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BERNSTEIN TYPE L_p INEQUALITIES FOR COMPOSITION OF POLYNOMIALS

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ABSTRACT. Let $P(z) = a_0 + \sum\limits_{v=\mu}^{n} a_v z^v \in \mathscr{P}_{n,\,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, then for $0 \leq p \leq \infty$, Gardner and Weems (J. Math. Anal. Appl. **219** (1998), 472-478) proved that

$$\|P'\|_p \leq \frac{n}{\|k^\mu + z\|_p} \|P\|_p.$$

In this note, we consider a more general class of polynomials $P \circ Q \in \mathscr{P}_{mn,\,\mu}$ defined by $(P \circ Q)(z) = P(Q(z))$, where $Q \in \mathscr{P}_m$ and provide an extension of the above inequality and related results.

1. INTRODUCTION

Let $\mathscr{P}_n$ be the linear space of all polynomials $P(z) = \sum\limits_{v=0}^{n} a_v z^v$ over $\mathbb{C}$ of degree at most n and $P'(z)$ be the derivative of $P(z)$. For $P \in \mathscr{P}_n$, we define

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{i\theta})| d\theta\right),$$

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \quad \text{for } 0 < p < \infty$$

and

$$\|P\|_\infty = \max_{|z|=1} |P(z)|$$

Notice that $\|P\|_0 = \lim_{p \to 0^+} \|P\|_p$ and $\|P\|_\infty = \lim_{p \to \infty} \|P\|_p$. For $1 \leq p \leq \infty$, $\|.\|_p$ is a norm and hence $\mathscr{P}_n$ is a normed linear space under $\|.\|_p$

The Bernstein inequality asserts that

$$\|P'\| \leq n\|P\| \tag{1.1}$$

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holds for every polynomial $P \in \mathscr{P}_n$. Well, it is clear that inequality (1.1) relates the supremum norm of a polynomial with its derivative and this inequality reduces to equality if and only if $P(z) = \alpha z^n$ for some complex constant α. Various analogues of this inequality are known in which the underlying unit circles, the maximum norms, and the family of functions are replaced by more general circles, norms and families of functions such as hyperholomorphic functions respectively. The inequality

$$|P'(z)| \geq n \min_{|z|=1} |P(z)| \tag{1.2}$$

holds for every polynomial $P \in \mathscr{P}_n$ which has all zeros in $|z| \leq 1$. This inequality is ascribed to Aziz and Dawood [2]. Bernstein-type inequalities are known on various regions of the complex plane and n-dimensional Euclidean space such as Quaternions (see [6]), for various norms such as weighted L_p norms (to be discussed further), and for many classes of functions such as polynomials with various constraints. Note that, Bernstein-type inequalities have their own intrinsic interest. Inequality (1.2) is sharp and equality holds when $P(z) = \alpha z^n$, where $|\alpha| = 1$ which has all its zeros at the origin, one would expect a relationship between the bound n and the distance of the zeros of the polynomial from the origin. This fact was observed as a refinement of Bernstein's inequality, conjectured by Erdös and later proved by Lax [10], can be provided under assumptions on the location of the zeros of a polynomial and states the following:

Theorem 1.1. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then*

$$\|P'\| \leq \frac{n}{2}\|P\|.$$

This inequality is sharp and equality holds if P has all of its zeros on $|z| = 1$. For polynomials of a complex variable, we also have the following more general result, due to Malik [11], which is one of the best-known polynomial inequality after the Bernstein inequality and will be useful in proving some of our results.

Theorem 1.2. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, then*

$$\|P'\| \leq \frac{n}{1+k}\|P\|.$$

Obviously, Theorem 1.1 follows from Theorem 1.2 when $k = 1$. Chan and Malik [3] introduced the class of polynomials of the form $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, where $1 \leq \mu \leq n$. Let us denote the linear space of all such polynomials as $\mathscr{P}_{n,\,\mu}$. Notice that $\mathscr{P}_{n,\,1} = \mathscr{P}_n$. In relation to $\mathscr{P}_{n,\,\mu}$, Chan and Malik [3] presented the following result.

Theorem 1.3. *If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \in \mathscr{P}_{n,\,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, then*

$$\|P'\| \leq \frac{n}{1+k^{\mu}}\|P\|.$$

2. Integral norm inequalities for $\mathscr{P}_n$

Zygmund's [13] well known result relating the integral norm of a polynomial and its derivative states that if $P \in \mathscr{P}_n$, then for $1 \leq p \leq \infty$

$$\|P'\|_p \leq n\|P\|_p.$$

DeBruijn [4] presented the extension of Theorem 1.1 to L_p norms by establishing the following:

Theorem 2.1. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then $1 \leq p \leq \infty$*

$$\|P'\|_p \leq \frac{n}{\|1+z\|_p}\|P\|_p.$$

Of course, Theorem 1.1 is the limiting case of Theorem 2.1 as $p \to \infty$. Rahman and Schmeisser [12] asserted that Theorem 2.1 in fact holds for $0 \leq p \leq \infty$.

Gardner and Weems [7, Corollary 2.2] showed that Theorem 1.3 can be extended to L_p inequalities where $0 \leq p \leq \infty$ and were able to prove the following result which contains an L_p extension of Theorem 1.2 as a special case. Of special interest, is the fact that this result also holds for L_p norms for all $1 \leq p \leq \infty$. In particular, they proved:

Theorem 2.2. *If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \in \mathscr{P}_{n,\,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, then for $0 \leq p \leq \infty$*

$$\|P'\|_p \leq \frac{n}{\|k^{\mu}+z\|_p}\|P\|_p.$$

In the present paper, we consider a more general class of polynomials $P \circ Q \in \mathscr{P}_{mn,\,\mu}$ defined by $(P \circ Q)(z) = P(Q(z))$, where $Q \in \mathscr{P}_m$ and produce the result mainly in the next section which in particular yields Theorem 2.2 and many other striking results as special cases. Note that

$$\|P \circ Q\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(Q(e^{i\theta}))|^p d\theta\right)^{\frac{1}{p}} \quad \text{for } 0 < p < \infty$$

and

$$\|P \circ Q\|_\infty = \max_{|z|=1} |P(Q(z))|.$$

3. Statement of Results

Our main result is the following:

Theorem 3.1. *Let $P \circ Q \in \mathscr{P}_{mn,\,\mu}$. If $S(z) = P(Q(z)) \neq 0$ for $|z| < k$, where $k \geq 1$ and $Q(z) \neq 0$ in $|z| > 1$ with $\min\limits_{|z|=1} |Q(z)| = A$, then for $0 \leq p \leq \infty$*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |P'(Q(e^{i\theta}))|^p d\theta\right)^{\frac{1}{p}} \leq \frac{n}{A\|k^\mu + z\|_p}\|P \circ Q\|_p.$$

***Remark* 3.1.** If we have $Q(z) = z$, then $Q(z) = 0$ in $|z| < 1$ with $\min\limits_{|z|=1} |Q(z)| = A = 1$. Therefore from Theorem 3.1, we have Theorem 2.2 as a special case.

Since Theorem 3.1 also holds for L_p norms for all $1 \leq p \leq \infty$. In particular, we have the following generalization of a result of Gardner and Weems [7, Corollary 2.3].

Corollary 3.1. *Let $P \circ Q \in \mathscr{P}_{mn,\,\mu}$. If $P(Q(z)) \neq 0$ for $|z| < k$, where $k \geq 1$ and $Q(z) \neq 0$ in $|z| > 1$ with $\min\limits_{|z|=1} |Q(z)| = A$, then for $1 \leq p \leq \infty$*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |P'(Q(e^{i\theta}))|^p d\theta\right)^{\frac{1}{p}} \leq \frac{n}{A\|k^\mu + z\|_p}\|P \circ Q\|_p.$$

***Remark* 3.2.** With $p = \infty$, we have $\left(\frac{1}{2\pi}\int\limits_0^{2\pi} |P'(Q(e^{i\theta}))|^p d\theta\right)^{\frac{1}{p}} \to \max\limits_{|z|=1} |P'(Q(z))|$ and if $Q(z) = z$ then $\min\limits_{|z|=1} |Q(z)| = A = 1$, Corollary 3.1 reduces to Theorem 1.3. With $p = \infty$, $Q(z) = z$ and $\mu = 1$, it reduces to Theorem 1.2. With $p = \infty$, $Q(z) = z$, $\mu = 1$ and $k = 1$, it reduces to Theorem 1.1. Finally if we fix $Q(z) = z$, $\mu = 1$ and $k = 1$, Corollary 3.1 reduces to Theorem 2.1.

***Remark* 3.3.** Interestingly, the result due to Dewan et al. [5, Theorem 1] also follows from Corollary 3.1 when $p = \infty$ and $\mu = 1$.

4. Lemmas

Definition 4.1. *For $\gamma = (\gamma_0, ..., \gamma_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum\limits_{v=0}^{n} a_v z^v$, define*

$$\Lambda_\gamma P(z) = \sum_{v=0}^{n} \gamma_v a_v z^v.$$

The operator Λ_γ is admissible if it preserves one of the following properties:
(a) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
(b) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

Lemma 4.1. *Let* $\phi(x) = \psi(\log x)$ *where* ψ *is a convex non-decreasing function on* $\mathbb{R}$. *Then for all* $P \in \mathscr{P}_n$ *and each admissible operator* Λ_γ

$$\int_0^{2\pi} \phi\left(|\Lambda_\gamma P(e^{i\theta})|\right)|d\theta \le \int_0^{2\pi} \phi\left(c(\gamma,n)|P(e^{i\theta})|\right)d\theta,$$

where $c(\gamma,n) = \max(|\gamma_0|,|\gamma_n|)$.

The proof of Lemma 4.1 was given by Arestov [1].

Lemma 4.2. *Let* $P \circ Q \in \mathscr{P}_{mn,\,\mu}$. *If* $S(z) = P(Q(z)) \neq 0$ *in* $|z| < k$, *where* $k \ge 1$, *then for* $|z| = 1$

$$k^{\mu}|S'(z)| \le |R'(z)|,$$

where $R(z) = z^{mn}\overline{S(\frac{1}{\bar z})} = z^{mn}\overline{P(Q(\frac{1}{\bar z}))}$.

Proof. Since $S(z) \neq 0$ in $|z| < k$, from Laguerre's Theorem [9] we have

$$\alpha S'(z) \neq zS'(z) - mnS(z) \tag{4.1}$$

for $|\alpha| < k$, $|z| < k$. Now choose the $\arg\alpha$ in (4.1) appropriately, then we get for any fixed z

$$|\alpha||S'(z)| \neq |zS'(z) - nS(z)|.$$

This gives for $|\alpha| < k$ and $|z| < k$

$$|\alpha||S'(z)| < |zS'(z) - mnS(z)| \tag{4.2}$$

because otherwise the inequality is violated for sufficiently small values of $|\alpha|$. Letting $|\alpha| \to k$ in (4.2), we have

$$k|S'(kz)| \le |kzS'(kz) - mnS(kz)| \tag{4.3}$$

for $|z| \le 1$. Since $a_1 = a_2 = ... = a_{\mu-1} = 0$, from (4.3) we get

$$k^{\mu}\left|\sum_{\nu=\mu}^{mn} \nu a_\nu (kz)^{\nu-1}\right| \le |kzS'(kz) - mnS(kz)| \tag{4.4}$$

for $|z| \le 1$. In fact (4.4) also holds for $|z| = 1$, replace z by z/k in (4.4), then we obtain for $|z| = 1$

$$k^{\mu}\left|\sum_{\nu=\mu}^{mn} \nu a_\nu (z)^{\nu-1}\right| \le |zS'(z) - mnS(z)|.$$

It can be easily verified for $|z| = 1$ that

$$|R'(z)| = |zS'(z) - mnS(z)|.$$

Consequently $k^{\mu}|S'(z)| \le |R'(z)|$ for $|z| = 1$. $\square$

5. Proofs of the theorems

Proof of Theorem 3.1. Since $S(z) \neq 0$ in $|z| < k$, from Laguerre's Theorem [9], we have

$$mnS(z) - (z-\alpha)S'(z) \neq 0$$

for $|z| < k$, $|\alpha| < k$. Therefore, setting $\alpha = -ze^{-i\zeta}$, $\zeta \in \mathbb{R}$, the operator Λ_γ

$$\Lambda_\gamma S(z) = (e^{i\zeta}+1)zS'(z) - mne^{i\zeta}S(z)$$

is admissible and thus by Lemma 4.1 with $\psi(x) = e^{px}$,

$$\int_0^{2\pi} \left|(e^{i\zeta}+1)S'(e^{i\theta}) - imne^{i\zeta}S(e^{i\theta})\right|^p d\theta \leq m^p n^p \int_0^{2\pi} \left|S(e^{i\theta})\right|^p d\theta$$

for $p > 0$. Then it is clear that

$$\int_0^{2\pi} \left|S'(e^{i\theta}) + e^{i\zeta}\left\{S'(e^{i\theta}) - imnS(e^{i\theta})\right\}\right|^p d\theta \leq m^p n^p \int_0^{2\pi} \left|S(e^{i\theta})\right|^p d\theta.$$

This gives

$$\int_0^{2\pi}\int_0^{2\pi} \left|S'(e^{i\theta}) + e^{i\zeta}\left\{S'(e^{i\theta}) - imnS(e^{i\theta})\right\}\right|^p d\theta d\zeta \leq 2\pi m^p n^p \int_0^{2\pi} \left|S(e^{i\theta})\right|^p d\theta. \quad (5.1)$$

Now

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \left|S'(e^{i\theta}) + e^{i\zeta}\left\{S'(e^{i\theta}) - imnS(e^{i\theta})\right\}\right|^p d\theta d\zeta \\
&= \int_0^{2\pi} \left|S'(e^{i\theta})\right|^p \int_0^{2\pi} \left|1 + e^{i\zeta}\left\{\frac{S'(e^{i\theta}) - imnS(e^{i\theta})}{S'(e^{i\theta})}\right\}\right|^p d\zeta d\theta \\
&= \int_0^{2\pi} \left|S'(e^{i\theta})\right|^p \int_0^{2\pi} \left|e^{i\zeta} + \left|\frac{S'(e^{i\theta}) - imnS(e^{i\theta})}{S'(e^{i\theta})}\right|\right|^p d\zeta d\theta \\
&= \int_0^{2\pi} \left|S'(e^{i\theta})\right|^p \int_0^{2\pi} \left|e^{i\zeta} + \left|\frac{R'(e^{i\theta})}{S'(e^{i\theta})}\right|\right|^p d\zeta d\theta,
\end{aligned}$$

where the function R is as defined in Lemma 4.2. With the help of Lemma 4.2, we have

$$\int_0^{2\pi}\int_0^{2\pi}\left|S'(e^{i\theta})+e^{i\zeta}\left\{S'(e^{i\theta})-imnS(e^{i\theta})\right\}\right|^p d\theta d\zeta \geq \int_0^{2\pi}\left|S'(e^{i\theta})\right|^p \int_0^{2\pi}\left|e^{i\zeta}+k^{\mu}\right|^p d\zeta d\theta. \tag{5.2}$$

Thus combining (5.1) and (5.2), we see that

$$\left(\int_0^{2\pi}\left|S'(e^{i\theta})\right|^p d\theta\right)\left(\int_0^{2\pi}\left|e^{i\zeta}+k^{\mu}\right|^p d\zeta\right) \leq 2\pi m^p n^p \int_0^{2\pi}\left|S(e^{i\theta})\right|^p d\theta.$$

Equivalently

$$\left(\int_0^{2\pi}\left|P'(Q(e^{i\theta}))Q'(e^{i\theta})\right|^p d\theta\right)\left(\int_0^{2\pi}\left|e^{i\zeta}+k^{\mu}\right|^p d\zeta\right) \leq 2\pi m^p n^p \int_0^{2\pi}\left|P(Q(e^{i\theta}))\right|^p d\theta.$$

Since all the zeros of $Q(z)$ lie in $|z| \leq 1$ and Q is of degree at most m, after using inequality (1.2) on the polynomial $Q(z)$ with $\min\limits_{|z|=1}|Q(z))| = A$, we have

$$\left(\int_0^{2\pi}\left|P'(Q(e^{i\theta}))\right|^p d\theta\right)\left(\int_0^{2\pi}\left|e^{i\zeta}+k^{\mu}\right|^p d\zeta\right) \leq \frac{2\pi n^p}{A^p}\int_0^{2\pi}\left|P(Q(e^{i\theta}))\right|^p d\theta.$$

Consequently,

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|P'(Q(e^{i\theta}))|^p d\theta\right)^{\frac{1}{p}} \leq \frac{n}{A\|k^{\mu}+z\|_p}\|P\circ Q\|_p.$$

This proves Theorem 3.1 for $0 < p < \infty$. The result holds for $p = 0$ and $p = \infty$ by letting $p \to 0^+$ and $p \to \infty$. □

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