 2.3.** *Let  $f \in \mathcal{N}(\alpha, c)$ . Assume that, for  $1 \in \partial D$ ,  $f$  has an angular limit  $f(1)$  at the point 1,  $f'(1) = (1-c)f(1)$ . Then we have the inequality*

$$\left| \left( \frac{zf'(z)}{f(z)} \right)' \right|_{z=1} \geq \frac{|c|}{1+\alpha} \left( 1 + \frac{2(|c|(1+\alpha) - |c_2|)^2}{(|c|(1+\alpha))^2 - |c_2|^2 + |(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|} \right). \quad (2.3)$$

*Proof.* Let  $w(z)$  be the same as in the proof of Theorem 2.1 and  $m(z) = z$ . By the maximum principle, for each  $z \in D$ , we have the inequality  $|w(z)| \leq |m(z)|$ . So,

$$\begin{aligned} \phi(z) &= \frac{w(z)}{m(z)} = \frac{1}{z} \left( \frac{p(z) - 1}{c(1+\alpha) + \alpha(p(z) - 1)} \right) \\ &= \frac{1}{z} \frac{c_2z + (2c_3 - c_2^2)z^2 + \dots}{c(1+\alpha) + \alpha(c_2z + (2c_3 - c_2^2)z^2 + \dots)} \\ &= \frac{c_2 + (2c_3 - c_2^2)z + \dots}{c(1+\alpha) + \alpha(c_2z + (2c_3 - c_2^2)z^2 + \dots)} \end{aligned}$$

is an analytic function in  $D$  and  $|\phi(z)| \leq 1$  for  $z \in D$ . In particular, we have

$$|\phi(0)| = \frac{|c_2|}{|c|(1+\alpha)} \leq 1 \quad (2.4)$$

and

$$|\phi'(0)| = \frac{|(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|}{|c(1+\alpha)|^2}.$$

The auxiliary function

$$\varphi(z) = \frac{\phi(z) - \phi(0)}{1 - \overline{\phi(0)}\phi(z)}$$

is analytic in  $D$ ,  $\varphi(0) = 0$ ,  $|\varphi(z)| < 1$  for  $|z| < 1$  and  $|\varphi(1)| = 1$ . From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1 + |\varphi'(0)|} &\leq |\varphi'(1)| = \frac{1 - |\phi(0)|^2}{|1 - \overline{\phi(0)}\phi(1)|^2} |\phi'(1)| \\
&\leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \{ |w'(1)| - |m'(1)| \} \\
&= \frac{1 + \frac{|c_2|}{|c|(1+\alpha)}}{1 - \frac{|c_2|}{|c|(1+\alpha)}} \left( \frac{|p'(1)|(1+\alpha)}{|c|} - 1 \right).
\end{aligned}$$

Since

$$\varphi'(z) = \frac{1 - |\phi(0)|^2}{(1 - \overline{\phi(0)}\phi(z))^2} \phi'(z)$$

and

$$|\varphi'(0)| = \frac{|\phi'(0)|}{1 - |\phi(0)|^2} = \frac{\frac{|(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|}{|c(1+\alpha)|^2}}{1 - \left(\frac{|c_2|}{|c|(1+\alpha)}\right)^2},$$

we obtain

$$\begin{aligned}
\frac{2}{1 + \frac{|(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|}{(|c|(1+\alpha))^2 - |c_2|^2}} &\leq \frac{|c|(1+\alpha) + |c_2|}{|c|(1+\alpha) - |c_2|} \left( \frac{|p'(1)|(1+\alpha)}{|c|} - 1 \right), \\
\frac{2 \left( (|c|(1+\alpha))^2 - |c_2|^2 \right)}{(|c|(1+\alpha))^2 - |c_2|^2 + |(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|} &\frac{|c|(1+\alpha) - |c_2|}{|c|(1+\alpha) + |c_2|} \\
&\leq \frac{|p'(1)|(1+\alpha)}{|c|} - 1
\end{aligned}$$

and

$$|p'(1)| \geq \frac{|c|}{1+\alpha} \left( 1 + \frac{2(|c|(1+\alpha) - |c_2|)^2}{(|c|(1+\alpha))^2 - |c_2|^2 + |(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|} \right). \quad \square$$

If  $f(z) - z$  has zeros different from  $z = 0$ , taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

**Theorem 2.4.** *Let  $f \in \mathcal{N}(\alpha, c)$  and  $a_1, a_2, \dots, a_n$  be the zeros of the function  $f(z) - z$  in  $D$  that are different from zero. Assume that, for  $1 \in \partial D$ ,  $f$  has an angular limit  $f(1)$  at the point 1,  $f'(1) = (1 - c)f(1)$ . Then we have the inequality*

$$\left| \left( \frac{zf'(z)}{f(z)} \right)' \right|_{z=1} \geq \frac{|c|}{1+\alpha} \left( 1 + \sum_{i=1}^n \frac{1-|a_i|^2}{|1-a_i|^2} \right. \\ \left. + \frac{2 \left( |c|(1+\alpha) \prod_{i=1}^n |a_i| - |c_2| \right)^2}{\left( |c|(1+\alpha) \prod_{i=1}^n |a_i| \right)^2 - |c_2|^2 + \prod_{i=1}^n |a_i| \left| (2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2 + c_2c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|} \right). \quad (2.5)$$

*Proof.* Let  $w(z)$  be as in (1.2) and  $a_1, a_2, \dots, a_n$  be the zeros of the function  $f(z) - z$  in  $D$  that are different from zero. Also, consider the function

$$B(z) = z \prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z}.$$

By the maximum principle for each  $z \in D$ , we have

$$|w(z)| \leq |B(z)|.$$

Consider the function

$$\Theta(z) = \frac{w(z)}{B(z)} = \left( \frac{p(z) - 1}{c(1+\alpha) + \alpha(p(z) - 1)} \right) \frac{1}{z \prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z}} \\ = \frac{c_2 z + (2c_3 - c_2^2) z^2 + \dots}{c(1+\alpha) + \alpha(c_2 z + (2c_3 - c_2^2) z^2 + \dots)} \frac{1}{z \prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z}} \\ = \frac{c_2 + (2c_3 - c_2^2) z + \dots}{c(1+\alpha) + \alpha(c_2 z + (2c_3 - c_2^2) z^2 + \dots)} \frac{1}{\prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z}}.$$

$\Theta(z)$  is analytic in  $D$  and  $|\Theta(z)| < 1$  for  $|z| < 1$ . In particular, we have

$$|\Theta(0)| = \frac{|c_2|}{|c|(1+\alpha) \prod_{i=1}^n |a_i|}$$

and

$$|\Theta'(0)| = \frac{\left| (2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2 + c_2c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|}{|c|^2(1+\alpha)^2 \prod_{i=1}^n |a_i|}.$$

The auxiliary function

$$\lambda(z) = \frac{\Theta(z) - \Theta(0)}{1 - \overline{\Theta(0)}\Theta(z)}$$

is analytic in  $D$ ,  $|\lambda(z)| < 1$  for  $|z| < 1$  and  $\lambda(0) = 0$ . Since  $f'(1) = (1-c)f(1)$ , we take  $|\lambda(1)| = 1$ . From (1.4), we obtain

$$\begin{aligned} \frac{2}{1+|\lambda'(0)|} &\leq |\lambda'(1)| = \frac{1-|\Theta(0)|^2}{|1-\overline{\Theta(0)}\Theta(1)|} |\Theta'(1)| \\ &\leq \frac{1+|\Theta(0)|}{1-|\Theta(0)|} (|w'(1)| - |B'(1)|). \end{aligned}$$

It can be seen that

$$|\lambda'(0)| = \frac{|\Theta'(0)|}{1-|\Theta(0)|^2}$$

and

$$\begin{aligned} |\varphi'(0)| &= \frac{\left| (2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2 + c_2c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|}{|c|^2(1+\alpha)^2 \prod_{i=1}^n |a_i|} \\ &\quad \cdot \frac{1}{1 - \left( \frac{|c_2|}{|c|(1+\alpha) \prod_{i=1}^n |a_i|} \right)^2} \\ &= \prod_{i=1}^n |a_i| \frac{\left| (2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2 + c_2c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|}{\left( |c|(1+\alpha) \prod_{i=1}^n |a_i| \right)^2 - |c_2|^2}. \end{aligned}$$

Also, we have

$$|B'(1)| = 1 + \sum_{i=1}^n \frac{1-|a_i|^2}{|1-a_i|^2}.$$

Therefore, we obtain

$$\begin{aligned} &\frac{2}{1 + \prod_{i=1}^n |a_i|} \frac{\left| (2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2 + c_2c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|}{\left( |c|(1+\alpha) \prod_{i=1}^n |a_i| \right)^2 - |c_2|^2} \\ &\leq \frac{|c|(1+\alpha) \prod_{i=1}^n |a_i| + |c_2|}{|c|(1+\alpha) \prod_{i=1}^n |a_i| - |c_2|} \left( \frac{|p'(1)|(1+\alpha)}{|c|} - 1 - \sum_{i=1}^n \frac{1-|a_i|^2}{|1-a_i|^2} \right), \\ &\frac{2 \left( \left( |c|(1+\alpha) \prod_{i=1}^n |a_i| \right)^2 - |c_2|^2 \right)}{\left( |c|(1+\alpha) \prod_{i=1}^n |a_i| \right)^2 - |c_2|^2 + \prod_{i=1}^n |a_i| \left| (2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2 + c_2c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|} \\ &\leq \frac{|c|(1+\alpha) \prod_{i=1}^n |a_i| + |c_2|}{|c|(1+\alpha) \prod_{i=1}^n |a_i| - |c_2|} \left( \frac{|p'(1)|(1+\alpha)}{|c|} - 1 - \sum_{i=1}^n \frac{1-|a_i|^2}{|1-a_i|^2} \right), \end{aligned}$$

$$\begin{aligned} & \frac{2 \left( |c|(1+\alpha) \prod_{i=1}^n |a_i| - |c_2| \right)^2}{\left( |c|(1+\alpha) \prod_{i=1}^n |a_i| \right)^2 - |c_2|^2 + \prod_{i=1}^n |a_i| \left| (2c_3 - c_2^2) c(1+\alpha) - \alpha c_2^2 + c_2 c(1+\alpha) \sum_{i=1}^n \frac{1-|a_i|^2}{a_i} \right|} \\ & \leq \frac{|p'(1)|(1+\alpha)}{|c|} - 1 - \sum_{i=1}^n \frac{1-|a_i|^2}{|1-a_i|^2} \end{aligned}$$

and so, we get inequality (2.5).  $\square$

If  $f(z) - z$  has no zeros different from  $z = 0$  in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

**Theorem 2.5.** *Suppose that  $f \in \mathcal{N}(\alpha, c)$ ,  $f(z) - z$  has no zeros in  $D$  except  $z = 0$  and  $c_2 > 0$ . Assume that, for  $1 \in \partial D$ ,  $f$  has an angular limit  $f(1)$  at the point 1,  $f'(1) = (1 - c)f(1)$ . Then we have the inequality*

$$\left| \left( \frac{zf'(z)}{f(z)} \right)'_{z=1} \right| \geq \frac{|c|}{1+\alpha} \left( 1 - \frac{2c_2(1+\alpha) \ln^2 \left( \frac{c_2}{|c|(1+\alpha)} \right)}{2c_2(1+\alpha) \ln \left( \frac{c_2}{|c|(1+\alpha)} \right) - \left| (2c_3 - c_2^2) c(1+\alpha) - \alpha c_2^2 \right|} \right). \quad (2.6)$$

*Proof.* Let  $c_2 > 0$  in the expression of the function  $f(z)$ . Having in mind the inequality (2.4) and that the function  $f(z) - z$  has no zeros in  $D$  except  $z = 0$ , we denote by  $\ln \phi(z)$  the analytic branch of the logarithm normed by the condition

$$\ln \phi(0) = \ln \left( \frac{c_2}{|c|(1+\alpha)} \right) < 0.$$

The auxiliary function

$$\Omega(z) = \frac{\ln \phi(z) - \ln \phi(0)}{\ln \phi(z) + \ln \phi(0)}$$

is analytic in the unit disc  $D$ ,  $|\Omega(z)| < 1$ ,  $\Omega(0) = 0$  and  $|\Omega(1)| = 1$  for  $1 \in \partial D$ .

From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |\Omega'(0)|} & \leq |\Omega'(1)| = \frac{|2 \ln \phi(0)|}{|\ln \phi(1) + \ln \phi(0)|^2} \left| \frac{\phi'(1)}{\phi(1)} \right| \\ & = \frac{-2 \ln \phi(0)}{\ln^2 \phi(0) + \arg^2 \phi(1)} \{ |w'(1)| - 1 \}. \end{aligned}$$

Replacing  $\arg^2 \phi(1)$  by zero, then

$$\frac{1}{1 - \frac{\frac{(2c_3 - c_2^2) c(1+\alpha) - \alpha c_2^2}{|c|(1+\alpha)^2}}{\frac{2c_2}{|c|(1+\alpha)} \ln \left( \frac{c_2}{|c|(1+\alpha)} \right)}}} \leq \frac{-1}{\ln \left( \frac{c_2}{|c|(1+\alpha)} \right)} \left\{ \frac{|p'(1)|(1+\alpha)}{|c|} - 1 \right\}$$

and

$$1 - \frac{2c_2(1+\alpha)\ln^2\left(\frac{c_2}{|c|(1+\alpha)}\right)}{2c_2(1+\alpha)\ln\left(\frac{c_2}{|c|(1+\alpha)}\right) - |(2c_3 - c_2^2)c(1+\alpha) - \alpha c_2^2|} \leq \frac{|p'(1)|(1+\alpha)}{|c|}.$$

Thus, we obtain the inequality (2.6).  $\square$

#### REFERENCES

- [1] T. Akyel and B. N. Örnek, *Sharpened forms of the Generalized Schwarz inequality on the boundary*, Proc. Indian Acad. Sci. (Math. Sci.), **126(1)** (2016) 69-78.
- [2] T. A. Azeroğlu and B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Variab. Elliptic Equa. **58** (2013) 571-577.
- [3] H. P. Boas, *Julius and Julia: Mastering the Art of the Schwarz lemma*, Amer. Math. Monthly **117** (2010) 770-785.
- [4] V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disc*, J. Math. Sci. **122** (2004) 3623-3629.
- [5] G. M. Golusin, *Geometric Theory of Functions of Complex Variable* [in Russian], 2nd edn., Moscow 1966.
- [6] H-P Li and M. Mateljević, *Boundary Schwarz lemma for harmonic and Pluriharmonic mappings in the unit ball*, Journal of Mathematical Inequalities, **16**(2022), 477-498.
- [7] M. Mateljević, N. Mutavdžić, *The Boundary Schwarz lemma for harmonic and pluriharmonic mappings and some generalisations*, arXiv:2111.02384 [math.CV]
- [8] M. Mateljević, *Rigidity of holomorphic mappings & Schwarz and Jack lemma*, In press.
- [9] M. Mateljević, N. Mutavdžić and B. N. Örnek, *Note on Some Classes of Holomorphic Functions Related to Jack's and Schwarz's Lemma*, Appl. Anal. Discrete Math. **16** (2022) 111-131.
- [10] P. R. Mercer, *Boundary Schwarz inequalities arising from Rogosinski's lemma*, Journal of Classical Analysis **12** (2018) 93-97.
- [11] P. R. Mercer, *An improved Schwarz Lemma at the boundary*, Open Mathematics **16** (2018) 1140-1144.
- [12] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc. **128** (2000) 3513-3517.
- [13] B. N. Örnek, *Sharpened forms of the Schwarz lemma on the boundary*, Bull. Korean Math. Soc., **50** (2013), 2053-2059.
- [14] B. N. Örnek and T. Düzenli, *Boundary Analysis for the Derivative of Driving Point Impedance Functions*, IEEE Transactions on Circuits and Systems II: Express Briefs **65(9)** (2018) 1149-1153.
- [15] B. N. Örnek and T. Düzenli, *On Boundary Analysis for Derivative of Driving Point Impedance Functions and Its Circuit Applications*, IET Circuits, Systems and Devices, **13(2)** (2019), 145-152.
- [16] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin. 1992.
- [17] H. Unkelbach, *Über die Randverzerrung bei konformer Abbildung*, Math. Z., **43**(1938), 739-742.

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