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SLANT LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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ABSTRACT. The aim of the present paper is to investigate geometric characteristics of slant lightlike submanifolds of semi-Riemannian product manifolds. We obtain characterization theorems for the existence of slant lightlike submanifolds of semi-Riemannian product manifolds. We also find a necessary and sufficient condition enabling the induced connection on slant lightlike submanifolds of semi-Riemannian product manifolds to be a metric connection. Then, we establish some results for the integrability of distributions associated with this class of lightlike submanifolds. Consequently, we investigate totally umbilical slant lightlike submanifolds of semi-Riemannian product manifolds. In particular, we prove that every totally umbilical slant lightlike submanifold of a semi-Riemannian product manifold is always totally geodesic.

1. INTRODUCTION

The concept of slant submanifolds arises from slant immersions, introduced by Chen [3] and has been studied extensively in [4]. Literature suggests that a variety of generalized classes of slant submanifolds have been investigated by Carriazo [2], Papaghiuc [12] and Sahin [13]. On the other hand, the study of slant submanifolds in contact geometry was introduced and developed by Lotta [10]- [11]. In the last two decades, the study of lightlike submanifolds is a topic of special interest for mathematicians and physicists. One may see that the geometry of lightlike submanifolds is significantly different from those of non-degenerate submanifolds. In the case of lightlike submanifolds, the tangent bundle is non-complementary with the normal bundle, which makes the theory of lightlike submanifolds more complicated and interesting than non-degenerate submanifolds. In recent years, the theory of lightlike submanifolds developed many potential applications in mathematical physics and relativity. For instance, the concept of lightlike submanifolds has been successfully employed in the study of black holes, asymptotically flat spacetimes, Killing horizon and electronic and radiation fields (see, [5] and [8]).

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Then, considering the effectiveness of the concept of lightlike submanifolds and interesting geometric features of slant submanifolds, Sahin [14], introduced the idea of slant lightlike submanifolds of indefinite almost Hermitian manifolds. Later, Sahin proved the existence of slant lightlike submanifolds in Sasakian manifolds in [15]. Afterwards, several other generalizations of slant lightlike submanifolds namely screen slant lightlike submanifolds, pointwise slant lightlike submanifolds, screen pseudo slant lightlike submanifolds and semi-slant lightlike submanifolds of indefinite Kaehler manifolds were considered and investigated by many others (for details, see [7], [16], [17], [18]).

It is interesting to note that semi-Riemannian product manifolds are a generalization of Riemannian product manifolds in the semi-Riemannian case and they have rich geometric properties. Therefore, it is interesting to study lightlike submanifolds of semi-Riemannian product manifolds. In this regard, the concept of *GCR*-lightlike submanifolds of semi-Riemannian product manifolds has been introduced and investigated by Kumar et al. [9]. But the concept of slant lightlike submanifolds of semi-Riemannian product manifolds is yet to be explored.

Therefore, in the present paper, we investigate the geometry of slant lightlike submanifolds of semi-Riemannian product manifolds and justify their existence by obtaining several characterization theorems. Then, we give a non-trivial example of slant lightlike submanifolds of semi-Riemannian product manifolds. We also find a necessary and sufficient condition for the induced connection on a slant lightlike submanifold of a semi-Riemannian product manifold to be a metric connection. Then, we establish some results for the integrability of distributions arising in this class of lightlike submanifolds. Finally, we investigate totally umbilical slant lightlike submanifolds of semi-Riemannian product manifolds and show that every totally umbilical slant lightlike submanifold of a semi-Riemannian product manifold is totally geodesic.

2. PRELIMINARIES

2.1. **Geometry of lightlike submanifolds**

Suppose we have a n - dimensional submanifold (K,g) of an $(m+n)$ real dimensional semi-Riemannian manifold $(\bar{K},\bar{g})$ such that $\bar{g}$ is a metric with constant index q satisfying $m,n \geq 1$, $1 \leq q \leq m+n-1$. If the metric $\bar{g}$ is degenerate on TK, then T_pK and $T_pK^{\perp}$ both are degenerate and there exists a radical (null) subspace $Rad(T_pK)$ such that $Rad(T_pK) = T_pK \cap T_pK^{\perp}$. If $Rad(TK) : p \in K \rightarrow Rad(T_pK)$ is a smooth distribution on K of rank $r(>0)$, $1 \leq r \leq n$, then K is known as an r-lightlike submanifold of $\bar{K}$ (see, [5]). Then the radical distribution $Rad(TK)$ of TK is defined as:

$$Rad(TK) = \cup_{p\in K}\{\xi \in T_pK | g(u,\xi) = 0,\ \forall\, u \in T_pK, \xi \neq 0\}.$$

Further, let $S(TK)$ be the screen distribution in TK such that $TK = Rad(TK) \perp$

$S(TK)$ and similarly let $S(TK^{\perp})$ be a screen transversal vector bundle in $TK^{\perp}$ such that $TK^{\perp} = Rad(TK) \perp S(TK^{\perp})$.
Moreover, there exists a local null frame $\{N_i\}$ of null sections with values in the orthogonal complement of $S(TK^{\perp})$ in $S(TK^{\perp})^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{2.1}$$

for any $i, j \in \{1, 2, .., r\}$, where $\{\xi_j\}$ is any local basis of $\Gamma(Rad(TK))$. It implies that $tr(TK)$ and $ltr(TK)$, respectively, are vector bundles in $T\bar{K}|_K$ and $S(TK^{\perp})^{\perp}$ with the property

$$tr(TK) = ltr(TK) \perp S(TK^{\perp})$$

and

$$T\bar{K}|_K = TK \oplus tr(TK) = S(TK) \perp (Rad(TK) \oplus ltr(TK)) \perp S(TK^{\perp}). \tag{2.2}$$

Let $\bar{\nabla}$ and ∇, respectively, denote the Levi-Civita connection on $\bar{K}$ and torsion-free linear connection on K. Then, the Gauss and Weingarten formulae are given as

$$\bar{\nabla}_Y Z = \nabla_Y Z + h^l(Y, Z) + h^s(Y, Z), \tag{2.3}$$

$$\bar{\nabla}_Y N = -A_N Y + \nabla^l_Y N + D^s(Y, N), \tag{2.4}$$

$$\bar{\nabla}_Y W = -A_W Y + D^l(Y, W) + \nabla^s_Y W, \tag{2.5}$$

where $Y, Z \in \Gamma(TK), N \in \Gamma(ltr(TK))$ and $W \in \Gamma(S(TK^{\perp}))$. Further by employing Eqs. (2.3) and (2.5), we derive

$$g(A_W Y, Z) = \bar{g}(h^s(Y, Z), W) + \bar{g}(Z, D^l(Y, W)). \tag{2.6}$$

Let us denote the projection morphism of TK on the screen distribution $S(TK)$ by η. It follows that

$$\nabla_Y \eta Z = \nabla^*_Y \eta Z + h^*(Y, \eta Z), \ \nabla_Y \xi = -A^*_\xi Y + \nabla^{*t}_Y \xi, \tag{2.7}$$

where $\{h^*(Y, \eta Z), \nabla^{*t}_Y \xi\} \in \Gamma(Rad(TK))$ and $\{\nabla^*_Y \eta Z, A^*_\xi Y\} \in \Gamma(S(TK))$. Further, employing Eqs. (2.4), (2.5) and (2.7), we attain

$$\bar{g}(h^l(Y, \eta Z), \xi) = g(A^*_\xi Y, \eta Z). \tag{2.8}$$

As $\bar{\nabla}$ is a metric connection on $\bar{K}$, for any $Y, Z, W \in \Gamma(TK)$, one has

$$(\nabla_Y g)(Z, W) = \bar{g}(h^l(Y, Z), W) + \bar{g}(h^l(Y, W), Z), \tag{2.9}$$

which implies that ∇ is not always a metric connection on K.

2.2. **Semi-Riemannian product manifolds**

Suppose that (K_1, g_1) and (K_2, g_2) are two m_1 and m_2-dimensional semi- Riemannian manifolds with constant index $q_1 > 0$ and $q_2 > 0$, respectively. Consider $\pi : K_1 \times K_2 \to K_1$ and $\sigma : K_1 \times K_2 \to K_2$ the projection maps given by $\pi(y, z) = y$ and

$\sigma(y,z) = z$, for any $(y,z) \in K_1 \times K_2$. We denote the product manifold by $(\bar{K}, \bar{g}) = (K_1 \times K_2, \bar{g})$, where

$$\bar{g}(Y,Z) = g_1(\pi_* Y, \pi_* Z) + g_2(\sigma_* Y, \sigma_* Z),$$

for any $Y, Z \in \Gamma(T\bar{K})$, where $*$ stands for the differential mapping. Then we have

$$\pi_*^2 = \pi_*, \ \sigma_*^2 = \sigma_*, \ \pi_* \sigma_* = \sigma_* \pi_*, \ \pi_* + \sigma_* = I,$$

where I is the identity map of $T(K_1 \times K_2)$. Thus $(\bar{K}, \bar{g})$ is an $(m_1 + m_2)$ - dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\bar{K} = K_1 \times K_2$ is characterized by K_1 and K_2, which are totally geodesic submanifolds of $\bar{K}$. Now if we put $F = \pi_* - \sigma_*$ we see that $F^2 = I$ and

$$\bar{g}(FY, Z) = \bar{g}(Y, FZ), \tag{2.10}$$

for any $Y, Z \in \Gamma(T\bar{K})$, where F is called an almost product structure on $K_1 \times K_2$. If we denote the Levi-Civita connection on $\bar{K}$ by $\bar{\nabla}$, then it can be seen that

$$(\bar{\nabla}_Y F)Z = 0, \tag{2.11}$$

for any $Y, Z \in \Gamma(T\bar{K})$, that is, F is parallel with respect to $\bar{\nabla}$.

3. Slant lightlike submanifolds

Firstly, we prove two essential lemmas following [14], for later use.

Lemma 3.1. *Consider an r-lightlike submanifold K of semi-Riemannian product manifold $\bar{K}$ with index $2q$ and $FRad(TK)$ a distribution on K with $Rad(TK) \cap FRad(TK) = \{0\}$. Then, $Fltr(TK)$ is a subbundle of $S(TK)$ such that $FRad(TK) \cap Fltr(TK) = \{0\}$.*

Proof. By the hypothesis, we have $FRad(TK) \subset S(TK)$. On the contrary, assume that $ltr(TK)$ is invariant. Choose $\xi \in \Gamma(Rad(TK))$ and $N \in \Gamma(ltr(TK))$, thus we have $1 = \bar{g}(\xi, N) = \bar{g}(F\xi, FN) = 0$ as $F\xi \in \Gamma(S(TK))$ and $FN \in \Gamma(ltr(TK))$, which leads to a contradiction. It implies that $ltr(TK)$ is not invariant w.r.t. F. Now $\bar{g}(F\xi, FN) = 0$, as $S(TK^{\perp})$ is orthogonal to $S(TK)$. But $\bar{g}(\xi, N) = \bar{g}(F\xi, FN) \neq 0$, for $\xi \in \Gamma(Rad(TK))$, which is again a contradiction. Therefore, it implies that FN does not belong to $S(TK^{\perp})$. Thus, we conclude that $Fltr(TK)$ is a distribution on K. Therefore, FN does not belong to $FRad(TK)$. Moreover, if $FN \in \Gamma(Rad(TK))$, then we have $F^2 N = N \in \Gamma(FRad(TK))$, which is not possible. On the other hand, FN does not belong to $FRad(TK)$. Hence, $Fltr(TK) \subset S(TK)$ and $Fltr(TK) \cap FRad(TK) = \{0\}$. □

Lemma 3.2. *For an r-lightlike submanifold K of a semi-Riemannian product manifold $\bar{K}$, with the assumption of Lemma 3.1 (provided, $r = q$), any complementary distribution to $FRad(TK) \oplus Fltr(TK)$ in $S(TK)$ must be Riemannian.*

Proof. Assume that $dim(\bar{K}) = m+n$ and $dim(K) = m$. From lemma 3.1, we have $Fltr(TK) \oplus FRad(TK) \subset S(TK)$. Now, assume D is the complementary distribution to $Fltr(TK) \oplus FRad(TK)$ in $S(TK)$. Take a local quasi-orthonormal field of frames on $\bar{K}$ along K written as $\{\xi_i, N_i, F\xi_i, FN_i, X_j, W_k\}$, for $i \in \{1,...,r\}$, $j \in \{3r+1,...,m\}, k \in \{r+1,...,n\}$, where $\{\xi_i\}$ and $\{N_i\}$, respectively, are lightlike bases of $Rad(TK)$ and $ltr(TK)$, whereas $\{F\xi_i, FN_i, X_j\}$ and $\{W_k\}$ are orthonormal basis of $S(TK)$ and $S(TK^{\perp})$, respectively. Then, we can construct orthonormal basis $\{U_1,...,U_{2r}, V_1,...,V_{2r}\}$ as follows.

$$U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1), \qquad U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1),$$
$$U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2), \qquad U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2),$$
$$\dots\dots\dots\dots \quad \dots\dots\dots\dots$$
$$U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r), \qquad U_{2r} = \frac{1}{\sqrt{2}}(\xi_r - N_r),$$
$$V_1 = \frac{1}{\sqrt{2}}(F\xi_1 + FN_1), \qquad V_2 = \frac{1}{\sqrt{2}}(F\xi_1 - FN_1),$$
$$V_3 = \frac{1}{\sqrt{2}}(F\xi_2 + FN_2), \qquad V_4 = \frac{1}{\sqrt{2}}(F\xi_2 - FN_2),$$
$$\dots\dots\dots\dots \quad \dots\dots\dots\dots$$
$$V_{2r-1} = \frac{1}{\sqrt{2}}(F\xi_r + FN_r), \qquad V_{2r} = \frac{1}{\sqrt{2}}(F\xi_r - FN_r),$$

for the basis $\{\xi_1,...,\xi_r, N_1,...,N_r, F\xi_1,...,F\xi_r, FN_1,...,FN_r\}$ of $Rad(TK) \oplus ltr(TK) \oplus FRad(TK) \oplus Fltr(TK)$. Clearly, Span$\{\xi_i, N_i, F\xi_i, FN_i\}$ is a non-degenerate space, thus we conclude that $Rad(TK) \oplus ltr(TK) \oplus FRad(TK) \oplus Fltr(TK)$ is a non-degenerate space with index $2r$ on $\bar{K}$. As $index(T\bar{K}) = index(Rad(TK) \oplus ltr(TK)) + index(FRad(TK) \oplus Fltr(TK)) + index(D \perp S(TK^{\perp}))$. Therefore, we obtain $2q = 2r + index\ (D \perp S(TK^{\perp}))$, which implies that $D \perp S(TK^{\perp})$ is Riemannian (provided, $r = q$), that is, $index(D \perp S(TK^{\perp})) = 0$. Hence, the result follows. □

We note that $Rad(TK)$ is degenerate in TK, therefore vectors of $Rad(TK)$ can not be used to study the angle between them. In this regard, Lemma 3.2 plays a crucial role in defining the angle between vectors. Thus, we define a slant lightlike submanifold of semi-Riemannian product manifolds following Sahin [14] as follows.

Definition 3.1. *A q-lightlike submanifold K of a semi-Riemannian product manifold $\bar{K}$ with index $2q$, is called a slant lightlike submanifold of $\bar{K}$, if*

(A) *$Rad(TK)$ is a distribution on K such that $FRad(TK) \cap Rad(TK) = \{0\}$.*

(B) *For each non-zero vector field Z tangent to D at $z \in U \subset K$, the angle $\theta(Z)$ between FZ and the vector space D_z is constant (known as slant angle), that is, it is independent of the choice of $z \in U \subset K$ and $Z \in D_z$, where D is complementary distribution to $FRad(TK) \oplus Fltr(TK)$ in the screen distribution $S(TK)$.*

Then, in view of Definition 3.1, TK of K is given as

$$TK = Rad(TK) \perp (FRad(TK) \oplus Fltr(TK)) \perp D. \tag{3.1}$$

Note: In the forthcoming part, we shall denote a slant lightlike submanifold by **s.l.s.** and a semi-Riemannian product manifold by $\bar{K}$, unless otherwise indicated.
For $Y \in \Gamma(TK)$, we have

$$FY = \phi Y + SY, \tag{3.2}$$

where $\phi Y \in \Gamma(TK)$ and $SY \in \Gamma(tr(TK))$. Similarly, for any $V \in \Gamma(tr(TK))$,

$$FV = tV + nV, \tag{3.3}$$

where $tV \in \Gamma(TK)$ and $nV \in \Gamma(tr(TK))$.
Consider P_1, P_2, P_3 and P_4 the projections of TK on $Rad(TK)$, $F(Rad(TK))$, $F(ltr(TK))$ and D, respectively. Then, for any $Y \in \Gamma(TK)$, we have

$$Y = P_1Y + P_2Y + P_3Y + P_4Y, \tag{3.4}$$

then applying F to Eq. (3.4), we obtain

$$FY = FP_1Y + FP_2Y + FP_3Y + FP_4Y, \tag{3.5}$$

which after using Eq. (3.2) yields that

$$FY = FP_1Y + FP_2Y + \phi P_4Y + SP_3Y + SP_4Y. \tag{3.6}$$

Moreover, Eq. (3.6) can be rewritten as

$$FY = \phi Y + SP_3Y + SP_4Y, \tag{3.7}$$

where $\phi Y = FP_1Y + FP_2Y + \phi P_4Y$.
Further differentiating Eq. (3.6) along with Eqs. (2.3)-(2.5), (3.2) and (3.3) and then considering the components on $Rad(TK), FRad(TK), Fltr(TK)$, $D, ltr(TK)$ and $S(TK^{\perp})$, respectively, we derive

$$\begin{aligned} P_1(\nabla_{Y_1}FP_1Y_2) + P_1(\nabla_{Y_1}FP_2Y_2) + P_1(\nabla_{Y_1}\phi P_4Y_2) = \\ P_1(A_{SP_3Y_2}Y_1) + P_1(A_{SP_4Y_2}Y_1) + FP_2\nabla_{Y_1}Y_2. \end{aligned} \tag{3.8}$$

$$\begin{aligned} P_2(\nabla_{Y_1}FP_1Y_2) + P_2(\nabla_{Y_1}FP_2Y_2) + P_2(\nabla_{Y_1}\phi P_4Y_2) = \\ P_2(A_{SP_3Y_2}Y_1) + P_2(A_{SP_4Y_2}Y_1) + FP_1\nabla_{Y_1}Y_2. \end{aligned} \tag{3.9}$$

$$\begin{aligned} P_3(\nabla_{Y_1}FP_1Y_2) + P_3(\nabla_{Y_1}FP_2Y_2) + P_3(\nabla_{Y_1}\phi P_4Y_2) = \\ P_3(A_{SP_3Y_2}Y_1) + P_3(A_{SP_4Y_2}Y_1) + Fh^l(Y_1, Y_2). \end{aligned} \tag{3.10}$$

$$\begin{aligned} P_4(\nabla_{Y_1} FP_1Y_2) + P_4(\nabla_{Y_1} FP_2Y_2) + P_4(\nabla_{Y_1} \phi P_4 Y_2) = \\ P_4(A_{SP_3Y_2}Y_1) + P_4(A_{SP_4Y_2}Y_1) + \phi P_4 \nabla_{Y_1} Y_2 + th^s(Y_1, Y_2). \end{aligned} \tag{3.11}$$

$$\begin{aligned} h^l(Y_1, FP_1Y_2) + h^l(Y_1, FP_2Y_2) + h^l(Y_1, \phi P_4 Y_2) = \\ SP_3 \nabla_{Y_1} Y_2 - \nabla^l_{Y_1} SP_3 Y_2 - D^l(Y_1, SP_4Y_2). \end{aligned} \tag{3.12}$$

$$\begin{aligned} h^s(Y_1, FP_1Y_2) + h^s(Y_1, FP_2Y_2) + h^s(Y_1, \phi P_4 Y_2) = \\ SP_4 \nabla_{Y_1} Y_2 - \nabla^s_{Y_1} SP_4 Y_2 - D^s(Y_1, SP_3Y_2) + nh^s(Y_1, Y_2). \end{aligned} \tag{3.13}$$

Lemma 3.3. *For a* **s.l.s.** *K of $\bar{K}$, one has $SP_4Y \in \Gamma(S(TK^\perp))$, for $Y \in \Gamma(TK)$.*

Proof. For $Y \in \Gamma(TK)$, we have $SP_4Y \in \Gamma(S(TK^\perp))$ if and only if $\bar{g}(SP_4Y, \xi) = 0$, for $\xi \in \Gamma(Rad(TK))$. Therefore, $\bar{g}(SP_4Y, \xi) = \bar{g}(FP_4Y - \phi P_4 Y, \xi) = \bar{g}(FP_4Y, \xi) = g(P_4Y, F\xi) = 0$ implies SP_4Y has no components in $ltr(TK)$. Hence the result follows. □

Note: From Lemma 3.3, we have $SD \subset S(TK^\perp)$, which implies that there exist $\mu \subset S(TK^\perp)$ such that $S(TK^\perp) = SD \perp \mu$.

Theorem 3.1. *(Existence Theorem) A q-lightlike submanifold K of $\bar{K}$ is* **s.l.s.**, *if and only if*

(i) *$Fltr(TK)$ is a distribution on K.*
(ii) $\phi^2 P_4 Z = \cos^2\theta(P_4Z)$ *for $Z \in \Gamma(TK)$.*

Proof. Assume that K be a **s.l.s.** of $\bar{K}$. Then from Lemma 3.1, we have $Fltr(TK)$ is also a distribution on K such that $Fltr(TK) \subset S(TK)$, which proves (i). On the other hand, the angle between D_z and FP_4Z is constant, thus we acquire

$$\cos\theta(P_4Z) = \frac{\bar{g}(FP_4Z, \phi P_4 Z)}{|FP_4Z||\phi P_4 Z|} = \frac{\bar{g}(P_4Z, F\phi P_4 Z)}{|P_4Z||\phi P_4 Z|} = \frac{\bar{g}(P_4Z, \phi^2 P_4 Z)}{|P_4Z||\phi P_4 Z|}. \tag{3.14}$$

Moreover, we also have

$$\cos\theta(P_4Z) = \frac{|\phi P_4 Z|}{|FP_4Z|}. \tag{3.15}$$

Thus from Eqs. (3.14) and (3.15), we derive

$$\cos^2\theta(P_4Z) = \frac{\bar{g}(P_4Z, \phi^2 P_4 Z)}{|P_4Z|^2}. \tag{3.16}$$

As $\theta(P_4Z)$ is constant, thus we have

$$\phi^2 P_4 Z = \cos^2\theta(P_4Z), \tag{3.17}$$

which proves (ii).
Conversely, suppose that K is a q-lightlike submanifold of $\bar{K}$ satisfying (i) and (ii). Then by (i), it follows that $FRad(TK)$ is a distribution on K. Further, Lemma 3.2

gives that the complementary distribution of $FRad(TK) \oplus Fltr(TK)$ in $S(TK)$ is Riemannian. Therefore

$$g(\phi P_4 Z, \phi P_4 Z) = g(\phi^2 P_4 Z, P_4 Z) = \cos^2\theta(P_4 Z) g(P_4 Z, P_4 Z), \tag{3.18}$$

for any $P_4 Z \in D_z$, which implies that

$$\cos^2\theta(P_4 Z) = \frac{g(\phi P_4 Z, \phi P_4 Z)}{g(P_4 Z, P_4 Z)}. \tag{3.19}$$

Hence, the proof is complete. □

Theorem 3.2. *(Existence Theorem) A q-lightlike submanifold K of $\bar{K}$ is* **s.l.s.***, if and only if*

(i) *$Fltr(TK)$ is a distribution on K.*
(ii) *$tSP_4Z = \sin^2\theta(P_4 Z)$, for every vector Z on K.*

Proof. Suppose that K is a **s.l.s.** of $\bar{K}$. Then employing Lemma 3.1, $Fltr(TK)$ is also a distribution on K such that $Fltr(TK) \subset S(TK)$, which proves (i). Further, applying F to Eq. (3.6) and using Eqs. (3.2) and (3.6), we acquire $Z = P_1 Z + P_2 Z + \phi^2 P_4 Z + S\phi P_4 Z + FSP_3 Z + tSP_4 Z + nSP_4 Z$. Then comparing the tangential components on both sides, we derive

$$Z = P_1 Z + P_2 Z + \phi^2 P_4 Z + P_3 Z + tSP_4 Z. \tag{3.20}$$

Further, employing Eq. (3.4), we get

$$P_4 Z = \phi^2 P_4 Z + tSP_4 Z. \tag{3.21}$$

Since K is **s.l.s.**, then using Theorem 3.1, we have $\phi^2 P_4 Z = \cos^2\theta P_4 Z$, which further gives $tSP_4 Z = \sin^2\theta(P_4 Z)$,which proves (ii).
Conversely, let K be a q-lightlike submanifold of $\bar{K}$ such that (i) and (ii) hold. By (ii), we have $tSP_4 Z = \sin^2\theta(P_4 Z)$ and further using Eq. (3.21), we obtain $\phi^2 P_4 Z = (1 - \sin^2\theta(P_4 Z)) = \cos^2(P_4 Z)$. Hence, the result follows by taking similar steps as in the proof of Theorem 3.1. □

Corollary 3.1. *For a* **s.l.s.** *K of $\bar{K}$, one has*

$$g(\phi P_4 Y_1, \phi P_4 Y_2) = \cos^2\theta g(P_4 Y_1, P_4 Y_2) \tag{3.22}$$

and

$$\bar{g}(SP_4 Y_1, SP_4 Y_2) = \sin^2\theta g(P_4 Y_1, P_4 Y_2), \tag{3.23}$$

for $Y_1, Y_2 \in \Gamma(TK)$.

Example 3.1. *Consider K a submanifold of the semi-Euclidean space $(R_2^{10}, \bar{g})$ given by the equations*

$$x^1 = u^1,\ x^2 = u^2,\ x^3 = u^1,\ x^4 = u^5,\ x^5 = u^4\sin\theta,$$
$$x^6 = u^3 k\sin\theta,\ x^7 = u^4\cos\theta,\ x^8 = u^3 k\cos\theta,\ x^9 = ku^4,\ x^{10} = u^3$$

where the signature of g is $(-,-,+,+,+,+,+,+,+,+)$ *with respect to the basis* $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10})$. *Then* TK *is spanned by* Z_1, Z_2, Z_3, Z_4, Z_5, *where*

$$Z_1 = \partial x_1 + \partial x_3, \quad Z_2 = \partial x_2, \quad Z_3 = k \sin\theta \partial x_6 + k \cos\theta \partial x_8 + \partial x_{10},$$

$$Z_4 = \sin\theta \partial x_5 + \cos\theta \partial x_7 + k \partial x_9, \; Z_5 = \partial x_4.$$

Clearly, K is a 1-lightlike submanifold as $\{Z_1\} \in \Gamma(Rad(TK))$. *Moreover,* $FZ_1 = Z_2 + Z_5$, *which implies that* $FRad(TK) = Span\{Z_2, Z_5\}$. *Choose* $D = Span\{Z_3, Z_4\}$, *which is Riemannian. Then, K is a* **s.l.s.** *with slant angle* $cos^{-1}(\frac{2k}{1+k^2})$ *with screen transversal bundle* $S(TK^{\perp})$ *spanned by*

$$W = -k \sin\theta \partial x_5 + \sin\theta \partial x_6 - k \cos\theta \partial x_7 + \cos\theta \partial x_8 + \partial x_9 - k \partial x_{10},$$

which is also Riemannian. Furthermore, $ltr(TK)$ *is spanned by*

$$N_1 = \frac{1}{2}\{-\partial x_1 + \partial x_3\}.$$

Therefore, we obtain $FN_1 = \frac{1}{2}\{-\partial x_2 + \partial x_4\} = \frac{1}{2}\{-Z_2 + Z_5\} \in \Gamma(Fltr(TK) \subseteq S(TK))$. *Hence K is a proper* **s.l.s.** *of* R_2^{10}.

Theorem 3.3. *Suppose that K is a proper* **s.l.s.** *of* $\bar{K}$. *Then the induced connection* ∇ *is a metric connection, if and only if,*

$$\nabla_Y F\xi \in \Gamma(FRad(TK)) \text{ and } th(Y, F\xi) = 0,$$

for $Y \in \Gamma(TK)$ *and* $\xi \in \Gamma(Rad(TK))$.

Proof. For $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$, employing Eq. (2.11), one has $\bar{\nabla}_Y \xi = \bar{\nabla}_Y F^2 \xi = F \bar{\nabla}_Y F\xi$. Further, using Eqs. (2.3) and (3.3), we acquire

$$\nabla_Y \xi + h(Y, \xi) = F\nabla_Y F\xi + th(Y, F\xi) + nh(Y, F\xi). \tag{3.24}$$

Then, equating the tangential components on both sides, we derive

$$\nabla_Y \xi = F\nabla_Y F\xi + th(Y, F\xi). \tag{3.25}$$

Hence, from Eq. (3.25), $\nabla_Y \xi \in \Gamma(Rad(TK))$ if and only if $\nabla_Y F\xi \in \Gamma(FRad(TK))$ and $th(Y, F\xi) = 0$, which gives the result. □

Next, we will examine some conditions for integrability of distributions associated with a **s.l.s.** of $\bar{K}$. Firstly, we present a basic lemma.

Lemma 3.4. *Consider a* **s.l.s.** *K of* $\bar{K}$, *then*

$$(\nabla_{Y_1} \phi) Y_2 = A_{SP_1 Y_2} Y_1 + A_{SP_4 Y_2} Y_1 + th(Y_1, Y_2) \tag{3.26}$$

and

$$\begin{aligned} S\nabla_{Y_1} Y_2 - h(Y_1, \phi Y_2) + Ch(Y_1, Y_2) = {} & D^s(Y_1, SP_3 Y_2) \\ & + D^l(Y_1, SP_4 Y_2) + \nabla^s_{Y_1} SP_4 Y_2 + \nabla^l_{Y_1} SP_3 Y_2, \end{aligned} \tag{3.27}$$

where

$$(\nabla_{Y_1}\phi)Y_2 = \nabla_{Y_1}\phi Y_2 - \phi\nabla_{Y_1}Y_2, \tag{3.28}$$

for any $Y_1, Y_2 \in \Gamma(TK)$.

Proof. Employing Eqs. (2.3)-(2.5), (3.2) and (3.3) and equating the tangential and transversal components, the proof follows. □

Theorem 3.4. *Assume that K is a* **s.l.s.** *of $\bar{K}$. Then the slant distribution D is integrable if and only if*

$$\nabla_{Z_1}\phi Z_2 - A_{SP_4Z_2}Z_1 - th(Z_1,Z_2) - \phi\nabla_{Z_2}Z_1 \in \Gamma(D),$$

for each $Z_1, Z_2 \in \Gamma(D)$.

Proof. For $Z_1, Z_2 \in \Gamma(D)$, employing Eqs. (3.26) and (3.28), we attain

$$\phi[Z_1,Z_2] = \nabla_{Z_1}\phi Z_2 - A_{SP_4Z_2}Z_1 - th(Z_1,Z_2) - \phi\nabla_{Z_2}Z_1,$$

which proves the assertion. □

Theorem 3.5. *Assume that K is a* **s.l.s.** *of $\bar{K}$. Then the anti-invariant distribution $Fltr(TK)$ is integrable if and only if*

$$A_{SP_3Y_2}Y_1 + th(Y_1,Y_2) + \phi\nabla_{Y_2}Y_1 = 0,$$

for $Y_1, Y_2 \in \Gamma(Fltr(TK))$.

Proof. For $Y_1, Y_2 \in \Gamma(Fltr(TK))$, employing Eqs. (3.26) and (3.28), we get

$$\phi[Y_1,Y_2] = -A_{SP_3Y_2}Y_1 - th(Y_1,Y_2) - \phi\nabla_{Y_2}Y_1,$$

which gives the result. □

Theorem 3.6. *Consider a* **s.l.s.** *K of $\bar{K}$, then $Rad(TK)$ is integrable if and only if*

(i) $P_1(\nabla_{\xi_1}FP_1\xi_2) = P_1(\nabla_{\xi_2}FP_1\xi_1)$ *and* $P_4(\nabla_{\xi_1}\phi P_4\xi_2) = P_4(\nabla_{\xi_2}\phi P_4\xi_1)$,

(ii) $h^l(\xi_1,FP_1\xi_2) = h^l(\xi_2,FP_1\xi_1)$ *and* $h^s(\xi_1,FP_1\xi_2) = h^s(\xi_2,FP_1\xi_1)$,

for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$.

Proof. Consider Eq. (3.8), for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$, we derive

$$P_1(\nabla_{\xi_1}FP_1\xi_2) = FP_2\nabla_{\xi_1}\xi_2. \tag{3.29}$$

Interchanging the role of ξ_1 and ξ_2, Eq. (3.29) yields

$$P_1(\nabla_{\xi_2}FP_1\xi_1) = FP_2\nabla_{\xi_2}\xi_1. \tag{3.30}$$

Then from Eqs. (3.29) and (3.30), we derive

$$P_1(\nabla_{\xi_1}FP_1\xi_2) - P_1(\nabla_{\xi_2}FP_1\xi_1) = FP_2[\xi_1,\xi_2]. \tag{3.31}$$

Now, for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$, using Eq. (3.11), we have

$$P_4(\nabla_{\xi_1}\phi P_4\xi_2) = \phi P_4\nabla_{\xi_1}\xi_2 + th^s(\xi_1,\xi_2). \tag{3.32}$$

By interchanging the role of ξ_1 and ξ_2 in Eq. (3.32), we get

$$P_4(\nabla_{\xi_2}\phi P_4\xi_1) = \phi P_4\nabla_{\xi_2}\xi_1 + th^s(\xi_2,\xi_1). \tag{3.33}$$

Further, from Eqs. (3.32) and (3.33), we obtain

$$P_4(\nabla_{\xi_1}\phi P_4\xi_2) - P_4(\nabla_{\xi_2}\phi P_4\xi_1) = \phi P_4[\xi_1,\xi_2]. \tag{3.34}$$

Next, consider Eq. (3.12), for $\xi_1,\xi_2 \in \Gamma(Rad(TK))$, we acquire

$$h^l(\xi_1, FP_1\xi_2) = SP_3\nabla_{\xi_1}\xi_2. \tag{3.35}$$

By interchanging the role of ξ_1 and ξ_2 in Eq. (3.35), we get

$$h^l(\xi_2, FP_1\xi_1) = SP_3\nabla_{\xi_2}\xi_1. \tag{3.36}$$

Then using Eqs. (3.35) and (3.36), we acquire

$$h^l(\xi_1, FP_1\xi_2) - h^l(\xi_2, FP_1\xi_1) = SP_3[\xi_1,\xi_2]. \tag{3.37}$$

Next, using Eq. (3.13), for $\xi_1,\xi_2 \in \Gamma(Rad(TK))$, we have

$$h^s(\xi_1, FP_1\xi_2) = SP_4\nabla_{\xi_1}\xi_2 + nh^s(\xi_1,\xi_2). \tag{3.38}$$

Then interchanging the role of ξ_1 and ξ_2 in Eq. (3.38), we get

$$h^s(\xi_2, FP_1\xi_1) = SP_4\nabla_{\xi_2}\xi_1 + nh^s(\xi_2,\xi_1). \tag{3.39}$$

Further from Eqs. (3.38) and (3.39), we attain

$$h^s(\xi_1, FP_1\xi_2) - h^s(\xi_2, FP_1\xi_1) = SP_4[\xi_1,\xi_2]. \tag{3.40}$$

Hence the proof follows from Eqs. (3.31), (3.34), (3.37) and (3.40). □

Theorem 3.7. *For a* **s.l.s.** *K of $\bar{K}$, the distribution $FRad(TK)$ is integrable if and only if*

(i) $P_2(\nabla_{\xi_1^*}FP_2\xi_2^*) = P_2(\nabla_{\xi_2^*}FP_2\xi_1^*)$ *and* $P_4(\nabla_{\xi_1^*}\phi P_4\xi_2^*) = P_4(\nabla_{\xi_2^*}\phi P_4\xi_1^*)$,

(ii) $h^l(\xi_1^*, FP_2\xi_2^*) = h^l(\xi_2^*, FP_2\xi_1^*)$ *and* $h^s(\xi_1^*, FP_2\xi_2^*) = h^s(\xi_2^*, FP_2\xi_1^*)$,

for $\xi_1^*,\xi_2^* \in \Gamma(FRad(TK))$.

Proof. From Eq. (3.9), for $\xi_1^*,\xi_2^* \in \Gamma(FRad(TK))$, we have

$$P_2(\nabla_{\xi_1^*}FP_2\xi_2^*) = FP_1\nabla_{\xi_1^*}\xi_2^*. \tag{3.41}$$

After interchanging ξ_1^* and ξ_2^*, Eq. (3.41) becomes

$$P_2(\nabla_{\xi_2^*}FP_2\xi_1^*) = FP_1\nabla_{\xi_2^*}\xi_1^*. \tag{3.42}$$

From Eqs. (3.41) and (3.42), we obtain

$$P_2(\nabla_{\xi_1^*}FP_2\xi_2^*) - P_2(\nabla_{\xi_2^*}FP_2\xi_1^*) = FP_1[\xi_1^*,\xi_2^*]. \tag{3.43}$$

Further using Eq. (3.11), for $\xi_1^*,\xi_2^* \in \Gamma(FRad(TK))$, we acquire

$$P_4(\nabla_{\xi_1^*}\phi P_4\xi_2^*) = \phi P_4\nabla_{\xi_1^*}\xi_2^* + th^s(\xi_1^*,\xi_2^*). \tag{3.44}$$

By interchanging the role of ξ_1^* and ξ_2^* in Eq. (3.44), we get

$$P_4(\nabla_{\xi_2^*}\phi P_4\xi_1^*) = \phi P_4\nabla_{\xi_2^*}\xi_1^* + th^s(\xi_2^*,\xi_1^*). \tag{3.45}$$

Using Eqs. (3.44) and (3.45), we obtain

$$P_4(\nabla_{\xi_1^*}\phi P_4\xi_2^*) - P_4(\nabla_{\xi_2^*}\phi P_4\xi_1^*) = \phi P_4[\xi_1^*,\xi_2^*]. \tag{3.46}$$

Next, for $\xi_1^*,\xi_2^* \in \Gamma(FRad(TK))$, using Eq. (3.12), we have

$$h^l(\xi_1^*,FP_2\xi_2^*) = SP_3\nabla_{\xi_1^*}\xi_2^*. \tag{3.47}$$

Interchanging ξ_1^* and ξ_2^*, Eq. (3.47) yields

$$h^l(\xi_2^*,FP_2\xi_1^*) = SP_3\nabla_{\xi_2^*}\xi_1^*. \tag{3.48}$$

Following Eqs. (3.47) and (3.48), we obtain

$$h^l(\xi_1^*,FP_2\xi_2^*) - h^l(\xi_2^*,FP_2\xi_1^*) = SP_3[\xi_1^*,\xi_2^*]. \tag{3.49}$$

Finally, using Eq. (3.13), for $\xi_1^*,\xi_2^* \in \Gamma(FRad(TK))$, we attain

$$h^s(\xi_1^*,FP_2\xi_2^*) = SP_4\nabla_{\xi_1^*}\xi_2^* + nh^s(\xi_1^*,\xi_2^*). \tag{3.50}$$

By interchanging ξ_1^* and ξ_2^* in Eq. (3.50), we have

$$h^s(\xi_2^*,FP_2\xi_1^*) = SP_4\nabla_{\xi_2^*}\xi_1^* + nh^s(\xi_2^*,\xi_1^*). \tag{3.51}$$

Then from Eqs. (3.50) and (3.51), we obtain

$$h^s(\xi_1^*,FP_2\xi_2^*) - h^s(\xi_2^*,FP_2\xi_1^*) = SP_4[\xi_1^*,\xi_2^*]. \tag{3.52}$$

Hence the proof follows from Eqs. (3.43), (3.46), (3.49) and (3.52). □

4. TOTALLY UMBILICAL SLANT LIGHTLIKE SUBMANIFOLDS

Definition 4.1. *[6] A lightlike submanifold (K,g) of a semi-Riemannian manifold $(\bar{K},\bar{g})$ is called totally umbilical, if there exist a transversal curvature vector field $H \in \Gamma(tr(TK))$ on K such that*

$$h(Y_1,Y_2) = \bar{g}(Y_1,Y_2)H,$$

for $Y_1,Y_2 \in \Gamma(TK)$. Using Eqs. (2.3) and (2.5), we say that K is totally umbilical, if and only if, there exist smooth vector fields $H^l \in \Gamma(ltr(TK))$ and $H^s \in \Gamma(S(TK^{\perp}))$ such that

$$h^l(Y_1,Y_2) = g(Y_1,Y_2)H^l,\ h^s(Y_1,Y_2) = g(Y_1,Y_2)H^s,\ D^l(Y_1,W) = 0,$$

for $Y_1,Y_2 \in \Gamma(TK)$ and $W \in \Gamma(S(TK^{\perp}))$. On the other hand, a lightlike submanifold is totally geodesic if $h(Y_1,Y_2) = 0$, for $Y_1,Y_2 \in \Gamma(TK)$. Thus, a lightlike submanifold is totally geodesic, if $H^l = 0$ and $H^s = 0$.

Theorem 4.1. *Consider K a totally umbilical* **s.l.s.** *of $\bar{K}$. Then at least one of the following statements is true:*

(a) *K is an anti-invariant submanifold.*

(b) $D = \{0\}$.
(c) *If K is a proper slant lightlike submanifold, then $H^s \in \Gamma(\mu)$.*

Proof. For a totally umbilical **s.l.s.** K of $\bar{K}$, using Definition 4.1 and Eq. (3.22), for $Z = P_4Z \in \Gamma(D)$, we have

$$h(\phi P_4Z, \phi P_4Z) = g(\phi P_4Z, \phi P_4Z)H. \tag{4.1}$$

Then, using Eq. (2.3), we obtain

$$\cos^2\theta g(P_4Z, P_4Z)H = \bar{\nabla}_{\phi P_4Z}\phi P_4Z - \nabla_{\phi P_4Z}\phi P_4Z, \tag{4.2}$$

which yields

$$\cos^2\theta g(P_4Z, P_4Z)H = F\bar{\nabla}_{\phi P_4Z}P_4Z - \bar{\nabla}_{\phi P_4Z}SP_4Z - \nabla_{\phi P_4Z}\phi P_4Z. \tag{4.3}$$

Further using Eqs. (2.3)-(2.5), we derive

$$\begin{aligned}\cos^2\theta g(P_4Z, P_4Z)H = &F\nabla_{\phi P_4Z}P_4Z + Fh^l(\phi P_4Z, P_4Z) + Fh^s(\phi P_4Z, P_4Z)\\ &+ A_{SP_4Z}\phi P_4Z - \nabla^s_{\phi P_4Z}SP_4Z - D^l(\phi P_4Z, SP_4Z)\\ &- \nabla_{\phi P_4Z}\phi P_4Z.\end{aligned}$$

Employing Eqs. (3.2), (3.3) and using Definition 4.1, we obtain

$$\begin{aligned}\cos^2\theta g(P_4Z, P_4Z)H = &\phi\nabla_{\phi P_4Z}P_4Z + S\nabla_{\phi P_4Z}P_4Z + g(\phi P_4Z, P_4Z)FH^l\\ &+ g(\phi P_4Z, P_4Z)tH^s + g(\phi P_4Z, P_4Z)nH^s + A_{SP_4Z}\phi P_4Z\\ &- \nabla^s_{\phi P_4Z}SP_4Z - D^l(\phi P_4Z, SP_4Z) - \nabla_{\phi P_4Z}\phi P_4Z.\end{aligned} \tag{4.4}$$

Then considering the inner product of Eq. (4.4) w.r.t. SP_4Z, we get

$$\begin{aligned}\cos^2\theta g(P_4Z, P_4Z)\bar{g}(H^s, SP_4Z) = &\bar{g}(S\nabla_{\phi P_4Z}P_4Z, SP_4Z)\\ &-\bar{g}(\nabla^s_{\phi P_4Z}SP_4Z, SP_4Z).\end{aligned} \tag{4.5}$$

Taking $Y_1 = Y_2 \in \Gamma(D)$ in Eq. (3.23) and then considering the covariant derivative w.r.t. ϕP_4Z, we derive

$$\bar{g}(\nabla^s_{\phi P_4Z}SP_4Z, SP_4Z) = \sin^2\theta g(\nabla_{\phi P_4Z}P_4Z, P_4Z). \tag{4.6}$$

Next using Eqs. (3.23) and (4.6) in Eq. (4.5), we obtain

$$\cos^2\theta g(P_4Z, P_4Z)\bar{g}(H^s, SP_4Z) = 0. \tag{4.7}$$

Thus Eq. (4.7) yields that either $P_4Z = 0$ or $\theta = \pi/2$ or $H^s \in \Gamma(\mu)$. Hence, the proof follows. □

Theorem 4.2. *Every totally umbilical proper* **s.l.s** *of $\bar{K}$ is totally geodesic.*

Proof. Since $\bar{K}$ is a semi-Riemannian product manifold, therefore for $Z = P_4Z \in \Gamma(D)$, from Eq. (2.11), we have $\bar{\nabla}_Z FZ = F\bar{\nabla}_Z Z$, which gives that

$$\begin{aligned}&\nabla_Z \phi P_4 Z + h^l(Z, \phi P_4 Z) + h^s(Z, \phi P_4 Z) - A_{SP_4 Z} Z + \nabla^s_Z SP_4 Z \\ &+ D^l(Z, SP_4 Z) = \phi \nabla_Z Z + S\nabla_Z Z + Fh^l(Z,Z) + th^s(Z,Z) + nh^s(Z,Z). \qquad (4.8)\end{aligned}$$

In view of Definition 4.1 and equating the tangential components on both sides of above equation, we derive

$$\nabla_Z \phi P_4 Z - A_{SP_4 Z} Z = \phi \nabla_Z Z + Fh^l(Z,Z) + th^s(Z,Z). \qquad (4.9)$$

Next taking the inner product of Eq. (4.9) w.r.t. $F\xi \in \Gamma(Rad(TK))$, we obtain

$$g(A_{SP_4 Z} Z, F\xi) + \bar{g}(h^l(Z,Z), \xi) = 0. \qquad (4.10)$$

Then employing Eq. (2.6), we have

$$\bar{g}(h^s(Z, F\xi), SP_4 Z) + \bar{g}(F\xi, D^l(Z, SP_4 Z)) + \bar{g}(h^l(Z,Z), \xi) = 0. \qquad (4.11)$$

In view of Definition 4.1, the above equation reduces to

$$\bar{g}(H^s, SP_4 Z) g(Z, F\xi) + \bar{g}(H^l, \xi) g(Z,Z) = 0. \qquad (4.12)$$

From Theorem 4.1, we have $H^s \in \Gamma(\mu)$, therefore from Eq. (4.12), we have

$$\bar{g}(H^l, \xi) g(Z,Z) = 0. \qquad (4.13)$$

As D is non-degenerate, therefore we obtain $\bar{g}(H^l, \xi) = 0$, which further gives

$$H^l = 0. \qquad (4.14)$$

Moreover, $H^s \in \Gamma(\mu)$ for a proper totally umbilical **s.l.s.** of $\bar{K}$. Therefore, equating the transversal components on both sides of Eq. (4.8), we have

$$S\nabla_Z Z + nh^s(Z,Z) = h^l(Z, \phi P_4 Z) + h^s(Z, \phi P_4 Z) + \nabla^s_Z SP_4 Z + D^l(X, SP_4 Z).$$

Then using Definition 4.1, we derive

$$S\nabla_Z Z + g(Z,Z) nH^s = g(Z, \phi P_4 Z) H^l + g(Z, \phi P_4 Z) H^s + \nabla^s_Z SP_4 Z.$$

On taking the inner product of the above equation w.r.t. FH^s, we obtain

$$g(Z,Z)\bar{g}(H^s, H^s) = \bar{g}(\nabla^s_Z SP_4 Z, FH^s). \qquad (4.15)$$

Furthermore, one has $\bar{\nabla}_Z FH^s = F\bar{\nabla}_Z H^s$ and it implies

$$\begin{aligned}-A_{FH^s} Z + \nabla^s_Z FH^s + D^l(Z, FH^s) = &- \phi A_{H^s} Z - SA_{H^s} Z \\ &+ t\nabla^s_Z H^s + n\nabla^s_Z H^s + FD^l(Z, H^s). \qquad (4.16)\end{aligned}$$

Since μ is invariant by taking the inner product of Eq. (4.16) w.r.t. $SP_4 Z$, we obtain

$$\bar{g}(\nabla^s_Z FH^s, SP_4 Z) = -\bar{g}(SA_{H^s} Z, SP_4 Z) = -\sin^2\theta g(A_{H^s} Z, P_4 Z). \qquad (4.17)$$

As $\bar{\nabla}$ is a metric connection, thus we have $(\bar{\nabla}_Z\bar{g})(SP_4Z, FH^s) = 0$, which implies that $\bar{g}(\nabla^s_Z SP_4Z, FH^s) = -\bar{g}(\nabla^s_Z FH^s, SP_4Z)$, therefore Eq. (4.17) becomes

$$\bar{g}(\nabla^s_Z SP_4Z, FH^s) = \sin^2\theta g(A_{H^s}Z, P_4Z). \tag{4.18}$$

Then using Eq. (4.18) in Eq. (4.15), we have

$$g(Z,Z)\bar{g}(H^s, H^s) = \sin^2\theta g(A_{H^s}Z, P_4Z). \tag{4.19}$$

Now, employing Eqs. (2.6), the above equation yields

$$g(Z,Z)\bar{g}(H^s, H^s) = \sin^2\theta g(Z,Z)\bar{g}(H^s, H^s), \tag{4.20}$$

which implies that

$$(1-\sin^2\theta)g(Z,Z)\bar{g}(H^s, H^s) = 0.$$

As K is a proper **s.l.s.**, therefore $\sin^2\theta \neq 1$ and from the non-degeneracy of D, we derive

$$H^s = 0. \tag{4.21}$$

Hence, the result follows from Eqs. (4.14) and (4.21). □

Theorem 4.3. *For a proper totally umbilical* **s.l.s.** *K of $\bar{K}$, ∇ is always a metric connection.*

Proof. The proof follows directly from Eqs. (2.9) and (4.14). □

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