FG-COUPLED FIXED POINT THEOREMS FOR CONTRACTIVE AND GENERALIZED QUASI-CONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, we prove FG-coupled fixed point theorems for different contractive mappings and generalized quasi- contractive mappings in partially ordered complete metric spaces. We prove the existence of FG-coupled fixed points of continuous as well as discontinuous mappings. Our first three results generalize the theorems of Gnana Bhaskar and Lakshmikantham[T. Gnana Bhasakar, V. Lakshmikantham; Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7) (2006) 1379-1393]. We give some examples to illustrate the results.

1. INTRODUCTION

Coupled fixed point problems became a new trend in non-linear analysis as a generalization of fixed point theory. In 1987 Dajun Guo and V. Lakshmikantham [6] introduced the concept of a coupled fixed point and proved some coupled fixed point theorems for operators satisfying the mixed monotone property. In [12] Ran and Reurings proved fixed point theorems for contractive mappings in partially ordered metric spaces. These theorems are very important in the sense that contractive conditions are not satisfied for all elements in the space, but only for elements which are comparable by the partial order. As an extension of this result Gnana Bhaskar and Lakshmikantham [5] proved coupled fixed point theorems in partially ordered metric space using the mixed monotone property. Thereafter by changing the spaces and using different contractions several authors have proved various coupled fixed point theorems [1, 2, 7, 8, 11–14].

This paper is focused on FG-coupled fixed points which are recently defined in [9] as a generalization of coupled fixed point in Cartesian product of two different spaces. Here we emphasize the existence of FG- coupled fixed points for continuous as well as discontinuous mappings. In section 2 we prove existence and uniqueness of FG-coupled fixed point theorems for contractive type mappings

and in section 3 we prove existence theorems for generalized quasi-contractive mappings in partially ordered metric spaces. Now we recall some definitions.

Definition 1.1 ([5]). *An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the map $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.*

Definition 1.2 ([9]). *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$. We say that F and G have the mixed monotone property if for any $x, y \in X$*
$x_1, x_2 \in X$, $x_1 \leq_{P_1} x_2$ implies $F(x_1, y) \leq_{P_1} F(x_2, y)$ and $G(y, x_1) \geq_{P_2} G(y, x_2)$
$y_1, y_2 \in Y$, $y_1 \leq_{P_2} y_2$ implies $F(x, y_1) \geq_{P_1} F(x, y_2)$ and $G(y_1, x) \leq_{P_2} G(y_2, x)$.

Definition 1.3 ([9]). An element $(x,y) \in X \times Y$ is said to be a FG- coupled fixed point if $F(x,y) = x$ and $G(y,x) = y$.

Note 1.1. If $(x,y) \in X \times Y$ is an FG- coupled fixed point then $(y,x) \in Y \times X$ is GF- coupled fixed point.

Note 1.2. The metric d on $X \times Y$ is defined by $d((x,y),(u,v)) = d_X(x,u) + d_Y(y,v)$ for all $(x,y),(u,v) \in X \times Y$.

Note 1.3. The partial order $\leq$ on $X \times Y$ is defined by for any $(x,y),(u,v) \in X \times Y$; $(u,v) \leq (x,y)$ if and only if $x \geq_{P_1} u$, $y \leq_{P_2} v$.

Note 1.4. $F^{n+1}(x,y) = F(F^n(x,y), G^n(y,x))$ and $G^{n+1}(y,x) = G(G^n(y,x), F^n(x,y))$ for every $n \in \mathbb{N}$ and $(x,y) \in X \times Y$.

2. FG-COUPLED FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS

Theorem 2.1. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two continuous mappings having the mixed monotone property. Assume that there exist $k, l, m, n \in [0,1); k + l < 1$ and $m + n < 1$ with*

$$d_X(F(x,y), F(u,v)) \leq k \, d_X(x,u) + l \, d_Y(y,v) \; \forall x \geq_{P_1} u, \; y \leq_{P_2} v \qquad (2.1)$$

$$d_Y(G(y,x), G(v,u)) \leq m \, d_Y(y,v) + n \, d_X(x,u) \; \forall x \leq_{P_1} u, \; y \geq_{P_2} v. \qquad (2.2)$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exists $(x,y) \in X \times Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. Suppose $x_0 \leq_{P_1} F(x_0, y_0) = x_1$ (say) and $y_0 \geq_{P_2} G(y_0, x_0) = y_1$ (say).
Define $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(y_n, x_n)$ for $n = 1, 2, 3...$
Then we get $F^{n+1}(x_0, y_0) = x_{n+1}$ and $G^{n+1}(y_0, x_0) = y_{n+1}$.
Using mathematical induction and the mixed monotone property of F and G we prove that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y.
For, suppose $x_0 \leq_{P_1} x_1$ and $y_0 \geq_{P_2} y_1$. We claim that $x_n \leq_{P_1} x_{n+1}$ and $y_n \geq_{P_2} y_{n+1}$ $\forall n \in \mathbb{N}$.

For $n = 1$, $x_2 = F(x_1, y_1) \geq_{P_1} F(x_0, y_1) \geq_{P_1} F(x_0, y_0) = x_1$ and
$y_2 = G(y_1, x_1) \leq_{P_2} G(y_1, x_0) \leq_{P_2} G(y_0, x_0) = y_1$.
Assume that the result is true for n=m. ie, $x_{m+1} \geq_{P_1} x_m$ and $y_m \geq_{P_2} y_{m+1}$.
Now consider,
$x_{m+2} = F(x_{m+1}, y_{m+1}) \geq_{P_1} F(x_m, y_{m+1}) \geq_{P_1} F(x_m, y_m) = x_{m+1}$
$y_{m+2} = G(y_{m+1}, x_{m+1}) \leq_{P_2} G(y_{m+1}, x_m) \leq_{P_2} G(y_m, x_m) = y_{m+1}$.
So the result is true for $\forall n \in \mathbb{N}$.
Now, to prove the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy, we consider two cases.
Case 1: $m + n \leq k + l$ We claim that, for $j \in \mathbb{N}$,

$$d_X(F^{j+1}(x_0, y_0), F^j(x_0, y_0)) \leq (k+l)^j [d_X(x_1, x_0) + d_Y(y_1, y_0)] \qquad (2.3)$$

$$d_Y(G^{j+1}(y_0, x_0), G^j(y_0, x_0)) \leq (k+l)^j [d_Y(y_1, y_0) + d_X(x_1, x_0)]. \qquad (2.4)$$

The proof is by mathematical induction.
For $j = 1$, consider

$$\begin{aligned}
d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\
&\leq k \, d_X(F(x_0, y_0), x_0) + l \, d_Y(G(y_0, x_0), y_0) \\
&\leq (k+l) \, [d_X(x_1, x_0) + d_Y(y_1, y_0)].
\end{aligned}$$

Similarly we prove that $d_Y(G^2(y_0, x_0), G(y_0, x_0)) \leq (k+l)[d_X(x_1, x_0) + d_Y(y_1, y_0)]$
ie, the result is true for $j = 1$.
Now assume that the claim is true for $j \leq t$, and prove it for $j = t + 1$.
Consider,

$$\begin{aligned}
&d_X(F^{t+2}(x_0, y_0), F^{t+1}(x_0, y_0)) \\
&= d_X(F(F^{t+1}(x_0, y_0), G^{t+1}(y_0, x_0)), F(F^t(x_0, y_0), G^t(y_0, x_0))) \\
&\leq k \, d_X(F^{t+1}(x_0, y_0), F^t(x_0, y_0)) + l \, d_Y(G^{t+1}(y_0, x_0), G^t(y_0, x_0)) \\
&\leq k(k+l)^t [d_X(x_1, x_0) + d_Y(y_1, y_0)] + l \, (k+l)^t [d_X(x_1, x_0) + d_Y(y_1, y_0)] \\
&= (k+l)^{t+1} [d_X(x_1, x_0) + d_Y(y_1, y_0)].
\end{aligned}$$

Similarly we get $d_Y(G^{t+2}(y_0, x_0), G^{t+1}(y_0, x_0)) \leq (k+l)^{t+1}[d_X(x_1, x_0) + d_Y(y_1, y_0)]$.
Thus the claim is true for all $j \in \mathbb{N}$.
Now we prove that $\{F^j(x_0, y_0)\}$ and $\{G^j(y_0, x_0)\}$ are Cauchy sequences in X and
Y respectively.
For $t > j$, consider

$$\begin{aligned}
&d_X(F^j(x_0, y_0), F^t(x_0, y_0)) \\
&\leq d_X(F^j(x_0, y_0), F^{j+1}(x_0, y_0)) + d_X(F^{j+1}(x_0, y_0), F^{j+2}(x_0, y_0)) + \ldots \\
&\quad + d_X(F^{t-1}(x_0, y_0), F^t(x_0, y_0)) \\
&\leq \left[(k+l)^j + (k+l)^{j+1} + \ldots + (k+l)^{t-1} \right] [d_X(x_1, x_0) + d_Y(y_1, y_0)]
\end{aligned}$$

$$\leq \frac{{\delta_1}^j}{1-\delta_1}\left[d_X(x_1,x_0)+d_Y(y_1,y_0)\right] \text{ where } \delta_1 = k+l < 1$$
$$\to 0 \text{ as } j \to +\infty.$$

That is, $\{F^j(x_0,y_0)\}$ is a Cauchy sequence in X.

Similarly we can prove that $\{G^j(y_0,x_0)\}$ is a Cauchy sequence in Y.

Case 2: $k+l < m+n$.

Now we claim that

$$d_X(F^{j+1}(x_0,y_0),F^j(x_0,y_0)) < (m+n)^j[d_X(x_1,x_0)+d_Y(y_1,y_0)] \qquad (2.5)$$

$$d_Y(G^{j+1}(y_0,x_0),G^j(y_0,x_0)) < (m+n)^j[d_Y(y_1,y_0)+d_X(x_1,x_0)]. \qquad (2.6)$$

For $j = 1$

$$\begin{aligned}
d_X(F^2(x_0,y_0),F(x_0,y_0)) &= d_X(F(F(x_0,y_0),G(y_0,x_0)),F(x_0,y_0))\\
&\leq k\, d_X(F(x_0,y_0),x_0)+l\, d_Y(G(y_0,x_0),y_0)\\
&\leq (k+l)\,[d_X(x_1,x_0)+d_Y(y_1,y_0)]\\
&< (m+n)\,[d_X(x_1,x_0)+d_Y(y_1,y_0)].
\end{aligned}$$

Similarly we can get $d_Y(G^2(y_0,x_0),G(y_0,x_0)) < (m+n)[d_Y(y_1,y_0)+d_X(x_1,x_0)]$.

Assume that the claim is true for $j \leq t$. Now we prove the claim for $j = t+1$.

Consider,

$$\begin{aligned}
&d_X(F^{t+2}(x_0,y_0),F^{t+1}(x_0,y_0))\\
&= d_X(F(F^{t+1}(x_0,y_0),G^{t+1}(y_0,x_0)),F(F^t(x_0,y_0),G^t(y_0,x_0)))\\
&\leq k\, d_X(F^{t+1}(x_0,y_0),F^t(x_0,y_0))+l\, d_Y(G^{t+1}(y_0,x_0),G^t(y_0,x_0))\\
&< k(m+n)^t[d_X(x_1,x_0)+d_Y(y_1,y_0)]+l\,(m+n)^t[d_X(x_1,x_0)+d_Y(y_1,y_0)]\\
&< (m+n)^{t+1}[d_X(x_1,x_0)+d_Y(y_1,y_0)].
\end{aligned}$$

Similarly we can prove that
$d_Y(G^{t+2}(y_0,x_0),G^{t+1}(y_0,x_0)) < (m+n)^{t+1}[d_Y(y_1,y_0)+d_X(x_1,x_0)]$. Hence the claim is true for all $j \in \mathbb{N}$.

Now we prove that $\{F^j(x_0,y_0)\}$ and $\{G^j(y_0,x_0)\}$ are Cauchy sequences in X and Y respectively.

For $t > j$, consider

$$\begin{aligned}
&d_X(F^t(x_0,y_0),F^j(x_0,y_0))\\
&\leq d_X(F^j(x_0,y_0),F^{j+1}(x_0,y_0))+d_X(F^{j+1}(x_0,y_0),F^{j+2}(x_0,y_0))+...\\
&\quad +d_X(F^{t-1}(x_0,y_0),F^t(x_0,y_0))\\
&< \left[(m+n)^j+(m+n)^{j+1}+...+(m+n)^{t-1}\right][d_X(x_1,x_0)+d_Y(y_1,y_0)]
\end{aligned}$$

$$\leq \frac{\delta_2^j}{1 - \delta_2}[d_X(x_1, x_0) + d_Y(y_1, y_0)] \text{ where } \delta_2 = m + n < 1$$
$$\to 0 \text{ as } j \to +\infty.$$

So, $\{F^j(x_0, y_0)\}$ is a Cauchy sequence in X.
Similarly we can prove that $\{G^j(y_0, x_0)\}$ is a Cauchy sequence in Y.
Since X and Y are complete metric spaces, we have $\lim_{j \to +\infty} F^j(x_0, y_0) = x$ and $\lim_{j \to +\infty} G^j(y_0, x_0) = y$, where $x \in X$ and $y \in Y$.
Now we prove that $F(x, y) = x$ and $G(y, x) = y$.
Consider,

$$\begin{aligned}
d_X(F(x, y), x) &= \lim_{j \to +\infty} d_X(F(F^j(x_0, y_0), G^j(y_0, x_0)), F^j(x_0, y_0)) \\
&= \lim_{j \to +\infty} d_X(F^{j+1}(x_0, y_0), F^j(x_0, y_0)) \\
&= 0.
\end{aligned}$$

Therefore $F(x, y) = x$. Similarly we can prove that $G(y, x) = y$. $\qquad \square$

Corollary 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist $k, l \in [0, 1)$, $k + l < 1$ with*

$$d(F(x, y), F(u, v)) \leq k\, d(x, u) + l\, d(y, v) \quad \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. Taking $X = Y$ and $F = G$ in Theorem 2.1 we obtain the result. $\qquad \square$

In the above corollary if we replace k and l by $\dfrac{k}{2}$ we get the results of Gnana Bhaskar and Lakshmikantham.

Corollary 2.2. *[5, Theorem 2.1] Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}\, [d(x, u) + d(y, v)] \quad \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

The following example illustrates Theorem 2.1

Example 2.1. *Let $X = (-\infty, 0]$ and $Y = [0, +\infty)$ with the usual order and the usual metric. Define $F : X \times Y \to X$ by $F(x,y) = \dfrac{x}{3} - \dfrac{y}{4}$ and $G : Y \times X \to Y$ by $G(y,x) = \dfrac{y}{8} - \dfrac{x}{6}$. Then it is easy to check that F satisfies (2.1) with $k = \dfrac{1}{3}, l = \dfrac{1}{4}$ and G satisfies (2.2) with $m = \dfrac{1}{8}, n = \dfrac{1}{6}$ and $(0,0)$ is the FG-coupled fixed point.*

Remark 2.1. It can be shown that the FG- coupled fixed point is unique provided that for every $(x,y), (x^*,y^*) \in X \times Y$, there exists a $(z_1, z_2) \in X \times Y$ that is comparable to (x,y) and (x^*,y^*). The result is proved in the following theorem.

Theorem 2.2. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two continuous mappings having the mixed monotone property. For every $(x,y), (x^*,y^*) \in X \times Y$ there exists a $(z_1, z_2) \in X \times Y$ that is comparable to (x,y) and (x^*,y^*). Assume that there exist $k,l,m,n \in [0,1)$; $k+l < 1$ and $m+n < 1$ with*

$$d_X(F(x,y), F(u,v)) \leq k\, d_X(x,u) + l\, d_Y(y,v) \quad \forall x \geq_{P_1} u,\ y \leq_{P_2} v$$

$$d_Y(G(y,x), G(v,u)) \leq m\, d_Y(y,v) + n\, d_X(x,u)\ \forall x \leq_{P_1} u,\ y \geq_{P_2} v.$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exists a unique FG-coupled fixed point.

Proof. The existence of a FG- coupled fixed point is follows from the proof of Theorem 2.1. Now we prove that if (x^*, y^*) is another FG- coupled fixed point then $d((x,y), (x^*, y^*)) = 0$ where $x = \lim_{j \to +\infty} F^j(x_0, y_0)$ and $y = \lim_{j \to +\infty} G^j(y_0, x_0)$.
Case 1: If (x,y) is comparable to (x^*, y^*) with respect to the ordering in $X \times Y$, then $(F^j(x,y), G^j(y,x)) = (x,y)$ is comparable to $(F^j(x^*,y^*), G^j(y^*,x^*)) = (x^*,y^*)$ for every $j = 1, 2, 3...$
If $m + n \leq k + l$, consider,

$$
\begin{aligned}
d((x,y), (x^*, y^*)) &= d_X(x, x^*) + d_Y(y, y^*) \\
&= d_X(F^j(x,y), F^j(x^*, y^*)) + d_Y(G^j(y,x), G^j(y^*, x^*)) \\
&\leq 2^j (k+l)^j [d_X(x, x^*) + d_Y(y, y^*)] \\
&\to 0 \text{ as } j \to +\infty.
\end{aligned}
$$

This implies that $d((x,y), (x^*, y^*)) = 0$.
Similarly for $k + l < m + n$ we get

$$
\begin{aligned}
d((x,y), (x^*, y^*)) &< 2^j (m+n)^j [d_X(x, x^*) + d_Y(y, y^*)] \\
&\to 0 \text{ as } j \to +\infty.
\end{aligned}
$$

Therefore $d((x,y), (x^*, y^*)) = 0$.
Case 2: If (x,y) is not comparable to (x^*, y^*), then there exist $(z_1, z_2) \in X \times Y$ such

that which is comparable to (x,y) and (x^*,y^*).

Without loss of generality, consider $m+n \leq k+l$, then

$d((x,y),(x^*,y^*))$

$= d((F^j(x,y),G^j(y,x)),(F^j(x^*,y^*),G^j(y^*,x^*)))$

$\leq d((F^j(x,y),G^j(y,x)),(F^j(z_1,z_2),G^j(z_2,z_1)))+$

$\quad d((F^j(z_1,z_2),G^j(z_2,z_1)),(F^j(x^*,y^*),G^j(y^*,x^*)))$

$= d_X(F^j(x,y),F^j(z_1,z_2))+d_Y(G^j(y,x),G^j(z_2,z_1))+d_X(F^j(z_1,z_2),F^j(x^*,y^*))+$

$\quad d_Y(G^j(z_2,z_1),G^j(y^*,x^*))$

$\leq 2^j(k+l)^j\{[d_X(x,z_1)+d_Y(y,z_2)]+[d_X(z_1,x^*)+d_Y(z_2,y^*)]\}$

$\to 0$ as $j \to +\infty$.

Therefore $d((x,y),(x^*,y^*)) = 0$. □

Corollary 2.3. *Let $(X,d,\leq)$ be a partially ordered complete metric space and $F: X \times X \to X$ be a continuous mapping having the mixed monotone property. For every $(x,y),(x^*,y^*) \in X \times X$ there exists a $(z_1,z_2) \in X \times X$ that is comparable to (x,y) and (x^*,y^*). Assume that there exists $k,l \in [0,1)$; $k+l < 1$ with*

$$d(F(x,y),F(u,v)) \leq k\, d(x,u) + l\, d(y,v) \quad \forall x \geq u,\ y \leq v.$$

If there exist $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$, then there exists a unique coupled fixed point.

Proof. Taking $X = Y$ and $F = G$ in Theorem 2.2 we get the result. □

If we substitute $\dfrac{k}{2}$ in place of k and l in the above corollary we get the following theorem of Gnana Bhaskar and Lakshmikantham.

Corollary 2.4. *[5, Theorem 2.4] Let $(X,d,\leq)$ be a partially ordered complete metric space and $F: X \times X \to X$ be a continuous mapping having the mixed monotone property. For every $(x,y),(x^*,y^*) \in X \times X$ there exists a $(z_1,z_2) \in X \times X$ that is comparable to (x,y) and (x^*,y^*). Assume that there exist $k,l \in [0,1)$; $k+l < 1$ with*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)] \quad \forall x \geq u,\ y \leq v.$$

If there exist $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$, then there exists a unique coupled fixed point.

We can replace the continuity of F and G by other conditions to get the existence of a FG- coupled fixed point, as shown in the following theorem.

Theorem 2.3. *Let $(X,d_X,\leq_{P_1})$ and $(Y,d_Y,\leq_{P_2})$ be two partially ordered complete metric spaces and $F: X \times Y \to X$, $G: Y \times X \to Y$ be two mappings satisfying the mixed monotone property. Assume that X and Y have the following property*

(i) *If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq_{P_1} x \; \forall n$*

(ii) *If a non-increasing sequence $\{y_n\} \to y$ then $y \leq_{P_2} y_n \; \forall n$.*

Also assume that there exist $k, l, m, n \in [0, 1)$ such that $k + l < 1$, $m + n < 1$ with

$$d_X(F(x,y), F(u,v)) \leq k \, d_X(x,u) + l \, d_Y(y,v) \quad \forall x \geq_{P_1} u, \; y \leq_{P_2} v \qquad (2.7)$$

$$d_Y(G(y,x), G(v,u)) \leq m \, d_Y(y,v) + n \, d_X(x,u) \quad \forall x \leq_{P_1} u, \; y \geq_{P_2} v. \qquad (2.8)$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exists $(x, y) \in X \times Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. Following as in the proof of Theorem 2.1, we get $\lim_{j \to +\infty} F^j(x_0, y_0) = x$ and $\lim_{j \to +\infty} G^j(y_0, x_0) = y$.

Now we have,

$$\begin{aligned}
d_X(F(x,y), x) &\leq d_X(F(x,y), F^{j+1}(x_0, y_0)) + d_X(F^{j+1}(x_0, y_0), x) \\
&= d_X(F(x,y), F(F^j(x_0, y_0), G^j(y_0, x_0))) + d_X(F^{j+1}(x_0, y_0), x) \\
&\leq k \, d_X(x, F^j(x_0, y_0)) + l \, d_Y(y, G^j(y_0, x_0)) + d_X(F^{j+1}(x_0, y_0), x) \\
&\to 0 \text{ as } j \to +\infty.
\end{aligned}$$

Therefore $x = F(x,y)$.

Similarly using (2.8) and $\lim_{j \to +\infty} G^j(y_0, x_0) = y$, we get $y = G(y,x)$. $\square$

Corollary 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property*

(i) *If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x \; \forall n$*

(ii) *If a non-increasing sequence $\{y_n\} \to y$ then $y \leq y_n \; \forall n$.*

Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X. Assume that there exist $k, l \in [0, 1)$, $k + l < 1$ such that

$$d(F(x,y), F(u,v)) \leq k \, d(x,u) + l \, d(y,v) \; \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. Taking $X = Y$ and $F = G$ in Theorem 2.3, we get the result. $\square$

Replace k and l by $\dfrac{k}{2}$ in the above corollary, we get the result of Gnana Bhaskar and Lakshmikantham.

Corollary 2.6. [5, Theorem 2.2] *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property*

(i) *If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x \; \forall n$*

(ii) *If a non-increasing sequence $\{y_n\} \to y$ then $y \leq y_n \; \forall n$.*

Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X. Assume that there exist $k \in [0,1)$ such that

$$d(F(x,y), F(u,v)) \leq \frac{k}{2} \left[d(x,u) + d(y,v) \right] \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Remark 2.2. In Theorem 2.3 if add the condition: for every $(x,y), (x^*, y^*) \in X \times Y$, there exists a $(z_1, z_2) \in X \times Y$ that is comparable to both (x,y) and (x^*, y^*), we get unique FG- coupled fixed point.

Remark 2.3. If we take $k = l = \dfrac{a}{2}$ and $m = n = \dfrac{b}{2}$ where $a, b \in [0,1)$ with $a + b < 1$ in Theorems 2.1, 2.2 and 2.3, we get Theorems 2.1, 2.2 and 2.3 respectively of [9].

Remark 2.4. If we take $k = m$ and $l = n$ in Theorems 2.1, 2.2 and 2.3, we get Theorems 2.4, 2.5 and 2.6 respectively of [9].

3. FG-COUPLED FIXED POINT THEOREMS FOR GENERALIZED QUASI-CONTRACTIONS

The concept of quasi-contraction was defined by Ciric [3] in 1974. A self mapping T on a metric space X is said to be a quasi-contraction iff there exist a number $h, 0 \leq h < 1$, such that

$$d(Tx, Ty) \leq h \ \max\{d(x,y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$. In 1979 K.M. Das and K.V. Naik [4] introduced the concept of quasi-contraction for two mappings. Inspired by this we generalize the concept of quasi-contraction to a mapping on a product space and prove the following theorems.

Theorem 3.1. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two continuous functions having the mixed monotone property. Assume that there exist $k, l \in \left[0, \dfrac{1}{2}\right)$ such that*

$$d_X(F(x,y), F(u,v)) \leq k \, M(x,y,u,v); \quad \forall x \geq_{P_1} u, \ y \leq_{P_2} v \tag{3.1}$$

$$d_Y(G(y,x), G(v,u)) \leq l \, N(y,x,v,u); \quad \forall x \leq_{P_1} u, \ y \geq_{P_2} v, \tag{3.2}$$

where
$M(x,y,u,v) = \max \left\{ d_X(x,u), d_X(x, F(x,y)), d_X(x, F(u,v)), d_X(u, F(u,v)), d_X(u, F(x,y)) \right\}$
$N(y,x,v,u) = \max \left\{ d_Y(y,v), d_Y(y, G(y,x)), d_Y(y, G(v,u)), d_Y(v, G(v,u)), d_Y(v, G(y,x)) \right\}.$
If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x,y)$ and $y = G(y,x)$.

Proof. As in Theorem 2.1, it can be proved that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y.

Now we claim that

$$d_X(F^{n+1}(x_0,y_0), F^n(x_0,y_0)) \leq \left(\frac{k}{1-k}\right)^n d_X(x_0,x_1) \tag{3.3}$$

$$d_Y(G^{n+1}(y_0,x_0), G^n(y_0,x_0)) \leq \left(\frac{l}{1-l}\right)^n d_Y(y_0,y_1). \tag{3.4}$$

The proof of the claim is by mathematical induction using (3.1) and (3.2).

For $n = 1$, consider

$$
\begin{aligned}
&d_X(F^2(x_0,y_0), F(x_0,y_0)) \\
&= d_X(F(F(x_0,y_0), G(y_0,x_0)), F(x_0,y_0)) \\
&\quad \leq k\, M(F(x_0,y_0), G(y_0,x_0), x_0, y_0) \\
&= k\, \max\Big\{ d_X(F(x_0,y_0),x_0), d_X(F(x_0,y_0),F^2(x_0,y_0)), d_X(F(x_0,y_0),F(x_0,y_0)), \\
&\qquad\qquad d_X(x_0,F(x_0,y_0)), d_X(x_0,F^2(x_0,y_0)) \Big\} \\
&= k\, \max\Big\{ d_X(x_0,F(x_0,y_0)), d_X(F(x_0,y_0),F^2(x_0,y_0)), d_X(x_0,F^2(x_0,y_0)) \Big\} \\
&\leq k\, \max\Big\{ d_X(x_0,F(x_0,y_0)), d_X(F(x_0,y_0),F^2(x_0,y_0)), \\
&\qquad\qquad d_X(x_0,F(x_0,y_0)) + d_X(F(x_0,y_0),F^2(x_0,y_0)) \Big\} \\
&= k\, [d_X(x_0,F(x_0,y_0)) + d_X(F(x_0,y_0),F^2(x_0,y_0))].
\end{aligned}
$$

That is,

$$
\begin{aligned}
d_X(F^2(x_0,y_0), F(x_0,y_0)) &\leq \frac{k}{1-k}\, d_X(x_0,F(x_0,y_0)) \\
&= \frac{k}{1-k}\, d_X(x_0,x_1).
\end{aligned}
$$

So the inequality (3.3) is true for $n = 1$.

Assume that the result is true for $n \leq m$, then check for $n = m+1$.

Consider,

$$
\begin{aligned}
&d_X(F^{m+2}(x_0,y_0), F^{m+1}(x_0,y_0)) \\
&= d_X(F(F^{m+1}(x_0,y_0), G^{m+1}(y_0,x_0)), F(F^m(x_0,y_0), G^m(y_0,x_0))) \\
&\leq k\, M(F^{m+1}(x_0,y_0), G^{m+1}(y_0,x_0), F^m(x_0,y_0), G^m(y_0,x_0)) \\
&= k\, \max\Big\{ d_X(F^{m+1}(x_0,y_0),F^m(x_0,y_0)), d_X(F^{m+1}(x_0,y_0),F^{m+2}(x_0,y_0)), \\
&\qquad d_X(F^{m+1}(x_0,y_0),F^{m+1}(x_0,y_0)), d_X(F^m(x_0,y_0),F^{m+1}(x_0,y_0)), \\
&\qquad d_X(F^m(x_0,y_0),F^{m+2}(x_0,y_0)) \Big\}
\end{aligned}
$$

$$= k \ \max \Big\{ d_X(F^{m+1}(x_0,y_0), F^m(x_0,y_0)), d_X(F^{m+1}(x_0,y_0), F^{m+2}(x_0,y_0)),$$
$$d_X(F^m(x_0,y_0), F^{m+2}(x_0,y_0)) \Big\}$$
$$\leq k \ \max \Big\{ d_X(F^{m+1}(x_0,y_0), F^m(x_0,y_0)), d_X(F^{m+1}(x_0,y_0), F^{m+2}(x_0,y_0)),$$
$$d_X(F^m(x_0,y_0), F^{m+1}(x_0,y_0)) + d_X(F^{m+1}(x_0,y_0), F^{m+2}(x_0,y_0)) \Big\}$$
$$= k \ [d_X(F^m(x_0,y_0), F^{m+1}(x_0,y_0)) + d_X(F^{m+1}(x_0,y_0), F^{m+2}(x_0,y_0))].$$

That is,

$$d_X(F^{m+2}(x_0,y_0), F^{m+1}(x_0,y_0)) \leq \frac{k}{1-k} \ d_X(F^{m+1}(x_0,y_0), F^m(x_0,y_0))$$
$$\leq \Big(\frac{k}{1-k} \Big)^{m+1} d_X(x_0,x_1).$$

That is, inequality (3.3) is true for all $n \in \mathbb{N}$.
Similarly we can prove the inequality (3.4).
Now for $m > n$, consider

$$d_X(F^m(x_0,y_0), F^n(x_0,y_0))$$
$$\leq d_X(F^m(x_0,y_0), F^{m-1}(x_0,y_0)) + d_X(F^{m-1}(x_0,y_0), F^{m-2}(x_0,y_0)) + ...$$
$$+ d_X(F^{n+1}(x_0,y_0), F^n(x_0,y_0))$$
$$\leq \left[\Big(\frac{k}{1-k} \Big)^{m-1} + \Big(\frac{k}{1-k} \Big)^{m-2} + ... + \Big(\frac{k}{1-k} \Big)^n \right] d_X(x_0,x_1)$$
$$\to 0 \text{ as } n \to +\infty.$$

That is, $\{F^n(x_0,y_0)\}$ is a Cauchy sequence in X.
Similarly we can prove that $\{G^n(y_0,x_0)\}$ is a Cauchy sequence in Y.
Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that
$\lim_{n \to +\infty} F^n(x_0,y_0) = x$ and $\lim_{n \to +\infty} G^n(y_0,x_0) = y$.
As in Theorem 2.1, using the continuity of F and G we can show that $x = F(x,y)$
and $y = G(y,x)$.   □

Corollary 3.1. *Let* $(X,d,\leq)$ *be a partially ordered complete metric space. Let* $F : X \times X \to X$ *be a continuous function having the mixed monotone property. Assume that there exists* $k \in \Big[0, \frac{1}{2} \Big)$ *such that*

$$d(F(x,y), F(u,v)) \leq k \ M(x,y,u,v); \ \ \forall x \geq u, \ y \leq v$$

where
$M(x,y,u,v) = \max \Big\{ d(x,u), d(x,F(x,y)), d(x,F(u,v)), d(u,F(u,v)), d(u,F(x,y)) \Big\}.$
If there exist $x_0, y_0 \in X$ *satisfying* $x_0 \leq F(x_0,y_0)$ *and* $y_0 \geq F(y_0,x_0)$ *then there exist* $(x,y) \in X \times X$ *such that* $x = F(x,y)$ *and* $y = F(y,x)$.

Proof. Taking $X = Y$ anf $F = G$ in Theorem 3.1 we get the result. $\square$

The following example illustrates Theorem 3.1.

Example 3.1. Let $X = [-1, 0]$, $Y = [0, 1]$ with the usual order and the usual metric. Define $F : X \times Y \to X$ by $F(x, y) = \dfrac{x}{3}$ and $G : Y \times X \to Y$ by $G(y, x) = \dfrac{y}{4}$. Then we can easily check that F satisfies inequality (3.1) with $k = \dfrac{1}{3}$ and G satisfies inequality (3.2) with $l = \dfrac{1}{4}$ and $(0, 0)$ is the FG- coupled fixed point.

Theorem 3.2. *Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces and $F : X \times Y \to X$, $G : Y \times X \to Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*
  (i) *If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq_{P_1} x \ \forall n$*
  (ii) *If a non-increasing sequence $\{y_n\} \to y$ then $y \leq_{P_2} y_n \ \forall n$.*
Also assume that there exist $k, l \in \left[0, \dfrac{1}{2}\right)$ such that

$$d_X(F(x, y), F(u, v)) \leq k\, M(x, y, u, v); \quad \forall x \geq_{P_1} u, \ y \leq_{P_2} v \qquad (3.5)$$

$$d_Y(G(y, x), G(v, u)) \leq l\, N(y, x, v, u); \quad \forall x \leq_{P_1} u, \ y \geq_{P_2} v \qquad (3.6)$$

where
$M(x, y, u, v) = \max\left\{ d_X(x, u), d_X(x, F(x, y)), d_X(x, F(u, v)), d_X(u, F(u, v)), d_X(u, F(x, y)) \right\}$
$N(y, x, v, u) = \max\left\{ d_Y(y, v), d_Y(y, G(y, x)), d_Y(y, G(v, u)), d_Y(v, G(v, u)), d_Y(v, G(y, x)) \right\}.$
If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following the proof of Theorem 3.1 we get $\lim_{n \to +\infty} F^n(x_0, y_0) = x$ and $\lim_{n \to +\infty} G^n(y_0, x_0) = y$.
Now, consider

$d_X(F(x, y), x)$

$\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x)$

$= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x)$

$\leq k\, M(x, y, F^n(x_0, y_0), G^n(y_0, x_0)) + d_X(F^{n+1}(x_0, y_0), x)$

$= k\, \max\left\{ d_X(x, F^n(x_0, y_0)), d_X(x, F(x, y)), d_X(x, F^{n+1}(x_0, y_0)), \right.$

$\qquad \left. d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)), d_X(F^n(x_0, y_0), F(x, y)) \right\} + d_X(F^{n+1}(x_0, y_0), x).$

That is, $d_X(F(x, y), x) \leq k\, d_X(x, F(x, y))$ as $n \to \infty$, which implies that $d_X(F(x, y), x) = 0$. Therefore $F(x, y) = x$.
Also by using inequality (3.6) and $\lim_{n \to +\infty} G^n(y_0, x_0) = y$, we get $y = G(y, x)$. $\square$

Corollary 3.2. *Let $(X, d, \leq)$ be a partially ordered complete metric space and $F : X \times Y \to X$ be a mapping having the mixed monotone property. Assume that X satisfies the following property*

(i) *If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x \ \forall n$*
(ii) *If a non-increasing sequence $\{y_n\} \to y$ then $y \leq y_n \ \forall n$.*

Also assume that there exists $k \in \left[0, \dfrac{1}{2}\right)$ such that

$$d(F(x, y), F(u, v)) \leq k \, M(x, y, u, v); \quad \forall x \geq u, \ y \leq v$$

where
$$M(x, y, u, v) = \max \Big\{ d(x, u), d(x, F(x, y)), d(x, F(u, v)), d(u, F(u, v)), d(u, F(x, y)) \Big\}.$$
If there exists $(x_0, y_0) \in X \times X$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. Setting $X = Y$ and $F = G$ in Theorem 3.2 we get the result. □

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