A KOROVKIN-TYPE APPROXIMATION THEOREM FOR POSITIVE LINEAR OPERATORS IN $H_\omega(K)$ VIA POWER SERIES METHOD

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ABSTRACT. The aim of this paper is to present Korovkin theorems for positive linear operators of two variables from $H_\omega(K)$ into $C_B(K)$ via the power series method. In addition, we give an example that our new approximation result works but its classical case does not work. Furthermore, we obtain the rate of convergence of these operators.

1. INTRODUCTION

The classical Korovkin theorem obtains the uniform convergence of the sequence of positive linear operators to the identity operator in the space of real-valued continuous functions, $C[0,1]$, by using only three functions [1, 17]. That is, these types of theorems exhibit a variety of test functions, which assume the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1953 for the functions $1, x$ and x^2 in $C[0,1]$ [16]. Due to the simplicity and efficiency of these theorems, Korovkin-type approximation theory is a popular and well studied area in approximation theory.

Using various types of convergence or by changing the test functions, many mathematicians have investigated the Korovkin-type approximation theorems for a sequence of positive linear operators defined on different spaces (see e.g. [5–10, 15, 18–22]).

In this paper, we develop the Korovkin type approximation theorem for a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$ with the use of the power series method which is also a member of the class of all continuous summability methods. This method includes the Abel method as well as the Borel method. The results presented in this paper are motivated by [10] and [11].

Let $I = [0, \infty)$. The norm on $C_B(I)$ can be defined by

$$\|f\|_{C_B} = \sup_{x \in I} |f(x)|, \quad f \in C_B(I).$$

It is well-known that $H_\omega(I) \subset C_B(I)$ for any $f \in H_\omega(I)$. We recall that the Korovkin-type theorem on the $H_\omega(I)$ space was given by Gadjiev and Çakar in [11].

Theorem 1.1. *Let (L_n) be a sequence of positive linear operators from $H_\omega(I)$ into $C_B(I)$. Then for any $f \in H_\omega(I)$,*

$$\lim_{n \to \infty} \|L_n(f) - f\|_{C_B} = 0$$

is satisfied if the following holds:

$$\lim_{n \to \infty} \left\| L_n\left(\left(\frac{t}{1+t}\right)^v; x\right) - \left(\frac{x}{1+x}\right)^v \right\|_{C_B} = 0, \quad v = 0, 1, 2, 3.$$

Similarly as in [11], let us introduce a space denoted by $H_\omega(K)$, where $K := I^2 = [0,\infty) \times [0,\infty)$. Let ω be a function of the type of the modulus of continuity such that the following conditions are satisfied:

(1) ω is nonnegative increasing function on K with respect to δ_1, δ_2,
(2) $\omega(\delta, \delta_1 + \delta_2) \leq \omega(\delta, \delta_1) + \omega(\delta, \delta_2)$,
(3) $\omega(\delta_1 + \delta_2, \delta) \leq \omega(\delta_1, \delta) + \omega(\delta_2, \delta)$,
(4) $\lim_{\delta_1, \delta_2 \to 0} \omega(f; \delta_1, \delta_2) = 0$.

Consider the space $H_\omega(K)$ of real-valued functions f defined on K satisfying the following condition:

$$|f(u,v) - f(x,y)| \leq \omega\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right).$$

Also we denote the space of functions f, which is bounded and continuous on K by $C_B(K)$. A norm in $C_B(K)$ can be defined as

$$\|f\|_{C_B} = \sup_{(x,y) \in K} |f(x,y)|, \quad f \in C_B(K).$$

It is obvious by (4) that any function in $H_\omega(K)$ is continuous on K. Morever, any function $f \in H_\omega(K)$ satisfies the inequality

$$|f(x,y)| \leq |f(0,0)| + \omega(1,1), \text{ for } x, y \geq 0$$

and therefore is bounded on K. So, $H_\omega(K) \subset C_B(K)$.

2. APPROXIMATION PROPERTIES ON $H_\omega(K)$ VIA THE POWER SERIES METHOD

In this section, we give, using the power series method, the Korovkin approximation theorem for the sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Let (p_k) be a real sequence with $p_0 > 0$ and $p_k \geq 0$ $(k \in \mathbb{N})$, such that the corresponding power series $p(t) = \sum_{k=0}^{\infty} p_k t^k$ has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} x_k p_k t^k = L$$

exists then we say that $x = (x_k)$ is convergent in the sense of power series method [12, 14]. It is well-known that the power series method is regular if and only if

$$\lim_{t \to R^-} \frac{p_k t^k}{p(t)} = 0, \text{ for each } k \in \mathbb{N} \tag{2.1}$$

holds [4]. Throughout the paper, we assume that the methods fulfill condition (2.1). Let $L : H_\omega(K) \to C_B(K)$ be a linear operator. Then L is called positive if $Lf \geq 0$ whenever $f \geq 0$. If L is a positive linear operator then $f \leq g$ implies that $Lf \leq Lg$ and $|f| \leq g$ implies $|Lf| \leq Lg$.

In this section we assume that (L_k) is a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$ such that

$$\sup_{0 < t < R} \frac{1}{p(t)} \sum_{k=0}^{\infty} \|L_k(1)\|_{C_B} p_k t^k < \infty. \tag{2.2}$$

Also, $V_t((.);x,y)$ defined by

$$V_t(f;x,y) := \frac{1}{p(t)} \sum_{k=0}^{\infty} L_k(f;x,y) p_k t^k.$$

This positive linear operator is well-defined from $H_\omega(K)$ into $B(K)$ as we can see from the following inequality

$$\|V_t((.);x,y)\|_{C_B} \leq \sup_{0 < t < R} \frac{1}{p(t)} \sum_{k=0}^{\infty} \|L_k(1)\|_{C_B} p_k t^k < \infty.$$

Observe that $V_t((.);x,y)$ is also a positive linear operator.

We can derive the following theorem using the above terminology.

Theorem 2.1. *Let (L_k) be positive linear operators from $H_\omega(K)$ into $C_B(K)$ such that (2.2) holds. Then, for any $f \in H_\omega(K)$,*

$$\lim_{t \to R^-} \|V_t(f) - f\|_{C_B} = 0 \tag{2.3}$$

is satisfied if the following holds:

$$\lim_{t \to R^-} \|V_t(f_i) - f_i\|_{C_B} = 0, \ i = 0, 1, 2, 3 , \tag{2.4}$$

where

$$f_0(u,v) = 1, \ f_1(u,v) = \frac{u}{1+u}, \ f_2(u,v) = \frac{v}{1+v}, \ f_3(u,v) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2.$$

Proof. Assume that (2.4) holds and let $f \in H_\omega(K)$. Then by the equality (2.4) and since f is continuous on K for every $\varepsilon > 0$ there exist real numbers $\delta_1, \delta_2 > 0$ such

that $|f(u,v) - f(x,y)| < \varepsilon$ holds for all $(u,v) \in K$ satisfying $\left| \frac{u}{1+u} - \frac{x}{1+x} \right| \leq \delta_1$ and $\left| \frac{v}{1+v} - \frac{y}{1+y} \right| \leq \delta_2$. Let

$$K_{\delta_1, \delta_2} = \left\{ (u,v) \in K : \left| \frac{u}{1+u} - \frac{x}{1+x} \right| \leq \delta_1 \text{ and } \left| \frac{v}{1+v} - \frac{y}{1+y} \right| \leq \delta_2 \right\}. \quad (2.5)$$

Hence we can obtain

$$|f(u,v) - f(x,y)| < \varepsilon + 2H\chi_{K/K_{\delta_1,\delta_2}}(u,v), \quad (2.6)$$

where χ_K denotes the characteristic function of the set K and $H := \|f\|_{C_B}$. We can also obtain,

$$\chi_{K/K_{\delta_1,\delta_2}}(u,v) \leq \frac{1}{\delta_1^2} \left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \frac{1}{\delta_2^2} \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2. \quad (2.7)$$

Combining (2.6) and (2.7) we get

$$|f(u,v) - f(x,y)| \leq \varepsilon + \frac{2H}{\delta^2} \left\{ \left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2 \right\} \quad (2.8)$$

where $\delta = \min\{\delta_1, \delta_2\}$. Using linearity and positivity of the operators $V_t((.);x,y)$ we observe that

$$\begin{aligned} |V_t(f;x,y) - f(x,y)| &\leq V_t(|f(u,v) - f(x,y)|;x,y) \\ &\quad + |f(x,y)| |V_t(f_0;x,y) - f_0(x,y)| \end{aligned}$$

for all $t \in (0,R)$. On the other hand, from the inequality (2.8), we can write

$$\begin{aligned} |V_t(f;x,y) - f(x,y)| &\leq \varepsilon V_t(f_0;x,y) \\ &\quad + \frac{2H}{\delta^2} \left[V_t\left(\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 ;x,y \right) \right. \\ &\quad \left. + V_t\left(\left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2 ;x,y \right) \right] \\ &\quad + H |V_t(f_0;x,y) - f_0(x,y)| . \end{aligned}$$

By making some calculations, we have

$$\begin{aligned} |V_t(f;x,y) - f(x,y)| &\leq \varepsilon + \left(\varepsilon + H + \frac{2H}{\delta^2} \right) |V_t(f_0;x,y) - f_0(x,y)| \\ &\quad + \frac{4H}{\delta^2} |V_t(f_1;x,y) - f_1(x,y)| \\ &\quad + \frac{4H}{\delta^2} |V_t(f_2;x,y) - f_2(x,y)| \\ &\quad + \frac{2H}{\delta^2} |V_t(f_3;x,y) - f_3(x,y)| . \end{aligned}$$

Then by taking the supremum for $(x, y) \in K$, we find

$$\|V_t(f) - f\|_{C_B} \leq L \{ \|V_t(f_0) - f_0\|_{C_B}$$
$$+ \|V_t(f_1) - f_1\|_{C_B} + \|V_t(f_2) - f_2\|_{C_B} \|V_t(f_3) - f_3\|_{C_B} \} + \varepsilon, \quad (2.9)$$

where $L = \max \{ \varepsilon + H + \frac{2H}{\delta^2}, \frac{4H}{\delta^2}, \frac{2H}{\delta^2} \}$. Hence it follows from (2.4) and (2.9) that

$$\lim_{t \to R^-} \|V_t(f) - f\|_{C_B} = 0.$$

As a result, the proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.1. *Now we exhibit an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but not satisfy the the condition of Theorem 1.1.*

Let $p_k = 1$, in this case $R = 1$ and $p(t) = \frac{1}{1-t}, t \in (-1, 1)$. Thus, the power series method corresponds to the Abel method. Consider the sequence (L_k) defined by $L_k:$ $H_\omega(K) \to C_B(K), L_k(f; x, y) = (1 + \alpha_k) B_k(f; x, y)$ where (B_k) are the Bleimann, Butzer and Hanh operators [3] defined by

$$B_k(f; x, y) = \frac{1}{(1+x)^k (1+y)^k} \sum_{m=0}^{k} \sum_{n=0}^{k} f\left(\frac{m}{k-m+1}, \frac{n}{k-n+1}\right) \binom{k}{m} \binom{k}{n} x^m y^n.$$

Take $(\alpha_k) = \left((-1)^k\right)$. Observe that (α_k) is Abel convergent to zero, but it is not convergent. From [3], we have the following

$$B_k(f_0; x, y) = 1,$$

$$B_k(f_1; x, y) = \frac{k}{(1+k)} \frac{x}{(1+x)},$$

$$B_k(f_2; x, y) = \frac{k}{(1+k)} \frac{y}{(1+y)},$$

$$B_k(f_3; x, y) = \frac{k(k-1)}{(k+1)^2} \frac{x^2}{(1+x)^2} + \frac{k}{(k+1)^2} \frac{x}{1+x} + \frac{k(k-1)}{(k+1)^2} \frac{y^2}{(1+y)^2} + \frac{k}{(k+1)^2} \frac{y}{1+y},$$

where

$$f_0(u, v) = 1, \ f_1(u, v) = \frac{u}{1+u}, \ f_2(u, v) = \frac{v}{1+v}, \ f_3(u, v) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2.$$

Thus, we obtain that

$$\lim_{t \to 1^-} (1-t) \left\| \sum_{k=0}^{\infty} (L_k(f_i; x, y) - f_i(x, y)) t^k \right\|_{C_B} = 0, \ i = 0, 1, 2, 3.$$

Therefore, by Theorem 2.1 we see that, for all $f \in H_\omega(K)$

$$\lim_{t \to 1^-} (1-t) \left\| \sum_{k=0}^{\infty} (L_k(f;x,y) - f(x,y)) t^k \right\|_{C_B} = 0.$$

However, since (α_k) is not convergent, (L_k) is not uniformly convergent to f.

3. RATE OF CONVERGENCE

We now study the rate of convergence of the sequence of positive linear operators examined in Theorem 2.1 by using the modulus of smoothness.

Let us introduce the following modulus of smoothness for the bivariate case similarly as in [13] (see, for details [2], Sec 2.3).

$$\varpi(f;\delta_1,\delta_2) = \sup \left\{ \left| f(u,v) - f(x,y) \right| : (u,v),(x,y) \in K \text{ and} \right.$$

$$\left. \left| \frac{u}{1+u} - \frac{x}{1+x} \right| \leq \delta_1, \left| \frac{v}{1+v} - \frac{y}{1+y} \right| \leq \delta_2 \right\}, \quad \delta_1, \delta_2 > 0. \text{ It is clear that if } f \in H_\omega(K)$$

then, we have

(1) $\lim_{\delta_1,\delta_2 \to 0} \varpi(f;\delta_1,\delta_2) = 0$,

(2) $|f(u,v) - f(x,y)| \leq \varpi(f;\delta_1,\delta_2) \left(1 + \frac{\left| \frac{u}{1+u} - \frac{x}{1+x} \right|}{\delta_1,} \right) \left(1 + \frac{\left| \frac{v}{1+v} - \frac{y}{1+y} \right|}{\delta_2} \right).$

Theorem 3.1. *Let (L_k) be positive linear operators from $H_\omega(K)$ into $C_B(K)$ such that (2.2) holds. If*

(1) $\lim_{t \to R^-} \|V_t(f_0) - f_0\|_{C_B} = 0$,

(2) $\lim_{t \to R^-} \varpi(f;\alpha(t),\beta(t)) = 0$,

Then, for all $f \in H_\omega(K)$ we have the following

$$\lim_{t \to R^-} \|V_t(f) - f\|_{C_B} = 0$$

where

$$\alpha(t) = \sqrt{V_t \left(\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 ; x,y \right)} \text{ and } \beta(t) = \sqrt{V_t \left(\left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2 ; x,y \right)}.$$

Proof. Let $f \in H_\omega(K)$. By using the linearity and positivity of $V_t((.);x,y)$ and for every $x,y \in K$ taking into account

$$|f(u,v) - f(x,y)| \leq \varpi(f;\delta_1,\delta_2) \left(1 + \frac{\left| \frac{u}{1+u} - \frac{x}{1+x} \right|}{\delta_1,} \right) \left(1 + \frac{\left| \frac{v}{1+v} - \frac{y}{1+y} \right|}{\delta_2} \right)$$

for all $t \in (0,R)$ and $\delta_1, \delta_2 > 0$ we observe

$$|V_t(f;x,y) - f(x,y)| \leq V_t(|f(u,v) - f(x,y)|;x,y) + H|V_t(f_0;x,y) - f_0(x,y)|.$$
$$(3.1)$$

By using the Cauchy-Schwartz inequality in (3.1) we have,

$$V_t(|f(u,v) - f(x,y)|;x,y) \leq$$

$$\leq V_t\left(\varpi(f;\delta_1,\delta_2)\left(1 + \frac{\left|\frac{u}{1+u} - \frac{x}{1+x}\right|}{\delta_1,}\right)\left(1 + \frac{\left|\frac{v}{1+v} - \frac{y}{1+y}\right|}{\delta_2}\right);x,y\right)$$

$$= \varpi(f;\delta_1,\delta_2)\left[V_t(f_0;x,y) + \frac{1}{\delta_1}V_t\left(\left|\frac{u}{1+u} - \frac{x}{1+x}\right|;x,y\right)\right.$$

$$+ \frac{1}{\delta_2}V_t\left(\left|\frac{v}{1+v} - \frac{y}{1+y}\right|;x,y\right)$$

$$+ \left.\frac{1}{\delta_1\delta_2}V_t\left(\left|\frac{u}{1+u} - \frac{x}{1+x}\right|\left|\frac{v}{1+v} - \frac{y}{1+y}\right|;x,y\right)\right]$$

$$\leq \varpi(f;\delta_1,\delta_2)\left[V_t(f_0;x,y)\right.$$

$$+ \frac{1}{\delta_1}\sqrt{V_t\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2;x,y\right)}\sqrt{V_t(f_0;x,y)}$$

$$+ \frac{1}{\delta_2}\sqrt{V_t\left(\left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2;x,y\right)}\sqrt{V_t(f_0;x,y)}$$

$$+ \frac{1}{\delta_1\delta_2}\sqrt{V_t\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2;x,y\right)}$$

$$\times \left.\sqrt{V_t\left(\left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2;x,y\right)}\right]. \tag{3.2}$$

Now, let

$$\delta_1 = \alpha(t) = \sqrt{V_t\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2;x,y\right)}, \quad \delta_2 = \beta(t) = \sqrt{V_t\left(\left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2;x,y\right)}$$

and (3.2). Hence, by taking the supremum over $(x,y) \in K$ on both sides of the inequality (3.1), we obtain

$$\|V_t(f) - f\| \leq \varpi(f;\alpha(t),\beta(t))\left[\|V_t(f_0)\| + 2\left\|\sqrt{V_t(f_0)}\right\| + 1\right] + H\|V_t(f_0) - f_0\|.$$

On the other hand, by taking $\|V_t((.))\|_{H_\omega(K) \to C_B(K)} \leq M$ and for all $t \in (0,R)$ we consider the following inequality

$$\|V_t(f) - f\| \leq \varpi(f;\alpha(t),\beta(t))(2M+1) + H\|V_t(f_0) - f_0\|$$

$$\leq N(\varpi(f;\alpha(t),\beta(t)) + \|V_t(f_0) - f_0\|)$$

where $N = \max_{(x,y)\in K}\{2M+1,H\}$. In conclusion, by the assumption of the theorem we have for all $f \in H_{\omega}(K)$ that

$$\lim_{t\to R^-}\|V_t(f)-f\|_{C_B} = 0.$$

So, the proof of the theorem is complete. $\qquad\qquad\qquad\qquad\qquad\square$

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