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ON NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR F-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this article, we investigate two new Hermite-Hadamard type inequalities for F-convex functions via fractional integrals. Some special cases are also discussed as a refinement of previously known results.

1. INTRODUCTION

Convexity theory gives an effective and powerful technique for studying a wide range of problems which arise in various branches of pure and applied mathematics. Many inequalities have been obtained for the class of convex functions, but among those one of the most prominent is the so called Hermite-Hadamard's inequality.Suppose that $f : I \subseteq R \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [17]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave (1.1).

Various refinements of Hermite Hadamard type inequality for convex functions and their variant forms are being obtained by many researchers, (see, [5], [6], [13], [18], [23]- [25]) and the references therein. In the last years, several extensions and generalizations have been considered for classical convexity, such as quasi-convex [4], pseudo-convex [14], strongly convex [20], $\varepsilon-$convex [11], $s-$convex [10], $h-$convex [29] and etc. A new concept of convexity was recently introduced by Samet [21] that depends on a certain function satisfying some axioms, and generalizes different types of convexity, including $\varepsilon-$convex functions, $\alpha-$convex functions, $h-$convex functions, and many others.

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Recall the family $\mathcal{F}$ of mappings $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,1]\to\mathbb{R}$ satisfying the following axioms:

(A1) If $u_i\in L^1(0,1)$, $i=1,2,3$, then for every $\lambda\in[0,1]$, we have

$$\int_0^1 F(u_1(t),u_2(t),u_3(t),\lambda)dt = F\left(\int_0^1 u_1(t)dt,\int_0^1 u_2(t)dt,\int_0^1 u_3(t)dt,\lambda\right).$$

(A2) For every $u\in L^1(0,1)$, $w\in L^\infty(0,1)$ and $(z_1,z_2)\in\mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t),w(t)z_1,w(t)z_2,t)dt = T_{F,w}\left(\int_0^1 w(t)u(t)dt,z_1,z_2\right),$$

where $T_{F,w}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function that depends on (F,w), and it is nondecreasing with respect to the first variable.

(A3) For any $(w,u_1,u_2,u_3)\in\mathbb{R}^4$, $u_4\in[0,1]$, we have

$$wF(u_1,u_2,u_3,u_4) = F(wu_1,wu_2,wu_3,u_4)+L_w$$

where $L_w\in\mathbb{R}$ is a constant that depends only on w.

Definition 1.1. *Let $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $a<b$, be a given function. We say that f is a convex function with respect to some $F\in\mathcal{F}$ (or $F-$convex function) iff*

$$F(f(tx+(1-t)y),f(x),f(y),t)\le 0,\quad (x,y,t)\in[a,b]\times[a,b]\times[0,1].$$

Remark 1.1. 1) Let $\varepsilon\ge 0$, and let $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $a<b$, be an ε-convex function, that is (see [11])

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)+\varepsilon,\ (x,y,t)\in[a,b]\times[a,b]\times[0,1].$$

Define the functions $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,1]\to\mathbb{R}$ by

$$F(u_1,u_2,u_3,u_4) = u_1-u_4u_2-(1-u_4)u_3-\varepsilon \tag{1.2}$$

and $T_{F,w}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$T_{F,w}(u_1,u_2,u_3) = u_1-\left(\int_0^1 tw(t)dt\right)u_2-\left(\int_0^1(1-t)w(t)dt\right)u_3-\varepsilon. \tag{1.3}$$

For

$$L_w=(1-w)\varepsilon, \tag{1.4}$$

it is clear that $F\in\mathcal{F}$ and

$$F(f(tx+(1-t)y),f(x),f(y),t) = f(tx+(1-t)y)-tf(x)-(1-t)f(y)-\varepsilon\le 0,$$

that is f is an $F-$convex function. Particularly, taking $\varepsilon=0$, we show that if f is a convex function then f is an $F-$convex function with respect to F defined above.

2) Let $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $a<b$, be an α-convex function,$\alpha\in(0,1]$, that is

$$f(tx+(1-t)y)\leq t^{\alpha}f(x)+(1-t^{\alpha})f(y),\ \ (x,y,t)\in[a,b]\times[a,b]\times[0,1].$$

Define the functions $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,1]\to\mathbb{R}$ by

$$F(u_1,u_2,u_3,u_4)=u_1-u_4^{\alpha}u_2-(1-u_4^{\alpha})u_3 \tag{1.5}$$

and $T_{F,w}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$T_{F,w}(u_1,u_2,u_3)=u_1-\left(\int_0^1 t^{\alpha}w(t)dt\right)u_2-\left(\int_0^1(1-t^{\alpha})w(t)dt\right)u_3. \tag{1.6}$$

For $L_w=0$, it is clear that $F\in\mathcal{F}$ and

$$F(f(tx+(1-t)y),f(x),f(y),t)=f(tx+(1-t)y)-t^{\alpha}f(x)-(1-t^{\alpha})f(y)\leq 0,$$

that is f is an $F-$convex function.

3) Let $h:J\to[0,\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0,1)\subseteq J$. Let $f:[a,b]\to[0,\infty)$, $(a,b)\in\mathbb{R}^2$, $a<b$, be an h-convex function, that is (see [29])

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y),\ \ (x,y,t)\in[a,b]\times[a,b]\times[0,1].$$

Define the functions $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,1]\to\mathbb{R}$ by

$$F(u_1,u_2,u_3,u_4)=u_1-h(u_4)u_2-h(1-u_4)u_3 \tag{1.7}$$

and $T_{F,w}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$T_{F,w}(u_1,u_2,u_3)=u_1-\left(\int_0^1 h(t)w(t)dt\right)u_2-\left(\int_0^1 h(1-t)w(t)dt\right)u_3. \tag{1.8}$$

For $L_w=0$, it is clear that $F\in\mathcal{F}$ and

$$F(f(tx+(1-t)y),f(x),f(y),t)=f(tx+(1-t)y)-h(t)f(x)-h(1-t)f(y)\leq 0,$$

that is f is an F-convex function.

In [21], the author established the following Hermite-Hadamard type inequalities using the new convexity concept:

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $a<b$, be an F-convex function, for some $F\in\mathcal{F}$. Suppose that $f\in L_1[a,b]$. Then*

$$F\left(f\left(\frac{a+b}{2}\right),\frac{1}{b-a}\int_a^b f(x)dx,\frac{1}{b-a}\int_a^b f(x)dx,\frac{1}{2}\right)\leq 0,$$

$$T_{F,1}\left(\frac{1}{b-a}\int_a^b f(x)dx,f(a),f(b)\right)\leq 0.$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [9, 12, 15, 19].

Definition 1.2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

It is remarkable that Sarikaya et al. [26] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.9}$$

with $\alpha > 0$.

Sarıkaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Liouville fractional integrals in [22].

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.10}$$

Budak et al. in [1], prove the following Hermite-Hadamard type inequalities for F-convex functions via fractional integrals:

Theorem 1.4. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a,b \in I^{\circ}$, $a < b$. Let F be linear with respect to the first three variables. If f is F-convex on $[a,b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{a^+}f(b), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{b^-}f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}dt \leq 0$$

and

$$T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right],f(a)+f(b),f(a)+f(b)\right)+\int\limits_{0}^{1}L_{w(t)}dt\leq 0,$$

where $w(t)=\alpha t^{\alpha-1}$.

Theorem 1.5. *Let $I\subseteq R$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$, $a<b$. Suppose that $|f'|$ is F-convex on $[a,b]$ for some $F\in\mathcal{F}$ and the function $t\in[0,1]\to L_{w(t)}$ belongs to $L^{1}[0,1]$, where $w(t)=|(1-t)^{\alpha}-t^{\alpha}|$. Then*

$$\begin{aligned}&T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right|,\left|f'(a)\right|,\left|f'(b)\right|,t\right)\\&+\int_{0}^{1}L_{w(t)}dt\leq 0.\end{aligned}\tag{1.11}$$

For the other papers on inequalities for F-convex functions, see [2, 3, 16, 27, 28].

2. Hermite-Hadamard type inequality involving fractional integrals

In this section, we establish some inequalities of Hermite-Hadamard type including fractional integrals via F-convex functions.

Theorem 2.1. *Let $I\subseteq R$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping on I°, $a,b\in I^{\circ}$, $a<b$. Let F be linear with respect to the first three variables. If f is F-convex on $[a,b]$, for some $F\in\mathcal{F}$, then we have*

$$\begin{aligned}&F\left(f\left(\frac{a+b}{2}\right),\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b),\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a),\frac{1}{2}\right)\\&+\int_{0}^{1}L_{w(t)}dt\leq 0.\end{aligned}\tag{2.1}$$

$$\begin{aligned}&T_{F,w}\left(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)\right],f(a)+f(b),f(a)+f(b)\right)\\&+\int\limits_{0}^{1}L_{w(t)}dt\leq 0\end{aligned}\tag{2.2}$$

where $w(t)=\alpha t^{\alpha-1}$.

Proof. Since f is F-convex, we have

$$F\left(f\left(\frac{x+y}{2}\right),f(x),f(y),\frac{1}{2}\right)\leq 0,\ x,y\in[a,b].$$

For
$$x=\frac{t}{2}a+\frac{2-t}{2}b, \text{ and } y=\frac{t}{2}b+\frac{2-t}{2}a,$$
we have
$$F\left(f\left(\frac{a+b}{2}\right),f\left(\frac{t}{2}a+\frac{2-t}{2}b\right),f\left(\frac{t}{2}b+\frac{2-t}{2}a\right),\frac{1}{2}\right)\leq 0,\ t\in[0,1].$$
Multiplying this inequality by $w(t)=\alpha t^{\alpha-1}$ and using axiom (A3), we get
$$F\left(\alpha t^{\alpha-1}f\left(\frac{a+b}{2}\right),\alpha t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right),\alpha t^{\alpha-1}f\left(fract2b+\frac{2-t}{2}a\right),\frac{1}{2}\right)$$
$$+L_{w(t)}\leq 0,$$
for $t\in[0,1]$. Integrating over $[0,1]$ with respect to the variable t and using axiom (A1), we obtain
$$F\left(f\left(\frac{a+b}{2}\right)\alpha\int_0^1 t^{\alpha-1}dt,\alpha\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt,\right.$$
$$\left.\alpha\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)dt,\frac{1}{2}\right)+\int_0^1 L_{w(t)}dt\leq 0.$$
Here we get,
$$\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt=\frac{2^\alpha}{(b-a)^\alpha}\int_a^b(b-x)^{\alpha-1}f(x)dx$$
$$=\frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)$$
and
$$\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)dt=\frac{2^\alpha}{(b-a)^\alpha}\int_a^b(x-a)^{\alpha-1}f(x)dx$$
$$=\frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a).$$
Using these equalities, we obtain
$$F\left(f\left(\frac{a+b}{2}\right),\frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b),\frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a),\frac{1}{2}\right)$$
$$+\int_0^1 L_{w(t)}dt\leq 0$$
which gives (2.1).

On the other hand, since f is F-convex, we have
$$F\left(f\left(\frac{t}{2}a+\frac{2-t}{2}b\right),f(a),f(b),\frac{t}{2}\right)\leq 0,\ t\in[0,1]$$

and

$$F\left(f\left(\frac{t}{2}b+\frac{2-t}{2}a\right), f(b), f(a), \frac{t}{2}\right) \leq 0,\ t\in[0,1].$$

Using the linearity of F, we get

$$F\left(f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{t}{2}b+\frac{2-t}{2}a\right), f(a)+f(b), f(a)+f(b), \frac{t}{2}\right) \leq 0,$$

for $t\in[0,1]$. Applying the axiom (A3) for $w(t)=\alpha t^{\alpha-1}$, we obtain

$$\begin{aligned}&F\left(\alpha t^{\alpha-1}\left[f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right], \alpha t^{\alpha-1}\left[f(a)+f(b)\right],\right.\\&\left.\alpha t^{\alpha-1}\left[f(a)+f(b)\right], \frac{t}{2}\right)+L_{w(t)} \leq 0,\end{aligned}$$

for $t\in[0,1]$. Integrating over $[0,1]$ and using axiom (A2), we have

$$\begin{aligned}&T_{F,w}\left(\int_0^1 \alpha t^{\alpha-1}\left[f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right]dt, f(a)+f(b), f(a)+f(b)\right)\\&+\int_0^1 L_{w(t)}dt \leq 0.\end{aligned}$$

This gives

$$\begin{aligned}&T_{F,w}\left(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a), f(a)+f(b), f(a)+f(b)\right)\\&+\int_0^1 L_{w(t)}dt \leq 0,\end{aligned}$$

which completes the proof. □

Corollary 2.1. *If we choose $F(u_1,u_2,u_3,u_4)=u_1-u_4u_2-(1-u_4)u_3-\varepsilon$ in Theorem 2.1 then the function f is ε-convex on $[a,b]$, $\varepsilon\geq 0$ and we have the inequality*

$$f\left(\frac{a+b}{2}\right)-\varepsilon \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right] \leq \frac{f(a)+f(b)}{2}+\frac{\varepsilon}{2}.$$

Proof. Using (1.4) with $w(t)=\alpha t^{\alpha-1}$, we have

$$\int_0^1 L_{w(t)}dt=\varepsilon\int_0^1 (1-\alpha t^{\alpha-1})dt=0. \tag{2.3}$$

Using (1.2), (2.1) and (2.3), we get

$$0 \geq F\left(f\left(\frac{a+b}{2}\right), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt$$
$$= f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right] - \varepsilon.$$

That is

$$f\left(\frac{a+b}{2}\right) - \varepsilon \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right].$$

On the other hand, using (1.3) with $w(t) = \alpha t^{\alpha-1}$, we have

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \alpha \left(\int_0^1 t^{\alpha} dt \right) u_2 - \alpha \left(\int_0^1 (1-t) t^{\alpha-1} dt \right) u_3 - \varepsilon \\ &= u_1 - \frac{\alpha u_2 + u_3}{\alpha+1} - \varepsilon \end{aligned} \tag{2.4}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence, from (2.2) and (2.4), we obtain

$$\begin{aligned} 0 \geq & T_{F,w}\left(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right], f(a)+f(b), f(a)+f(b) \right) \\ & + \int_0^1 L_{w(t)} dt \\ = & \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right] \\ & - \frac{1}{\alpha+1} \left[\alpha (f(a)+f(b)) + (f(a)+f(b)) \right] - \varepsilon \\ = & \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right] - (f(a)+f(b)) - \varepsilon. \end{aligned}$$

Thus we get the inequality

$$\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right] \leq f(a) + f(b) + \varepsilon$$

and thus the proof is completed. □

Remark 2.1. If we take $\varepsilon = 0$ in Corollary 2.1, then f is convex and we have the inequality (1.9).

Corollary 2.2. *If we choose $F(u_1,u_2,u_3,u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3$ in Theorem 2.1, then the function f is h-convex on $[a,b]$ and we have the inequality*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \\ &\leq \alpha\left(\int_0^1 [h(t)+h(1-t)]\, t^{\alpha-1} dt\right)\frac{f(a)+f(b)}{2}.\end{aligned}$$

Proof. Using (1.4) and (2.1) with $L_{w(t)} = 0$, we have

$$\begin{aligned}0 \geq\, & F\left(f\left(\frac{a+b}{2}\right), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a), \frac{1}{2}\right) \\ &+ \int_0^1 L_{w(t)} dt \\ =\, & f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right].\end{aligned}$$

That is

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right].$$

On the other hand, using (1.8) and (2.2) with $w(t) = \alpha t^{\alpha-1}$, we obtain

$$\begin{aligned}0 \geq\, & T_{F,w}\left(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right], f(a)+f(b), f(a)+f(b)\right) \\ &+ \int_0^1 L_{w(t)} dt \\ =\, & \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \\ &- \alpha\left[\int_0^1 h(t)t^{\alpha-1}dt + \int_0^1 h(1-t)t^{\alpha-1}dt\right][f(a)+f(b)] \\ =\, & \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \\ &- \alpha\left(\int_0^1 [h(t)+h(1-t)]\, t^{\alpha-1}dt\right)[f(a)+f(b)],\end{aligned}$$

i.e.

$$\begin{aligned}&\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \\ &\qquad \leq \alpha\left(\int_0^1 [h(t)+h(1-t)]\, t^{\alpha-1}dt\right)[f(a)+f(b)]\end{aligned}$$

and thus the proof is completed. □

Theorem 2.2. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a, b \in I^{\circ}$, $a < b$. Let F be linear with respect to the first three variables. If f is F-convex on $[a,b]$, for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{a^{+}}^{\alpha} f\left(\frac{a+b}{2}\right), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b^{-}}^{\alpha} f\left(\frac{a+b}{2}\right), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0 \tag{2.5}$$

and

$$T_{F,w}\left(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^{-}}^{\alpha} f\left(\frac{a+b}{2}\right)\right], f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0$$

where $w(t) = \alpha t^{\alpha-1}$.

Proof. We can prove this theorem in a way similar to the proof of Theorem 2.1. Since f is F-convex, we have

$$F\left(f\left(\frac{x+y}{2}\right), f(x), f(y), \frac{1}{2}\right) \leq 0, \ x, y \in [a,b]$$

For

$$x = \frac{1+t}{2}a + \frac{1-t}{2}b, \text{ and } y = \frac{1+t}{2}b + \frac{1-t}{2}a,$$

we have

$$F\left(f\left(\frac{a+b}{2}\right), f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right), f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right), \frac{1}{2}\right) \leq 0, \ t \in [0,1].$$

Multiplying this inequality by $w(t) = \alpha t^{\alpha-1}$ and using axiom (A3),and then integrating the result over $[0,1]$, we obtain (by using axiom (A1))

$$F\left(f\left(\frac{a+b}{2}\right)\alpha \int_0^1 t^{\alpha-1} dt, \alpha \int_0^1 t^{\alpha-1} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt, \right.$$
$$\left. \alpha \int_0^1 t^{\alpha-1} f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt, \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

This gives

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{a^{+}}^{\alpha} f\left(\frac{a+b}{2}\right), \frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b^{-}}^{\alpha} f\left(\frac{a+b}{2}\right), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

The proof of (2.5) is completed.

On the other hand, since f is F-convex, we have

$$F\left(f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right), f(a), f(b), \frac{1+t}{2}\right)\leq 0,\ t\in[0,1]$$

and

$$F\left(f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right), f(b), f(a), \frac{1+t}{2}\right)\leq 0,\ t\in[0,1].$$

Using the linearity of F, we get

$$F\left(f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right), f(a)+f(b), f(a)+f(b), \frac{1+t}{2}\right)\leq 0,$$

for $t\in[0,1]$. Applying the axiom (A3) for $w(t)=\alpha t^{\alpha-1}$, we obtain

$$\begin{aligned}&F\left(\alpha t^{\alpha-1}\left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right],\right.\\&\left.\alpha t^{\alpha-1}\left[f(a)+f(b)\right], \alpha t^{\alpha-1}\left[f(a)+f(b)\right], \frac{1+t}{2}\right)+L_{w(t)}\leq 0,\end{aligned}$$

for $t\in[0,1]$. Integrating over $[0,1]$ and using axiom (A2), we have

$$\begin{aligned}&T_{F,w}\left(\int_0^1\alpha t^{\alpha-1}\left[f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right]dt, f(a)+f(b), f(a)+f(b)\right)\\&+\int_0^1 L_{w(t)}dt\leq 0.\end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.3. *If we choose* $F(u_1,u_2,u_3,u_4)=u_1-u_4u_2-(1-u_4)u_3-\varepsilon$ *in Theorem 2.2, then the function* f *is* ε*-convex on* $[a,b]$, $\varepsilon\geq 0$ *and we have the inequality*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)-\varepsilon&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\\&\leq\frac{f(a)+f(b)}{2}+\frac{\varepsilon}{2}.\end{aligned}$$

Remark 2.2. If we take $\varepsilon=0$ in Corollary 2.1, then f is convex and we have the following inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\leq\frac{f(a)+f(b)}{2}$$

which was given by Dragomir in [7].

One can also find the same inequality in [8].

Corollary 2.4. *If we choose $F(u_1,u_2,u_3,u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3$ in Theorem 2.1, then the function f is h-convex on $[a,b]$ and we have the inequality*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]$$
$$\leq \alpha\left(\int_0^1 [h(t)+h(1-t)]\, t^{\alpha-1}dt\right)\frac{f(a)+f(b)}{2}.$$

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