

FINITE BETWEENNESS RELATIONAL STRUCTURES ARE NOT FRAÏSSÉ

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ABSTRACT. We consider the class of finite sets with ternary relations satisfying certain axioms of betweenness. We show this class and a related larger class are not Fraïssé, correcting a claim in [2]. We also consider restricted Ramsey properties of these classes.

Definition 0.1. A countable or finite set X together with a ternary relation $R \subseteq X^3$ is called a betweenness structure if it satisfies the following four axioms

- (1) (Symmetry) : $(a, b, c) \in R \Rightarrow (c, b, a) \in R$
- (2) $\forall a, b \in X (a, a, b) \in R$
- (3) (Minimality) : $(a, b, a) \in R \Rightarrow a = b$
- (4) Transitivity
 - (a) (Weak Transitivity) $(a, x, b) \wedge (x, z, b) \in R \Rightarrow (a, z, b) \in R$.
 - (b) (Strong Transitivity) $(a, x, b) \wedge (a, y, b) \wedge (x, z, y) \in R \Rightarrow (a, z, b) \in R$.

Note that 4a is indeed weaker and it is a consequence of 4b taking $x = y$.

Let us denote the class of finite sets with a betweenness relation \mathcal{C} and let us denote the class of finite sets with a ternary relation satisfying 1-4a but not necessarily 4b by \mathcal{C}_w . This and related relational classes were introduced in [1] and further studied in [2] from a categorical point of view. In this note we consider Ramsey like properties of this class and show that neither \mathcal{C} nor \mathcal{C}_w is a Fraïssé class. Recall [4]

Definition 0.2. A class of finite relational structures \mathcal{A} is said to be a Fraïssé class if it satisfies the following

- (1) It has countably many mutual non-isomorphic structures.
- (2) It is closed under taking isomorphisms.
- (3) It is closed under taking substructures.
- (4) It satisfies the amalgamation property. I.e., whenever $A, B_1, B_2 \in \mathcal{A}$ and are such that there are embeddings $f_i : A \rightarrow B_i$ for $i = 1, 2$, there is a $C \in \mathcal{A}$ and

2020 Mathematics Subject Classification. Primary: 05D10 08A02; Secondary: 03C15 06A75.

Key words and phrases. $\square[\kappa]$, Fraïssé class, Ramsey class, betweenness relation.

embeddings $g_i : B_i \rightarrow C$ such that the resulting diagram commutes. That is $g_1 \circ f_1 = g_2 \circ f_2$.

A related weaker property than the amalgamation property is the *joint embedding property*: For all $B_1, B_2 \in \mathcal{A}$ there is a $C \in \mathcal{A}$ and embeddings $g_i : B_i \rightarrow C$ for $i = 1, 2$.

It is rather straightforward to show that both C and C_w satisfy (JEP) and the other axioms 1-3. However, we have the following

Theorem 0.1. *Neither C nor C_w satisfies the amalgamation property, so neither is a Fraïssé class.*

Proof. We give two examples. The first will show that the class of finite betweenness structures does not satisfy the amalgamation property and hence is not a Fraïssé class. The second example will show the same class of finite ternary relations satisfying only the weaker transitivity axiom 4a also does not satisfy the amalgamation property, hence is also not a Fraïssé class.

Example 1: Let

- $(A, R_A) := (\{x_1, x_2, c, y\}; \emptyset)$,
- $(B_1, R_{B_1}) = (\{x_1, x_2, c, y, a, b\}; (a, x_1, b), (a, x_2, b), (a, y, b))$,
- $(B_2, R_{B_2}) = (\{x_1, x_2, c, y, x\}; (x_1, x, x_2), (x, c, y))$.

Each of these families generates a finite betweenness relational structure by closing under the axioms 1-4 and let the embeddings $f_1 : (A, R_A) \rightarrow (B_1, R_{B_1})$ and $f_2 : (A, R_A) \rightarrow (B_2, R_{B_2})$ be the inclusion mappings $f_1 = f_2 = id_A$.

Next, consider an amalgamation of $(A, R_A), (B_1, R_{B_1}), (B_2, R_{B_2})$, that is embeddings $g_1 : B_1 \rightarrow C$ and $g_2 : B_2 \rightarrow C$ with $g_1 \circ f_1 = g_2 \circ f_2$. WLOG we may assume that $g_1 = id_{B_1}$ so it follows that $g_1 \circ f_1 = g_2 \circ f_2 = id_A$. So the structure (C, R_C) must satisfy

$$\{(a, x_1, b), (a, x_2, b), (a, y, b), (x_1, x, x_2), (x, c, y)\} \subseteq R_C.$$

Then by the transitivity axiom 4b it follows that $(a, x, b) \in R_C$. Again by transitivity it follows from $(a, x, b), (a, y, b)(x, c, y) \in F_C$ that also $(a, c, b) \in R_C$. But $\{a, c, b\} \subset B_1$ while $(a, c, b) \notin R_{B_1}$, hence no such amalgamation is possible.

Next we consider the class of finite structures with ternary relations satisfying axioms 1-3 and only weak transitivity 4a: $(a, x, b) \wedge (a, c, x)$ or $(b, c, x) \Rightarrow (a, c, b)$. And we show that this class is also not Fraïssé by giving an example showing that the amalgamation property fails.

Example 2: Let

- $(A, R_A) := (\{a, c, m, x_1\}; \emptyset)$,
- $(B_1, R_{B_1}) := (\{a, c, m, x_1, b\}; (a, m, b), (x_1, m, b), (a, x_1, b))$,

$$\bullet (B_2; R_{B_2}) := \left(\{a, c, m, x_1, x\}; (a, c, x), (x_1, x, m) \right).$$

As in the previous examples we consider the relational structures generated by closing under axioms 1-3 and 4a and consider the obvious embeddings $f_1 = f_2 = id_A$. We note that B_1 and B_2 are transitively closed and that $(A; R_A)$ is embedded in each. As before we assume (C, R_C) is an amalgamation of B_1 and B_2 and WLOG that $g_1 = id_{B_1}$ and so both $g_1 \circ f_1 = g_2 \circ f_2 = id_A$. Then (C, R_C) will have

$$\{(a, m, b), (x_1, m, b), (a, x_1, b), (a, c, x), (x_1, x, m)\} \subseteq R_C$$

and in particular $(x_1, m, b), (x_1, x, m) \in R_C$ implies that $(x_1, x, b) \in R_C$. Together with $(a, x_1, b) \in R_C$ we may conclude that $(a, x, b) \in R_C$. Finally, using this with $(a, c, x) \in R_C$ we may conclude that $(a, c, b) \in R_C$ which is a contradiction since $(a, c, b) \notin R_{B_1}$. \square

This immediately allows us to conclude that neither C nor C_w are Ramsey, even if we augment them with linear orders. This follows from the classical result of Nešetřil and Rödl that Ramsey implies Fraïssé [5], but we present the argument to exhibit the counterexample.

Corollary 0.1. *Neither C nor C_w are Ramsey classes.*

Proof. Let A, B_1, B_2 be as in Example 1 (or Example 2 if we consider C_w). Let $B = B_1 \dot{\cup} B_2$. Given a C define a coloring of any $A' \in \binom{C}{A}$ to be 1 if A' is a substructure in some $B'_1 \in \binom{C}{B_1}$, to be 2 if $A' \in \binom{C}{A}$ s.t. A' is a substructure in some $B'_2 \in \binom{C}{B_2}$, and to be 0 otherwise. This is a well defined map since A' can not be colored by 1 and 2 simultaneously, because otherwise this would be an amalgamation. Then any $B' \in \binom{C}{B}$ will not be monochromatic since it contains disjoint copies of B_1 and B_2 . \square

The structure A witnessing the failure of the Ramsey property has 4 elements. It is natural to ask if a particular class has a restricted form of Ramsey. E.g., given a class of finite structures \mathcal{M} and given $A \in \mathcal{M}$, \mathcal{M} is said to be A -Ramsey if for all $B \in \mathcal{M}$ and all $k \in \omega$, there is a $C \in \mathcal{M}$ such that

$$C \rightarrow (B)_k^A.$$

I.e., for all partitions of the copies of A inside of C into k pieces, there is a homogeneous copy of B inside C , i.e., a copy of B all whose copies of A lie inside one piece of the partition. In the above counterexamples A had at least 4 elements and, moreover, these examples show that even if we enrich the structures with linear orders, then the resulting classes are not Ramsey. The next result shows that there is an A with only 3 elements witnessing the failure of A -Ramsey. However, this example exploits the Sierpiński type partition that fails if we enrich the structure with linear orders.

Example 0.1. Let $A = \{x, y, z\}$ with the betweenness relation R_A generated by one triple (x, y, z) . And let $B = \{a, b, c, d, e\}$ with the betweenness relation R_B generated by the triples $\{(a, b, c), (b, c, d), (c, d, e), (d, e, a)\}$. Then for no betweenness structure C do we have $C \rightarrow (B)_2^A$.

Proof. Let (C, R_C) be a betweenness structure and fix a linear order of C . Partition the copies of A in C into two colors. For a particular copy of $A \cong \{f(x), f(y), f(z)\}$, put it into P_1 if $f(x) \leq_C f(y) \leq_C f(z)$ or $f(x) \geq_C f(y) \geq_C f(z)$ and put it into P_0 otherwise.

Now, for any copy of B in C , without loss of generality we may assume that $B \subseteq C$. So, if it is 1-homogeneous we must have $a \leq_C b \leq_C c \leq_C d \leq_C e \leq_C a$ or $a \geq_C b \geq_C c \geq_C d \geq_C e \geq_C a$ which is impossible for distinct points. On the other hand, if it were 0-homogeneous, then since b cannot lie between a and c this would imply that either $b <_C \min\{a, c\}$ or $b >_C \max\{a, c\}$.

Case 1. $b <_C \min\{a, c\}$. In this case, again because of 0-homogeneity, we must have that $c >_C \max\{b, d\}$ which then implies in order that $d <_C \min\{c, e\}$ and $e >_C \max\{d, a\}$ which finally must imply that $a <_C \min\{e, b\}$. But this contradicts our assumption that $b <_C \min\{a, c\}$.

Case 2. The alternative that $b >_C \max\{a, c\}$ leads to a similar contradiction.

This completes the proof. \square

Finally we show that the class of betweenness relations is point-Ramsey. We do not know if it is A -Ramsey for 2-point sets A , or if, when enriched with linear orders, it is A -Ramsey for structures with 3 elements.

Theorem 0.2. *The betweenness classes C and C_w are point-Ramsey.*

Proof. The proof is fairly standard and based on the existence of hypergraphs with large chromatic number:

Definition 0.3. *A hypergraph (X, \mathcal{M}) has chromatic number $> k$ if every partition of X into $\leq k$ parts contains a monochromatic $M \in \mathcal{M}$. The chromatic number of (X, \mathcal{M}) is the least such k and is denoted $\chi(X, \mathcal{M})$.*

The key theorem is due to Erdős and Hajnal, [3].

Theorem 0.3. *Let $2 \leq l, k, n \in \mathbb{N}$. There is a hypergraph (X, \mathcal{M}) which is n -uniform (i.e., $M \subseteq [X]^n$), has chromatic number $> k$ and is such that (X, \mathcal{M}) has girth at least l . That is, it does not contain cycles of length less than l .*

The only consequence of no short cycles we need is that a hypergraph with no cycles of length 3 has the property that any pair of hyperedges $M_1, M_2 \in \mathcal{M}$ intersect as sets in at most one point.

To apply this result, let's suppose that $(B, R_B) \in C$ is given. Let $n = |B|$ and fix $k \geq 2$. Let $l = 3$. Let (X, \mathcal{M}) be the hypergraph given by the Theorem. For all $M \in \mathcal{M}$, fix a bijection from B to M and let R_M be the resulting betweenness

relation on M which is the image of R_B under the bijection. Since any two elements of \mathcal{M} overlap in at most a point, it is easy to check that

$$R = \bigcup \{R_M : M \in \mathcal{M}\}$$

is a betweenness relation on X . Now, since (X, \mathcal{M}) has chromatic number greater than k it easily follows that

$$(X, R_M) \rightarrow (B)_k^1$$

completing the proof. □

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(Received: November 07, 2021)

(Revised: August 01, 2023)

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