SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.19.02.03
Vol.19 (32), No.2 (2023), 171–182

WEIGHT DEPENDENT CONVOLUTION ON BEURLING SPACES AND MULTIPLIERS

ABUDULAÏ ISSA AND YAOGAN MENSAH

ABSTRACT. In this paper, some properties of a generalized translation operator are obtained. A weight dependent convolution product on Beurling spaces is studied. A convolution theorem related to a weight Fourier transform is obtained. Multipliers for the pair $(\mathcal{L}^1_\omega(G), \mathcal{L}^p_\omega(G))$ are introduced.

1. INTRODUCTION

Weighted Lebesgue spaces endowed with the classical convolution have been intensively studied; see for instance [2, 3, 8, 9, 11] and references therein. In [10], Mahmoodi introduced a generalized convolution product which is related to a Beurling weight function and he studied the representations of a derived group algebra. In [6], Issa and Mensah studied multipliers of the latter group algebra. Some results in the two references [6, 10] generalize results on the classical convolution product. The present article overlaps the reference [7].

The purpose of this article is to continue the study of the generalized convolution product. Mainly, we consider the aforementioned convolution product on Beurling spaces. We use the property of stabilization of Beurling spaces by a specific translation-type operator to study the properties of the weight dependent convolution. We also introduced multipliers and linked them to the convolution product.

The rest of the paper is organized as follows. In Section 2 we recall some facts about classical convolution in Lebesgue spaces. Section 3 contains results about the translation-type operators Γ^s_ω. Section 4 is devoted to the properties of the weighted convolution in the Beurling spaces. As an application, we introduce a class of multipliers in Section 5.

2. ON SOME FACTS ABOUT THE CLASSICAL CONVOLUTION

This section has borrowed a lot from [4, 9]. Some of the results mentionned here will find their generalization in Section 4.

Let G be a locally compact (Hausdorff) group with a fixed left Haar measure. Denote by $L^p(G)$, $1 \leq p < \infty$, the space of p-integrable complex functions on G and

2010 *Mathematics Subject Classification.* 43A15, 43A20,43A22, 22D15.
Key words and phrases. weight, Beurling space, convolution, Banach algebra, multiplier.

by $L^\infty(G)$ the space of measurable functions that are bounded almost everywhere. These spaces are endowed respectively with the following norms under which they are Banach spaces :

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}, 1 \leq p < \infty$$

and

$$\|f\|_\infty = \sup \mathrm{ess}|f|.$$

If $f, g \in L^1(G)$, the convolution product of f and g is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy.$$

It is well-known that the convolution product is commutative, that is $f * g = g * f$ for all $f, g \in L^1(G)$, if and only if, the group G is an abelian group. The following inequality holds and it gives $L^1(G)$ a Banach algebra structure :

$$\forall f, g \in L^1(G),\ \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

The convolution is extended to L^p-functions. The following facts hold.

- Let $1 \leq p \leq \infty$. If $f \in L^1(G)$ and $g \in L^p(G)$, then $f * g \in L^p(G)$ and
$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$
- Suppose G is unimodular. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(G)$ and $g \in L^q(G)$, then $f * g \in \mathcal{C}_0(G)$ and
$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q,$$
where $\mathcal{C}_0(G)$ is the space of complex continuous functions that tend to zero at infinity.
- Suppose G is unimodular. Let $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^p(G)$ and $g \in L^q(G)$, then $f * g \in L^r(G)$ and
$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

.

3. PROPERTIES OF THE Γ_ω^s OPERATORS

A Beurling weight on G is any continuous map $\omega : G \longrightarrow (0, \infty)$ such that

$$\omega(xy) \leq \omega(x)\omega(y),\ \omega(x) \geq 1, \text{ and } \omega(e) = 1,$$

where e is the neutral element of G. The Beurling space $L^p_\omega(G)$ is defined by

$$L^p_\omega(G) = \left\{ f : G \to \mathbb{C} : \int_G |f(x)|^p \omega(x) dx < \infty \right\}, 1 \leqslant p < +\infty. \tag{3.1}$$

The mapping $f \longmapsto \|f\|_{p,\omega} = \left(\int_G |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}}$ is a norm on $L^p_\omega(G)$. In [6]

Issa and Mensah introduced a generalized translation operator Γ_ω^s defined by

$$\Gamma_\omega^s f(x) = \frac{\tau_s M_\omega f(x)}{\omega(x)}, \tag{3.2}$$

where M_ω is the multiplication operator defined by $(M_\omega f)(x) = \omega(x) f(x)$, and τ_s is the translation operator defined by $(\tau_s f)(x) = f(s^{-1}x)$. If $\omega \equiv 1$, then Γ_ω^s is the translation operator τ_s. That is why Γ_ω^s is considered as a generalized translation operator.

We start with a density result which is of independent interest.

Theorem 3.1. *Let G be a locally compact group. The space $L^1_\omega(G)$ is a dense subspace of $\left(L^1(G), \|\cdot\|_1\right)$.*

Proof. The space $\mathcal{C}_c(G)$ of complex continuous functions with compact support is known to be dense in $L^1(G)$. Let $f \in L^1_\omega(G)$. Then one has

$$\int_G |f(x)|\omega(x)dx < \infty.$$

The hypothesis $1 \leqslant \omega$ implies $\int_G |f(x)|dx \leqslant \int_G |f(x)|\omega(x)dx$. Thus $L^1_\omega(G) \subset L^1(G)$. Also, if $f \in \mathcal{C}_c(G)$, then $f\omega \in \mathcal{C}_c(G)$. Therefore $f \in L^1_\omega(G)$ by the inclusion $\mathcal{C}_c(G) \subset L^1(G)$. Hence $\mathcal{C}_c(G) \subset L^1_\omega(G)$. We have

$$\mathcal{C}_c(G) \subset L^1_\omega(G) \subset L^1(G).$$

Finally, taking the closures with respect to the topology of $L^1(G)$, we obtain

$$\overline{\mathcal{C}_c(G)} \subset \overline{L^1_\omega(G)} \subset L^1(G).$$

This implies $\overline{L^1_\omega(G)} = L^1(G)$ since $\mathcal{C}_c(G)$ is dense in $L^1(G)$. Hence, the space $L^1_\omega(G)$ is a dense subspace of $\left(L^1(G), \|\cdot\|_1\right)$. □

Theorem 3.2. *Let G be a locally compact group. Let $1 \leq p < \infty$. Let $f \in L^p_\omega(G)$ and $s \in G$. Then*

$$[\omega(s)]^{\frac{1-p}{p}} \|f\|_{p,\omega} \leqslant \|\Gamma_\omega^s f\|_{p,\omega} \leqslant \left[\omega(s^{-1})\right]^{\frac{p-1}{p}} \|f\|_{p,\omega}. \tag{3.3}$$

Proof. Let $f \in L^p_\omega(G)$. For all $s \in G$, one has

$$\begin{aligned}
\|\Gamma_\omega^s f\|_{p,\omega}^p &= \int_G |\Gamma_\omega^s f(x)|^p \omega(x)dx \\
&= \int_G |f(s^{-1}x)|^p \left(\frac{\omega(s^{-1}x)}{\omega(x)}\right)^p \omega(x)dx \\
&= \int_G |f(x)|^p \left(\frac{\omega(x)}{\omega(sx)}\right)^p \omega(sx)dx \\
&= \int_G |f(x)|^p \left(\frac{\omega(x)}{\omega(sx)}\right)^{p-1} \omega(x)dx.
\end{aligned}$$

However, $\omega(x) = \omega(s^{-1}sx) \leqslant \omega(s^{-1})\omega(sx)$. Therefore, $\dfrac{\omega(x)}{\omega(sx)} \leqslant \omega(s^{-1})$. Then

$$\begin{aligned}\|\Gamma^s_\omega f\|^p_{p,\omega} &\leqslant \int_G |f(x)|^p \left[\omega(s^{-1})\right]^{p-1} \omega(x)dx \\ &\leqslant \left[\omega(s^{-1})\right]^{p-1} \int_G |f(x)|^p \omega(x)dx = \left[\omega(s^{-1})\right]^{p-1} \|f\|^p_{p,\omega}.\end{aligned}$$

Thus $\|\Gamma^s_\omega f\|_{p,\omega} \leqslant \left[\omega(s^{-1})\right]^{\frac{p-1}{p}} \|f\|_{p,\omega}$.

On the other hand,

$$\begin{aligned}\|\Gamma^s_\omega f\|^p_{p,\omega} &= \int_G |\Gamma^s_\omega f(x)|^p \omega(x)dx \\ &= \int_G |f(s^{-1}x)|^p \left(\frac{\omega(s^{-1}x)}{\omega(x)}\right)^p \omega(x)dx \\ &= \int_G |f(s^{-1}x)|^p \left(\frac{\omega(s^{-1}x)}{\omega(x)}\right)^{p-1} \omega(s^{-1}x)dx.\end{aligned}$$

However, $\omega(s)\omega(s^{-1}x) \geqslant \omega(x)$. Therefore, $\dfrac{\omega(s^{-1}x)}{\omega(x)} \geqslant [\omega(s)]^{-1}$. Then

$$\begin{aligned}\|\Gamma^s_\omega f\|^p_{p,\omega} &\geqslant \int_G |f(s^{-1}x)|^p [\omega(s)]^{1-p} \omega(s^{-1}x)dx \\ &\geqslant [\omega(s)]^{1-p} \int_G |f(s^{-1}x)|^p \omega(s^{-1}x)dx \\ &\geqslant [\omega(s)]^{1-p} \|f\|^p_{p,\omega}.\end{aligned}$$

Thus, $\|\Gamma^s_\omega f\|_{p,\omega} \geqslant [\omega(s)]^{\frac{1-p}{p}} \|f\|_{p,\omega}$. Finally,

$$[\omega(s)]^{\frac{1-p}{p}} \|f\|_{p,\omega} \leqslant \|\Gamma^s_\omega f\|_{p,\omega} \leqslant \left[\omega(s^{-1})\right]^{\frac{p-1}{p}} \|f\|_{p,\omega}. \qquad \square$$

As a consequence of the above theorem, we deduce respectively the stability of $L^p_\omega(G)$ under the action of the operator Γ^s_ω and the uniform continuity of Γ^s_ω as in the following two corollaries.

Corollary 3.1. *Let G be a locally compact group. Let $1 \leq p < \infty$. Then $f \in L^p_\omega(G)$ if and only of $\Gamma^s_\omega f \in L^p_\omega(G)$.*

Proof. From Theorem 3.2, we deduce that $\|f\|_{p,\omega} < \infty$ if and only if $\|\Gamma^s_\omega f\|_{p,\omega} < \infty$. $\square$

Corollary 3.2. *Let G be a locally compact group. Let $1 \leq p < \infty$. Fix $s \in G$. The map $f \longmapsto \Gamma^s_\omega f$ is uniformly continuous from $L^p_\omega(G)$ into $L^p_\omega(G)$.*

Proof. Let $f,g \in L^p_\omega(G)$. From Theorem 3.2, we have

$$\|\Gamma^s_\omega f - \Gamma^s_\omega g\|_{p,\omega} = \|\Gamma^s_\omega (f-g)\|_{p,\omega} \leqslant C\|f-g\|_{p,\omega}$$

where $C = \left[\omega(s^{-1})\right]^{\frac{p-1}{p}}$. Thus the map $f \longmapsto \Gamma^s_\omega f$ is uniformly continuous. □

4. THE WEIGHT DEPENDENT CONVOLUTION

In [10], Mahmoodi introduced a generalized convolution product $*_\omega$ that depends on the weight ω. This convolution product is defined by

$$f *_\omega g(x) = \int_G f(y)g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy \tag{4.1}$$

under the assumption that the latter integral exists. One may observe that

$$f *_\omega g(x) = \int_G f(y)\Gamma^y_\omega g(x)\omega(y)dy.$$

If $\omega \equiv 1$, then the usual convolution is recovered :

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy. \tag{4.2}$$

Through the rest of the paper, we denote by $\mathcal{L}^p_\omega(G)$ the space $(L^p_\omega(G), \|\cdot\|_{p,\omega})$ as far as the product $*_\omega$ is considered.

In the following theorem, we provide a sufficient condition under which the space $\mathcal{L}^p_\omega(G)$ is a Banach algebra.

Theorem 4.1. *Let G be a locally compact group. Let $p > 1$. The space $\mathcal{L}^p_\omega(G)$ is a Banach algebra if $\omega * \omega \leqslant \omega$.*

Proof. The Beurling space $(L^p_\omega(G), \|\cdot\|_{p,\omega})$ is known to be a Banach space. Thus, it suffices to show that

$$\|f *_\omega g\|_{p,\omega} \leqslant \|f\|_{p,\omega}\|g\|_{p,\omega}, \forall f,g \in \mathcal{L}^p_\omega(G)$$

under the hypothesis $\omega * \omega \leqslant \omega$. Let $f,g \in \mathcal{L}^p_\omega(G)$.

$$\begin{aligned} f *_\omega g(x) &= \int_G f(y)g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy \\ &= \int_G f(y)g(y^{-1}x)\left[\omega(y^{-1}x)\omega(y)\right]^{\frac{1}{p}}\frac{1}{\omega(x)}\left[\omega(y^{-1}x)\omega(y)\right]^{\frac{p-1}{p}}dy. \end{aligned}$$

Now, we use the Hölder's inequality to get

$$\begin{aligned}
&|f *_\omega g(x)| \leqslant \\
&\leqslant \left(\int_G |f(y)|^p \omega(y)|g(y^{-1}x)|^p \omega(y^{-1}x)dy\right)^{\frac{1}{p}} \left(\int_G \frac{1}{(\omega(x))^{\frac{p}{p-1}}}(\omega(y^{-1}x)\omega(y))dy\right)^{\frac{p-1}{p}} \\
&= \left(\int_G |f(y)|^p \omega(y)|g(y^{-1}x)|^p \omega(y^{-1}x)dy\right)^{\frac{1}{p}} \frac{1}{\omega(x)} \left(\int_G \left(\omega(y^{-1}x)\omega(y)\right)dy\right)^{\frac{p-1}{p}} \\
&= \left(\int_G |f(y)|^p \omega(y)|g(y^{-1}x)|^p \omega(y^{-1}x)dy\right)^{\frac{1}{p}} \frac{1}{\omega(x)} \left[W(x)\right]^{\frac{p-1}{p}}
\end{aligned}$$

where we have set $W = \omega * \omega$. It follows that

$$\begin{aligned}
&\int_G |(f *_\omega g)(x)|^p \,(\omega(x))^p W^{1-p}(x)dx \leqslant \\
&\leqslant \left(\int_G |f(y)|^p \omega(y)dy\right)\left(\int_G |g(y^{-1}x)|^p \omega(y^{-1}x)dx\right) \\
&= \|f\|^p_{p,\omega}\|g\|^p_{p,\omega}.
\end{aligned}$$

Moreover, $W = \omega * \omega \leq \omega \Rightarrow W^{1-p} \geq w^{1-p} \Rightarrow \omega^p W^{1-p} \geq \omega$. Hence,

$$\int_G |(f *_\omega g)(x)|^p \omega(x)dx \leqslant \int_G |(f *_\omega g)(x)|^p \,(\omega(x))^p W^{1-p}(x)dx.$$

$$\text{Thus,} \quad \|f *_\omega g\|_{p,\omega} \leqslant \|f\|_{p,\omega}\|g\|_{p,\omega}.$$

□

Remark 4.1. The fact that $\mathcal{L}^1_\omega(G)$ is a Banach algebra was proved in [10].

In what follows, we set $\check{\omega}(y) = \omega(y^{-1})$, $y \in G$.

Theorem 4.2. *Let G be a locally compact group. Let $p > 1$ and q be such that $\frac{1}{p} + \frac{1}{q}$=1. If $f\check{\omega}^{\frac{1}{q}} \in \mathcal{L}^1_\omega(G)$ and $g \in \mathcal{L}^p_\omega(G)$, then $f *_\omega g \in \mathcal{L}^p_\omega(G)$ and*

$$\|f *_\omega g\|_{p,\omega} \leq \|f\check{\omega}^{\frac{1}{q}}\|_{1,\omega}\|g\|_{p,\omega}.$$

Proof. Let f, g and ω be as in the hypothesis. We have

$$\begin{aligned}
\|f *_\omega g\|_{p,\omega} &= \left\|\int_G f(y)\Gamma^y_\omega g(\cdot)\omega(y)dy\right\|_{p,\omega} \\
&\leq \int_G |f(y)|\|\Gamma^y_\omega g(\cdot)\|_{p,\omega}\omega(y)dy \text{ (Minkowski's inequality)} \\
&\leq \int_G |f(y)|[\omega(y^{-1})]^{\frac{p-1}{p}}\|g\|_{p,\omega}\omega(y)dy \text{ (Theorem 3.2)} \\
&\leq \|f\check{\omega}^{\frac{1}{q}}\|_{1,\omega}\|g\|_{p,\omega}.
\end{aligned}$$

□

Theorem 4.3. *Let G be a locally compact unimodular group. Let $p > 1$. If $f \in \mathcal{L}^p_\omega(G)$, $g \in \mathcal{L}^1_\omega(G)$ and $g\check{\omega} \in \mathcal{L}^1_\omega(G)$, then $f *_\omega g \in \mathcal{L}^p_\omega(G)$ and*

$$\|f *_\omega g\|_{p,\omega} \le \|g\check{\omega}\|_{1,\omega}^{1-\frac{1}{p}} \|g\|_{1,\omega}^{\frac{1}{p}} \|f\|_{p,\omega}.$$

Proof. Let $p > 1$ and let q be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

$$\begin{aligned}
|(f *_\omega g)(x)| &\leqslant \int_G |f(y)||g(y^{-1}x)| \frac{\omega(y)\omega(y^{-1}x)}{\omega(x)} dy \\
&\leqslant \int_G |f(y)|\Gamma^y_\omega g(x)|\omega(y) dy \\
&\leqslant \int_G |f(y)||\Gamma^y_\omega g(x)\omega(y)|^{\frac{1}{p}} |\Gamma^y_\omega g(x)\omega(y)|^{\frac{1}{q}} dy \\
&\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \left(\int_G |\Gamma^y_\omega g(x)\omega(y) dy|\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \left(\int_G |\frac{g(y^{-1}x)\omega(y^{-1}x)}{\omega(x)} \omega(y)| dy\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \left(\int_G |\frac{g(y)\omega(y)}{\omega(x)} \omega(xy^{-1})| dy\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \left(\int_G |g(y)\omega(y)\omega(y^{-1})| dy\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \left(\int_G |g(y)\omega(y^{-1})\omega(y)| dy\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \|g\check{\omega}\|_{1,\omega}^{\frac{1}{q}} \\
&\leqslant ((|g| *_\omega |f|^p)(x))^{\frac{1}{p}} \|g\check{\omega}\|_{1,\omega}^{\frac{1}{q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_G |(f *_\omega g)(x)|^p \omega(x) dx &\leqslant \int_G (|g| *_\omega |f|^p)(x) \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \omega(x) dx \\
&\leqslant \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \int_G (|g| *_\omega |f|^p)(x)\omega(x) dx \\
&\leqslant \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \|g\|_{1,\omega} \||f|^p\|_{1,\omega} \\
&\leqslant \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \|g\|_{1,\omega} \|f\|_{p,\omega}^p.
\end{aligned}$$

Hence, $\|f *_\omega g\|_{p,\omega} \le \|g\check{\omega}\|_{1,\omega}^{1-\frac{1}{p}} \|g\|_{1,\omega}^{\frac{1}{p}} \|f\|_{p,\omega}$. □

We recall the following fact which we may use in the sequel.

Theorem 4.4. *[5, Corollary 12.5] Let $f_1, f_2, ..., f_n$ be nonnegative functions in $L^1(G)$ and let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then,*

$$f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n} \in L^1(G)$$

$$\text{and } \int_G f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n} \leqslant \|f_1\|_1^{\alpha_1} \|f_2\|_1^{\alpha_2} \cdots \|f_n\|_1^{\alpha_n}.$$

Theorem 4.5. *Let p and q be real numbers such that*

$$1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} > 1.$$

Let $r = \dfrac{pq}{p+q-pq}$, so that $\dfrac{1}{p} + \dfrac{1}{q} - \dfrac{1}{r} = 1$. Let G be a locally compact unimodular group. Let $f \in \mathcal{L}^p_{\omega^p}(G)$ and $g \in \mathcal{L}^q_{\omega^q}(G)$. Then $f *_\omega g \in L^r(G)$ and

$$\|f *_\omega g\|_r \leqslant \|f\|_{p,\omega^p} \|g\|_{q,\omega^q}.$$

Proof. Set $A = |(f *_\omega g)(x)|$. We have

$$\begin{aligned}
A &\leqslant \int_G |f(y)g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}|dy \\
&\leqslant \int_G |f(y)g(y^{-1}x)\omega(y^{-1}x)\omega(y)|dy \\
&= \int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right]^{\frac{1}{r}} |g(y)\omega(y)|^{1-\frac{q}{r}} |f(y^{-1}x)\omega(y^{-1}x)|^{1-\frac{p}{r}} dy \\
&= \int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right]^{\frac{1}{r}} |g^q(y)\omega^q(y)|^{\frac{1}{q}-\frac{1}{r}} |f^p(y^{-1}x)\omega^p(y^{-1}x)|^{\frac{1}{p}-\frac{1}{r}} dy.
\end{aligned}$$

Since $\frac{1}{p} - \frac{1}{r} = \frac{q-1}{q}$ and $\frac{1}{q} - \frac{1}{r} = \frac{p-1}{p}$, by applying Theorem 4.4, one has

$$\int_G |f(y^{-1}x)g(y)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}|dy \leqslant \left[|\int_G f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|dy\right]^{\frac{1}{r}} \times$$
$$\times \left(\int_G |g^q(y)\omega^q(y)|dy\right)^{\frac{p-1}{p}} \times \left(\int_G |f^p(y^{-1}x)\omega^p(y^{-1}x)|dy\right)^{\frac{q-1}{q}}.$$ It follows that

$$|(f *_\omega g)(x)|^r \leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}} (\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} \int_G |f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|dy.$$

Set $B = \displaystyle\int_G |(f *_\omega g)(x)|^r dx$. Then,

$$
\begin{aligned}
B &\leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}} (\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} \int_G\int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right] dydx \\
&\leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}} (\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} \left[\int_G\left(\int_G |f(y^{-1}x)|^p\omega^p(y^{-1}x)dx\right)|g(y)|^q\omega^q(y)dy\right] \\
&= (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}} (\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} (\|f\|_{p,\omega^p})^p \left[\int_G g^q(y)\omega^q(y)|dx\right] \\
&= (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}} (\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} (\|f\|_{p,\omega^p})^p (\|g\|_{q,\omega^q})^q \\
&\leqslant (\|f\|_{p,\omega^p})^{p+\frac{pr(q-1)}{q}} (\|g\|_{q,\omega^q})^{q+\frac{qr(p-1)}{p}} = \|f\|_{p,\omega^p}^r\|g\|_{q,\omega^q}^r.
\end{aligned}
$$

Thus, $\|f *_\omega g\|_r \leqslant \|f\|_{p,\omega^p}\|g\|_{q,\omega^q}$. □

Theorem 4.6. *Let p and q be real numbers such that $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$, and let $r = \frac{pq}{p+q-pq}$, so that $\frac{1}{p}+\frac{1}{q}-\frac{1}{r} = 1$. Let G be a locally compact unimodular group. Let $f \in \mathcal{L}^p_{\omega^{p+1}}(G)$ and $g \in \mathcal{L}^q_{\omega^{q+1}}(G)$. Then $f *_\omega g \in \mathcal{L}^r_\omega(G)$ and*

$$\|f *_\omega g\|_{r,\omega} \leqslant \|f\|_{p,\omega^{p+1}}\|g\|_{q,\omega^{q+1}}.$$

Proof. In the proof of Theorem 4.5 we established that

$$|(f *_\omega g)(x)|^r \leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\int_G |f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|dy.$$

Set $A = \int_G |(f *_\omega g)(x)|^r\omega(x)dx$. We have

$$
\begin{aligned}
A &\leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\int_G\int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right] dy\omega(x)dx \\
&\leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\left[\int_G\left(\int_G |f(y^{-1}x)|^p\omega^p(y^{-1}x)\omega(x)dx\right)|g(y)|^q\omega^q(y)dy\right] \\
&= (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\left[\int_G\left(\int_G |f(x)|^p\omega^p(x)\omega(yx)dx\right)|g(y)|^q\omega^q(y)dy\right] \\
&\leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\left[\int_G\left(\int_G |f(x)|^p\omega^{p+1}(x)dx\right)|g(y)|^q\omega^{q+1}(y)dy\right] \\
&= (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} \left(\|f\|_{p,\omega^{p+1}}\right)^p \left[\int_G g^q(y)\omega^{q+1}(y)|dx\right] \\
&= (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}} \left(\|f\|_{p,\omega^{p+1}}\right)^p \left(\|g\|_{q,\omega^{q+1}}\right)^q \\
&\leqslant \left(\|f\|_{p,\omega^{p+1}}\right)^{p+\frac{pr(q-1)}{q}}\left(\|g\|_{q,\omega^{q+1}}\right)^{q+\frac{qr(p-1)}{p}} = \|f\|_{p,\omega^{p+1}}^r\|g\|_{q,\omega^{q+1}}^r
\end{aligned}
$$

because $\|f\|_{p,\omega^p} \leqslant \|f\|_{p,\omega^{p+1}}$ for $f \in \mathcal{L}^p_{\omega^{p+1}}(G)$ and $\|g\|_{q,\omega^q} \leqslant \|g\|_{q,\omega^{q+1}}$ for $g \in \mathcal{L}^q_{\omega^{q+1}}(G)$. Thus $\|f *_\omega g\|_{r,\omega} \leqslant \|f\|_{p,\omega^{p+1}}\|g\|_{q,\omega^{q+1}}$. □

Now, we look for a convolution theorem. Let us denote by $\widehat{G}$ the unitary dual of the group G, that is the set of unitary irreducible representations of G. For a class $\pi \in \widehat{G}$, denote by H_π its representation Hilbert space. Define the weighted Fourier transform $\mathcal{F}_\omega(f)$ of a function $f \in \mathcal{L}^1_\omega(G)$ by

$$\mathcal{F}_\omega(f)(\pi) = \int_G \pi(x)^* f(x)\omega(x)dx, \pi \in \widehat{G}.$$

Here, $\pi(x)^*$ denotes the adjoint of the operator $\pi(x)$.

Theorem 4.7. *Let G be a locally compact group. If $f, g \in \mathcal{L}^1_\omega(G)$, then $\forall \pi \in \widehat{G}$,*

$$\mathcal{F}_\omega(f *_\omega g)(\pi) = \mathcal{F}_\omega(g)(\pi)\mathcal{F}_\omega(f)(\pi).$$

Proof.

$$\begin{aligned}
\mathcal{F}_\omega(f *_\omega g)(\pi) &= \int_G (f *_\omega g)(x)\pi(x)^*\omega(x)dx \\
&= \iint_{G\times G} \pi(x)^* f(y)g(y^{-1}x)\omega(y^{-1}x)\omega(y)dydx \\
&= \iint_{G\times G} \pi(x)^* f(y)g(y^{-1}x)\omega(y^{-1}x)\omega(y)dydx \\
&= \iint_{G\times G} \pi(yx)^* f(y)g(x)\omega(x)\omega(y)dydx \\
&= \iint_{G\times G} \pi(x)^*\pi(y)^* f(y)g(x)\omega(x)\omega(y)dydx \\
&= \int_G \pi(x)^* \left(\int_G \pi(y)^* f(y)\omega(y)dy\right) g(x)\omega(x)dx \\
&= \int_G \pi(x)^* g(x)\omega(x)dx \int_G \pi(y)^* f(y)\omega(y)dy.
\end{aligned}$$

Thus $\mathcal{F}_\omega(f *_\omega g)(\pi) = \mathcal{F}_\omega(g)(\pi)\mathcal{F}_\omega(f)(\pi)$. □

5. MULTIPLIERS FOR THE PAIR $(\mathcal{L}^1_\omega(G), \mathcal{L}^p_\omega(G))$

According to Corollary 3.1, $f \in \mathcal{L}^p_\omega(G)$ if and only if $\Gamma^s_\omega f \in \mathcal{L}^p_\omega(G)$. Therefore we are able to define multipliers in our framework.

Definition 5.1. *A linear operator $T : \mathcal{L}^1_\omega(G) \longrightarrow \mathcal{L}^p_\omega(G)$ is said to be a* multiplier *for the pair $(\mathcal{L}^1_\omega(G), \mathcal{L}^p_\omega(G))$ if T commutes with every operator Γ^s_ω, $s \in G$. That is,*

$$T\Gamma^s_\omega = \Gamma^s_\omega T.$$

We denote by $\mathcal{M}^{1,p}_\omega(G)$ the set of multipliers for the pair $(\mathcal{L}^1_\omega(G), \mathcal{L}^p_\omega(G))$. Let us mention that the multipliers of the space $\mathcal{L}^1_\omega(G)$ have been studied in [6]. Therefore, one may assume here that $p > 1$.

For $f \in \mathcal{L}^p_\omega(G)$ and $h \in \mathcal{L}^q_\omega(G)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we set

$$\langle f, h\rangle_\omega = \int_G f(x)h(x^{-1})\omega(x)dx.$$

Theorem 5.1. *Let $T \in \mathcal{M}^{1,p}_\omega(G)$. Then $\forall f, h \in \mathcal{L}^1_\omega(G)$,*

$$T(f *_\omega h) = f *_\omega Th.$$

Proof. Let $T \in \mathcal{M}^{1,p}_\omega(G)$. Let $f, h \in \mathcal{L}^1_\omega(G)$ and $\xi \in \mathcal{L}^q_\omega(G)$. Let

$$T^* : \mathcal{L}^q_\omega(G) \longrightarrow (\mathcal{L}^1_\omega(G))^*$$

be the adjoint operator of T. Then

$$\begin{aligned}
\langle f *_\omega Th, \xi\rangle_\omega &= \int_G (f *_\omega Th)(x)\xi(x^{-1})\omega(x)dx \\
&= \iint_{G\times G} f(y)Th(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}\xi(x^{-1})\omega(x)dydx \\
&= \iint_{G\times G} f(y)\Gamma^y_\omega Th(x)\omega(x)\omega(y)\xi(x^{-1})dxdy \quad \text{(Funibi's Theorem)} \\
&= \iint_{G\times G} f(y)T\Gamma^y_\omega h(x)\omega(x)\omega(y)\xi(x^{-1})dxdy \\
&= \int_G \langle T\Gamma^y_\omega h, \xi\rangle_\omega f(y)\omega(y)dy \\
&= \int_G \langle \Gamma^y_\omega h, T^*\xi\rangle_\omega f(y)\omega(y)dy \\
&= \iint_{G\times G} \Gamma^y_\omega h(x)T^*\xi(x^{-1})\omega(x)\omega(y)f(y)dxdy \\
&= \iint_{G\times G} f(y)h(y^{-1}x)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}T^*\xi(x^{-1})\omega(x)dydx \\
&= \int_G (f*_\omega h)(x)T^*\xi(x^{-1})\omega(x)dx \\
&= \langle f*_\omega h, T^*\xi\rangle_\omega = \langle T(f*_\omega h), \xi\rangle_\omega.
\end{aligned}$$

Since ξ is arbitrary, then $T(f *_\omega h) = f *_\omega Th$. □

6. Conclusion

Some properties of a generalized translation operator are obtained. Using these properties, a convolution product on Beurling spaces have been studied. A convolution theorem related to a weight Fourier transform is obtained. Multipliers for the pair $(\mathcal{L}^1_\omega(G), \mathcal{L}^p_\omega(G))$ are introduced. As a perspective of this work, it would be interesting to characterize the multipliers as weight convolution operators by the means of the weight Fourier transform. It would also be interesting to study the case $p = \infty$.

REFERENCES

[1] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Univ. Press, United Kingdom, 2001.

[2] A. Benazzouz, *Contribution à l'analyse harmonique des algèbres de Beurling généralisées*, Thèse de 3ème cycle, Faculté des Sciences de Rabat, 1986.

[3] A. Bourouihiya, *Beurling weighted spaces, product-convolution operators and the tensor product of frames*, PhD thesis, University of Maryland, USA (2006).

[4] G. B. Folland, *A course in abstract harmonic analysis*, CRC Press, Inc, 1995.

[5] E. Hewitt K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, (1979).

[6] A. Issa, Y. Mensah, *Multipliers on weighted group algebras*, Gulf J. Math., 8(2), (2020), 35-45.

[7] A. Issa, Y. Mensah, *On Beurling spaces provided with a weight dependent convolution*, Functional Analysis, vol. 1, (2022) ID 6.

[8] E. Kaniuth, *A course in commutative Banach algebras*, Springer-Verlag, (2009).

[9] A. EL Kinani, A. Roukbi, A. Benazzouz, *Structure d'algèbre de Banach sur l'espace à poids* $L^p_\omega(G)$, Le Mathematiche, vol. LXIV (2009), Fasc. I, 179-193.

[10] A. Mahmoodi, *A new convolution on Beurling algebras*, Math. Sci., vol. 3, No 9, (2009) 63-82.

[11] H. Reiter, J. D. Stegeman, *Classical harmonic analysis and locally compact groups*, London Math. Society Monographs, vol. 22, Clarendon Press, Oxford, (2000).

(Received: August 23, 2021)
(Revised: May 13, 2023)

Abudulaï Issa
University of Lomé
Department of Mathematics
1 BP 1515 Lomé 01, Togo
e-mail: *issaabudulai13@gmail.com*
and
Yaogan Mensah
University of Lomé
Department of Mathematics
1 BP 1515 Lomé 01, Togo
and ICMPA-Unesco Chair
University of Abomey-Calavi, Benin
e-mail: *mensahyaogan2@gmail.com*