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# EXCEPTIONAL VALUES OF $p$-ADIC DERIVATIVES A SURVEY WITH SOME IMPROVEMENTS

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*In memory of Abdelbaki Boutabaa*

Abstract. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field of characteristic 0 and let $f$ be a meromorphic function in $\mathbb{K}$ admitting primitives. We show that $f$ has no value taken finitely many times provided an additional hypothesis is satisfied: either $f$ has finitely many poles of order $\geq 3$, or $f$ has two perfectly branched values, or the logarithm of the number of poles in the disk of center 0 and diameter $r$ is bounded by $O(\text{Log}(r))$ $(r > 1)$. We make the conjecture: all additional hypotheses are superfluous.

## 1. Introduction and main results

Let $f$ be a complex transcendental meromorphic function that admits primitives. Thanks to the Nevanlinna theory, it is known that for $f$ there exists at most one value $b$ taken finitely many times [8]. Consider now a transcendental meromorphic function $f$ in an algebraically closed complete ultrametric field $\mathbb{K}$ of characteristic 0 [1], [9]. It is well known that a transcendental meromorphic function $f$ can admit at most one value $b$ taken finitely many times [7]. But suppose now that $f$ admits primitives. In this survey, we recall two hypotheses proving that $f$ admits no value $b$ taken finitely many times. In both hypotheses, we assume that $f$ admits primitives. This suggests that if a transcendental meromorphic function $f$ in the field $\mathbb{K}$ admits primitives, then $f$ has no value taken finitely many times.

Many important results are due to Jean-Paul Bézivin [2], [3], [4].

**Notation and definitions:** We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ (i.e. the field of functions of the form $\dfrac{f}{g}$, with $f,\ g \in \mathcal{A}(\mathbb{K})$).

Given two meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{K})$ we will denote by $W(f,g)$ the Wronskian of $f$ and $g$: $f'g - fg'$.

Given $f \in \mathcal{M}(\mathbb{K})$ and $b \in \mathbb{K}$, $b$ is called *an exceptional value for* $f$ if $f - b$ has no zero in $\mathbb{K}$ and *a quasi-exceptional value for* $f$ if $f - b$ has finitely many zeros in $\mathbb{K}$.

Here, Log is the Neperian logarithm and we denote by $e$ the number such that $\text{Log}(e) = 1$ and Exp is the Archimedean exponential function.

The following theorem is well known [7]:

**Theorem 0:** *Let $f \in \mathcal{M}(\mathbb{K})$. Then $f$ has at most one quasi-exceptional value in $\mathbb{K}$. Moreover, if $f \in \mathcal{A}(\mathbb{K})$, then $f$ has no quasi-exceptional value.*

The following theorem 1 is esential to prove the main results that follow.

**Theorem 1 [2]:** *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $W(f,g)$ is a non-identically zero polynomial. Then both $f,\ g$ are polynomials.*

**Remark:** In Archimedean analysis, Theorem 1 does not hold. For example, take $f(x) = Exp(x),\ g(x) = Exp(-x)$. Then $W(f,g) = 2$. We can also consider $f(x) = xExp(x),\ g(x) = Exp(-x)$. Then $W(f,g) = 2x+1$.

**Theorem 2:** *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have finitely many poles of order $\geq 3$ and admit primitives. Then $f$ has no quasi-exceptional value.*

**Corollary:** *Let $F \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have finitely many multiple poles. Then $F'$ has no quasi-exceptional value.*

**Definition:** Let $f \in \mathcal{M}(\mathbb{K})$ and $b \in \mathbb{K}$. Then $b$ is called *a perfectly branched value of $f$* if all zeros of $f-b$ are multiple except maybe finitely many. Moreover, $b$ is called *a totally branched value of $f$* [6] if all zeros of $f-b$ are multiple, without exception.

**Theorem 3:** *Let $f \in \mathcal{M}(\mathbb{K})$ admit primitives. If $f$ has two perfectly branched values then, $f$ has no quasi-exceptional value. Moreover, if $f$ has one totally branched value, then $f$ has no exceptional value.*

**Notation:** Let $f \in \mathcal{M}(d(0,R^-))$. For each $r \in ]0,R[$, we denote by $s(r,f)$ the number of zeros of $f$ in $d(0,r)$, each counted with its multiplicity and we set $t(r,f) = s(r,\frac{1}{f})$.

Let $f \in \mathcal{A}(\mathbb{K})$. We can factor $f$ in the form $\overline{f}\widetilde{f}$ where the zeros of $\overline{f}$ are the distinct zeros of $f$ each with order 1. Moreover, if $f(0) \neq 0$ we can take $\overline{f}(0) = 1$ and if $f(0) = 0$, we can take $\overline{f}$ so that $(\overline{f})'(0) = 1$.

**Theorem 4:** *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ admit primitives and also satisfy $\text{Log}(t(r,f)) \leq O(\text{Log}(r))$. Then $f$ has no quasi-exceptional value.*

**Example 1:** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $|a_n| \leq |a_{n+1}|$ and $\lim\limits_{n\to+\infty} |a_n| = +\infty$ and let $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{b_n}{(x-a_n)^{s_n}}$ with $|b_n| \leq 1,\ s_n\ \geq 2\ \forall n$ and $s_n = 2\ \forall n \geq t$. Then the function $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{b_n}{(x-a_n)^{s_n}}$ admits primitives and has no quasi-exceptional value by Theorem 2.

**Example 2:** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $|a_n| < |a_{n+1}|$ and $\lim_{n\to+\infty} |a_n| = +\infty$ and suppose that $\text{Log}(n) = O(\text{Log}|a_n|)$. Then the function $f(x) = \sum_{n=0}^{\infty} \frac{b_n}{(x-a_n)^{s_n}}$ with $|b_n| \leq 1,\ s_n \geq 2\ \forall n$, admits primitives and has no quasi-exceptional value by Theorem 4.

**Example 3:** Let $h \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ be a function having only zeros of order 1 and let $P(x) \in \mathbb{K}[x]$. Let $f(x) = \frac{P(x)}{(h(x))^2}$. Then $f$ has no primitive.

Indeed, suppose that $f$ has a primitive $F = \frac{U}{V}$ where $U$ and $V$ lie in $\mathcal{A}(\mathbb{K})$ and have no common zeros. Since the zeros of $h$ are of order 1, it is seen that all zeros of $V$ are of order 1 and are all the zeros of $h$. Consequently, $\widetilde{V} = 1$, $\overline{V} = V$ and $F' = \frac{U'V - UV'}{V^2}$ admits no simplification. Therefore $U'V - UV' = P$. But then, by Theorem 1, $U$ and $V$ are polynomials and $V^2 = h^2$, a contradiction to the hypothesis $h \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$.

**Remark:** In Example 3, the function $f$ certainly has residues different from 0 because if all residues were null, the function then would have primitives [7].

Now, by Theorems 2 , 3 and 4 the following conjecture appears likely:

**Conjecture:** *A transcendental meromorphic function in $\mathbb{K}$ admitting primitives has no quasi-exceptional value.*

## 2. The proofs

**Notation:** Let $f \in \mathcal{M}(\mathbb{K})$, let $a \in \mathbb{K}$ and let $r > 0$. Then $|f(x)|$ has a limit when $|x-a|$ tends to $r$ (while remaining different from $r$) which is denoted by $\varphi_{a,r}(f)$. Particularly, if $a = 0$ we put $\lim_{\substack{|x|\to r\\ |x|\neq r}} |f(x)| = |f|(r)$.

The following proposition 1 is well known in ultrametric analysis [7].

**Proposition 1:** *Let $f \in \mathcal{M}(\mathbb{K})$. For each $n \in \mathbb{N}$ and for all $r \in ]0, R[$, we have*

$$|f^{(n)}|(r) \leq |n!| \frac{|f|(r)}{r^n}.$$

**Proposition 2:** *Let $h,\ l \in \mathcal{A}(\mathbb{K})$ be such that $h'l - hl' = c \in \mathbb{K}$, with $h$ non-affine. Then $c = 0$ and $\frac{h}{l}$ is a constant.*

Suppose $c \neq 0$. If $h(a) = 0$, then $l(a) \neq 0$. Next, $h$ and $l$ satisfy

$$\frac{h''}{h} = \frac{l''}{l} \tag{1}$$

Remark first that since $h$ is not affine, $h''$ is not identically zero. Next, every zero of $h$ or $l$ of order $\geq 2$ is a trivial zero of $h'l-hl'$, which contradicts $c\neq 0$. So we can assume that all zeros of $h$ and $l$ are of order 1.

Now suppose that a zero $a$ of $h$ is not a zero of $h''$. Since $a$ is a zero of $h$ of order 1, $\frac{h''}{h}$ has a pole of order 1 at $a$ and so does $\frac{l''}{l}$, hence $l(a)=0$, a contradiction. Consequently, each zero of $h$ is a zero of order 1 of $h$ and is a zero of $h''$ and hence, $\frac{h''}{h}$ is an element $\phi$ of $\mathcal{M}(\mathbb{K})$ that has no pole in $\mathbb{K}$. Therefore $\phi$ lies in $\mathcal{A}(\mathbb{K})$.

The same holds for $l$ and so, $l''$ is of the form $\psi l$ with $\psi\in\mathcal{A}(\mathbb{K})$. But since $\frac{h''}{h}=\frac{l''}{l}$, we have $\phi=\psi$.

Now, suppose $h,\ l$ belong to $\mathcal{A}(\mathbb{K})$. Since $h''$ is of the form $\phi h$ with $\phi\in\mathcal{A}(\mathbb{K})$, we have $|h''|(r)=|\phi|(r)|h|(r)$. But by Proposition 1, we know that $|h''|(r)\leq\frac{1}{r^2}|h|(r)$, a contradiction when $r$ tends to $+\infty$. Consequently, $c=0$. But then $h'l-hl'=0$ implies that the derivative of $\frac{h}{l}$ is identically zero, hence $\frac{h}{l}$ is constant, which ends the proof.

**Corollary 2.a :** *Let $h,\ l\in\mathcal{A}(\mathbb{K})$ with coefficients in $\mathbb{Q}$, also be entire functions in $\mathbb{C}$, with $h$ non-affine. If $h'l-hl'$ is a constant $c$, then $c=0$.*

**Proposition 3:** *Let $\psi\in\mathcal{M}(\mathbb{K})$ and let $(\mathcal{E})$ be the differential equations $y^{(n)}-\psi y=0$. Let $E$ be the sub-vector space of $\mathcal{M}(\mathbb{K})$ of the solutions of $(\mathcal{E})$.*
*If $n=1$, then the dimension of $E$ is at most 1.*
*If $\psi$ belongs to $\mathcal{A}(\mathbb{K})$, then $E=\{0\}$.*

*Proof.* In each case, we assume that $(\mathcal{E})$ admits a non-identically zero solution $h$. Then $h^{(n)}$ may not be identically zero.

Suppose first that $n=1$. Suppose that $g\in E$. Let $u=\frac{h}{g}$. Since $h'=\psi h$ we have $u'g+ug'=\psi ug$ therefore $u\frac{g'}{g}=u\psi=u'+u\frac{g'}{g}$ and hence $u'=0$ i.e. $u$ is a constant. Consequently, $E$ is at most of dimension 1.

Suppose now that $\psi$ lies in $\mathcal{A}(\mathbb{K})$. Then $|\psi|(r)=\frac{|h^{(n)}|(r)}{|h|(r)}$ is an increasing function in $r$ in $]0,+\infty[$, a contradiction to the inequality $\frac{|h^{(n)}|(r)}{|h|(r)}\leq\frac{1}{r^n}$ coming from Proposition 1. $\square$

**Proof of Theorem 1 [2]**

First, by Proposition 2 we check that the claim is satisfied when $W(f,g)$ is a polynomial of degree 0. Now, suppose the claim holds when $W(f,g)$ is a polynomial of certain degree $n$. We will show it for $n+1$. Let $f,g \in \mathcal{A}(\mathbb{K})$ be such that $W(f,g)$ is a non-identically zero polynomial $P$ of degree $n+1$

Thus, by the hypothesis, we have $f'g - fg' = P$, hence $f"g - fg" = P'$. We can extract $g'$ and get $g' = \frac{(f'g-P)}{f}$. Now consider the function $Q = f"g' - f'g"$ and replace $g'$ by what we just found: we can get $Q = f'(\frac{(f"g-fg")}{f}) - \frac{Pf"}{f}$.

Now, we can replace $f"g - fg"$ by $P'$ and obtain $Q = \frac{(f'P'-Pf")}{f}$. Thus, in that expression of $Q$, we can write $|Q|(R) \leq \dfrac{|f|(R)|P|(R)}{R^2|f|(R)}$, hence $|Q|(R) \leq \frac{|P|(R)}{R^2}$ $\forall R > 0$. But by definition, $Q$ belongs to $\mathcal{A}(\mathbb{K})$. Consequently, $Q$ is a polynomial of degree $t \leq n-1$.

Now, suppose $Q$ is not identically zero. Since $Q = W(f', g')$ and since $\deg(Q) < n$, by the induction hypothesis $f'$ and $g'$ are polynomials and so are $f,g$. Finally, suppose $Q = 0$. Then $P'f' - Pf" = 0$ and therefore $f', P$ are two solutions of the differential equation of order 1 for meromorphic functions in $\mathbb{K}:$ $(\mathcal{E})$ $y' = \psi y$ with $\psi = \frac{P'}{P}$, whereas $y$ belongs to $\mathcal{A}(\mathbb{K})$. By Proposition 3, the space of solutions of $(\mathcal{E})$ is of dimension 0 or 1. Consequently, there exists $\lambda \in \mathbb{K}$ such that $f' = \lambda P$, hence $f$ is a polynomial. The same holds for $g$. This ends the proof of Theorem 1.

**Proposition 4:** *Let $U,V \in \mathcal{A}(\mathbb{K})$ have no common zero and let $f = \dfrac{U}{V}$. If $f'$ has finitely many zeros, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\widetilde{V}$.*

*Proof.* If $V$ is a constant, the statement is obvious. So, we assume that $V$ is not a constant. Now $\widetilde{V}$ divides $V'$ and hence $V'$ factorizes in the way $V' = \widetilde{V}Y$ with $Y \in \mathcal{A}(\mathbb{K})$. Then no zero of $Y$ can be a zero of $V$. Consequently, we have

$$f'(x) = \frac{U'V - UV'}{V^2} = \frac{U'\overline{V} - UY}{\overline{V}^2\widetilde{V}}.$$

The two functions $U'\overline{V} - UY$ and $\overline{V}^2\widetilde{V}$ have no common zero since neither have $U$ and $V$. So, the zeros of $f'$ are those of $U'\overline{V} - UY$ which therefore has finitely many zeros and consequently is a polynomial $P$, hence $U'V - UV' = P\widetilde{V}$. □

**Proof of Theorem 2:**

*Proof.* Suppose that $f$ admits a quasi-exceptional value. Without loss of generality, we can assume that this value is 0. Let $F$ be a primitive of $f$ and let $F = \dfrac{U}{V}$, with $U$, $V \in \mathcal{A}(\mathbb{K})$, having no common zero. By Proposition 4, there exists a polynomial $P$ such that $U'V - UV' = P\widetilde{V}$. But since $f$ has finitely many poles of order $\geq 3$, $F$ has finitely many poles of order $\geq 2$ hence $\widetilde{V}$ has finitely many zeros, hence it is a polynomial. But then $P\widetilde{V}$ is a polynomial and then, by Theorem 1, both $U$, $V$ are polynomials, therefore $F \in \mathbb{K}(x)$ a contradiction. □

**Notation:** Given $r>0$, we denote by $d(0,r)$ the disk $\{x\in\mathbb{K}\mid |x|\leq r\}$. Given $f\in\mathcal{M}(\mathbb{K})$, we denote by $Z(r,f)$ the counting function of the zeros of $f$ in the disk $d(0,r)$, counting multiplicity, and by $\overline{Z}(r,f)$ the counting function of the zeros of $f$ in the disk $d(0,r)$, ignoring multiplicity. Next we put $N(r,f)=Z(r,\frac{1}{f})$, $T(r,f)=\max(Z(r,f),N(r,f))$ and $\overline{N}(r,f)=\overline{Z}(r,\frac{1}{f})$.

Let us now recall a simplified version of the Second Main Theorem [5], [7]:

**Second Main Theorem:** *Let $f\in\mathcal{M}(\mathbb{K})$ and let $\alpha_1,\dots,\alpha_q\in\mathbb{K}$, with $q\geq 2$. Then* $(q-1)T(r,f)\leq\sum_{j=1}^{q}\overline{Z}(r,f-\alpha_j)+\overline{N}(r,f)-\log r+O(1)\ \forall r\in I.$

**Proof of Theorem 3** Suppose that $f$ has two perfectly branched values $a$ and $b$ and a quasi-exceptional value $c$. Since $f$ admits primitives, $N(r,f)$ satisfies $\overline{N}(r,f)\leq\frac{N(r,f)}{2}+o(T(r,f))$ hence by the second Main Theorem, we have

$$2T(r,f)\leq\frac{(Z(r,f-a)+Z(r,f-b)+N(r,f))}{2}+o(T(r,f))$$

hence $2T(r,f)\leq\frac{3T(r,f)}{2}+o(T(r,f))$, a contradiction.

Suppose now that $f$ has one totally branched values $a$ and an exceptional value $c$. Since $f$ admits primitives, by the second Main Theorem, now we have

$$T(r,f)\leq\frac{Z(r,f-a)+N(r,f)}{2}-\log(r)+O(1)$$

hence $T(r,f)\leq\frac{2T(r,f)}{2}-\log(r)+O(1)$, a contradiction.

**Notation:** For each $n\in\mathbb{N}^*$, we set $\lambda_n=\max\{\frac{1}{|k|},\ 1\leq k\leq n\}$. Given positive integers $n,\ q$, we denote by $C_n^q$ the binomial coefficient $\dfrac{n!}{q!(n-q)!}$.

**Remark:** For every $n\in\mathbb{N}^*$, we have $\lambda_n\leq n$ because $k|k|\geq 1\ \forall k\in\mathbb{N}$. The equality holds for all $n$ of the form $p^h$.

**Proposition 5:** *Let $U,\ V\in\mathcal{A}(d(0,R^-))$. Then for all $r\in]0,R[$ and $n\geq 1$ we have*

$$|U^{(n)}V-UV^{(n)}|(r)\leq|n!|\lambda_n\frac{|U'V-UV'|(r)}{r^{n-1}}.$$

*More generally, given $j,\ l\in\mathbb{N}$, we have*

$$|U^{(j)}V^{(l)}-U^{(l)}V^{(j)}|(r)\leq|(j!)(l!)|\lambda_{j+l}\frac{|U'V-UV'|(r)}{r^{j+l-1}}.$$

*Proof.* Set $g = \dfrac{U}{V}$ and $f = g'$. Applying Proposition 1 to $f$ for $k-1$, we obtain

$$|g^{(k)}|(r) = |f^{(k-1)}|(r) \le |(k-1)!|\frac{|f|(r)}{r^{k-1}} = |(k-1)!|\frac{|U'V - UV'|(r)}{|V^2|(r)r^{k-1}}.$$

As in the proof of Proposition 1, we set $U = V\big(\dfrac{U}{V}\big)$. By Leibniz formula again, now we can obtain

$$U^{(n)} = \sum_{q=1}^{n} C_n^q V^{(n-q)}\Big(\frac{U}{V}\Big)^{(q)} + V^{(n)}\Big(\frac{U}{V}\Big)$$

hence

$$U^{(n)} - V^{(n)}\Big(\frac{U}{V}\Big) = \sum_{q=1}^{n} C_n^q V^{(n-q)}\Big(\frac{U}{V}\Big)^{(q)}. \qquad (1)$$

Now we have

$$\Big|\Big(\frac{U}{V}\Big)^{(q)}\Big|(r) = |g^{(q)}|(r) \le |(q-1)!|\frac{|U'V - UV'|(r)}{|V^2|(r)r^{q-1}}$$

and

$$|V^{(n-q)}|(r) \le |(n-q)!|\frac{|V|(r)}{r^{n-q}}.$$

Consequently, the general term in (1) is upper bounded as

$$\Big|C_n^q V^{(n-q)}\Big(\frac{U}{V}\Big)^{(q)}\Big|(r) \le \frac{|(n!)((n-q)!)((q-1)!)|}{|(q!)((n-q)!)|}\frac{|U'V - UV'|(r)}{|V|(r)r^{n-1}} \le \lambda_n\frac{|n!||U'V - UV'|(r)}{|V|(r)r^{n-1}}.$$

Therefore by (1) we obtain

$$\Big|U^{(n)} - V^{(n)}\Big(\frac{U}{V}\Big)\Big|(r) \le |n!|\lambda_n\frac{|U'V - UV'|(r)}{|V|(r)r^{n-1}}$$

and finally

$$\Big|U^{(n)}V - V^{(n)}U\Big|(r) \le |n!|\lambda_n\frac{|U'V - UV'|(r)}{r^{n-1}}.$$

We can now generalize the first statement. Set $P_j = U^{(j)}V - UV^{(j)}$. By induction, we can show the following equality that already holds for $l \le j$:

$$U^{(j)}V^{(l)} - U^{(l)}V^{(j)} = \sum_{h=0}^{l} C_l^h(-1)^h P_{j+h}^{(l-h)}.$$

Then, the second statement follows by applying the first. □

**Proposition 6:** *Let $U,V \in \mathcal{A}(\mathbb{K})$ and let $r,\ R \in ]0,+\infty[$ satisfy $r < R$. For all $x,y \in \mathbb{K}$ with $|x| \le R$ and $|y| \le r$, we have the inequality:*

$$|U(x+y)V(x) - U(x)V(x+y)| \le \frac{R|U'V - UV'|(R)}{e(\mathrm{Log}R - \mathrm{Log}r)}.$$

*Proof.* By Taylor's formula at the point $x$, we have

$$U(x+y)V(x) - U(x)V(x+y) = \sum_{n\ge 0} \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n.$$

Now, by Proposition 5, we have

$$\Big|\frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n\Big| \le \lambda_n \frac{|U'V - UV'|(R)}{R^{n-1}} r^n$$
$$= \lambda_n R|U'V - UV'|(R)(\frac{r}{R})^n.$$

As remarked above, we have $\lambda_n \le n$. Hence one has

$$\lim_{n\to+\infty} \lambda_n \Big(\frac{r}{R}\Big)^n = 0.$$

Consequently, on one hand $\lim\limits_{n\to+\infty} \Big|\frac{U^{(n)}(x)V(x) - U(x)V^{(n}(x)}{n!} y^n\Big| = 0$, on the other hand, we can define $B = \max_{n\ge 1}\{\lambda_n \Big(\frac{r}{R}\Big)^n\} R|U'V - UV'|(R)$ and we have $|U(x+y)V(x) - U(x)V(x+y)| \le B$. Now, we can check that the function $h$ defined in $]0,+\infty[$ as $h(t) = t\Big(\frac{r}{R}\Big)^t$ reaches its maximum at the point $u = \dfrac{1}{e(\mathrm{Log}R - \mathrm{Log}r)}$. Consequently, $B \le \dfrac{1}{e(\mathrm{Log}R - \mathrm{Log}r)}$ and therefore

$$|U(x+y)V(x) - U(x)V(x+y)| \le \frac{R|U'V - UV'|(R)}{e(\mathrm{Log}R - \mathrm{Log}r)}. \qquad \square$$

**Notation:** Let $D = d(a,s)$ and let $H(D)$ be the $\mathbb{K}$-algebra of analytic elements on $d(a,s)$, i.e. the $\mathbb{K}$-Banach space of converging power series converging in $d(a,s)$ [9]. Given $b \in d(a,s)$ and $r \in ]0,s]$, then $|f(x)|$ has a limit whenever $|x-b|$ tends to $r$, with $|x-b| \ne r$ and we denote by $\varphi_{b,r}(f)$ the number $\lim\limits_{\substack{|x-b|tor,\\ |x-b|\ne r}} |f(x)|$ [6], [7].

Given $f \in \mathcal{M}(\mathbb{K})$ and $r > 0$, we denote by $s(r,f)$ the number of zeros of $f$ in the disk $d(0,r)$, each counted with its multiplicity and we put $t(r,f) = s(r,\frac{1}{f})$.

Finally we denote by $\beta(r,f)$ the number of multiple poles of $f$, each counted with its multiplicity.

**Schwarz Lemma** [6] *Let $D = d(a,s)$ and let $f$ be a power series converging in the disk $d(a,s)$ and having at least (resp. at most) $q$ zeros in $d(a,r)$ with $q > 0$ and $0 < r < s$. Then we have $\frac{\varphi_{a,s}(f)}{\varphi_{a,r}(f)} \geq \left(\frac{s}{r}\right)^q$, (resp. $\frac{\varphi_{a,s}(f)}{\varphi_{a,r}(f)} \leq \left(\frac{s}{r}\right)^q$).*

**Schwarz Corollary:** *Let $f \in \mathcal{A}(\mathbb{K})$. The following two statements are equivalent:*
*$f$ is a polynomial of degree $q$,*
*there exists $q \in \mathbb{N}$ such that $\frac{|f|(r)}{r^q}$ has a finite limit when $r$ tends to $+\infty$.*

**Proposition 7:** *Let $f \in \mathcal{M}(\mathbb{K})$ be such that for some $c, q \in ]0,+\infty[$, $t(r,f)$ satisfies $t(r,f) \leq cr^q$ in $[1,+\infty[$. If $f'$ has finitely many zeros, then $f \in \mathbb{K}(x)$ .*

*Proof.* Suppose $f'$ has finitely many zeros and set $f = \frac{U}{V}$. If $V$ is a constant, the statement is immediate. So, we suppose $V$ is not a constant and hence it admits at least one zero $a$. By Proposition 4, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\widetilde{V}$. Next, we take $r, R \in [1,+\infty[$ such that $|a| < r < R$ and $x \in d(0,R),\ y \in d(0,r)$. By Proposition 5 we have

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\mathrm{Log}R - \mathrm{Log}r)}.$$

Notice that $U(a) \neq 0$ because $U$ and $V$ have no common zero. Now set $l = \max(1,|a|)$ and take $r \geq l$. Putting $c_1 = \frac{1}{e|U(a)|}$, we have

$$|V(a+y)| \leq c_1 \frac{R|P|(R)|\widetilde{V}|(R)}{\mathrm{Log}R - \mathrm{Log}r}.$$

Then taking the supremum of $|V(a+y)|$ inside the disk $d(0,r)$, we can derive

$$|V|(r) \leq c_1 \frac{R|P|(R)|\widetilde{V}|(R)}{\mathrm{Log}R - \mathrm{Log}r}. \tag{1}$$

Let us apply Schwarz Lemma, by taking $R = r + \frac{1}{r^q}$, after noticing that the number of zeros of $\widetilde{V}(R)$ is bounded by $s(r,V)$. So, we have

$$|\widetilde{V}|(R) \leq \left(1 + \frac{1}{r^{q+1}}\right)^{\beta((r+\frac{1}{r^q}),V)} |\widetilde{V}|(r). \tag{2}$$

Now, due to the hypothesis: $s(r,V) = t(r,f) \leq cr^q$ in $[1,+\infty[$, we have

$$\left(1 + \frac{1}{r^{q+1}}\right)^{\beta((r+\frac{1}{r^q}),V)} \leq \left(1 + \frac{1}{r^{q+1}}\right)^{[c(r+\frac{1}{r^q})^m]} = \tag{3}$$

$$\mathrm{Exp}\left[c(r + \frac{1}{r^q})^q \mathrm{Log}(1 + \frac{1}{r^{q+1}})\right].$$

The function $h(r)=c(r+\frac{1}{r^m})^m\text{Log}(1+\frac{1}{r^{m+1}})$ is continuous on $]0,+\infty[$ and equivalent to $\dfrac{c}{r}$ when $r$ tends to $+\infty$. Consequently, it is bounded on $[l,+\infty[$. Therefore, by (2) and (3) there exists a constant $M>0$ such that, for all $r\in[l,+\infty[$ by (3) we obtain

$$|\widetilde{V}|(r+\frac{1}{r^q})\leq M|\widetilde{V}|(r). \tag{4}$$

On the other hand,

$$\text{Log}\Big(r+\frac{1}{r^q}\Big)-\text{Log}r=\text{Log}\Big(1+\frac{1}{r^{q+1}}\Big)$$

clearly satisfies an inequality of the form

$$\text{Log}\Big(1+\frac{1}{r^{q+1}}\Big)\geq\frac{c_2}{r^{q+1}}$$

in $[l,+\infty[$ with $c_2>0$. Moreover, we can obviously find positive constants $c_3,\ c_4$ such that

$$(r+\frac{1}{r^q})|P|\Big(r+\frac{1}{r^q}\Big)\leq c_3r^{c_4}.$$

Consequently, by (1) and (4) we can find positive constants $c_5,\ c_6$ such that $|V|(r)\leq c_5r^{c_6}|\widetilde{V}|(r)\ \forall r\in[l,+\infty[$. Thus, writing again $V=\overline{V}\widetilde{V}$, we have $|\overline{V}|(r)|\widetilde{V}|(r)\leq c_5r^{c_6}|\widetilde{V}|(r)$ and hence

$$|\overline{V}|(r)\leq c_5r^{c_6}\ \forall r\in[l,+\infty[.$$

Consequently, by Schwarz Corollary $\overline{V}$ is a polynomial of degree $\leq c_6$ and hence it has finitely many zeros and so does $V$. But then, by Theorem 2, $f$ must be a rational function. □

**Corollary 7.a:** *Let $f$ be a meromorphic function on $\mathbb{K}$ such that, for some $c,q\in]0,+\infty[$, $t(r,f)$ satisfies $t(r,f)\leq cr^q$ in $[1,+\infty[$. If for some $b\in\mathbb{K}$ $f'-b$ has finitely many zeros, then $f$ is a rational function.*

*Proof.* Suppose $f'-b$ has finitely many zeros. Then $f-bx$ satisfies the same hypothesis as $f$, hence it is a rational function and so is $f$. □

Theorem 4 is now a simple corollary of Corollary 7.a:

**Proof of Theorem 4**

*Proof.* Indeed, since $f$ admits primitives, all poles are multiple, and given a primitive $F$ of $f$, we have $t(r,F)\leq t(r,f)$. Consequently, by the hypothesis we have $\text{Log}(t(r,F))\leq O(\text{Log}(r))$ and hence, thanks to Corollary 7.a, $F'$ has no quasi-exceptional value. □

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