SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.17.01.04
Vol.17 (30), No.1 (2021), 37 – 44

NECESSARY AND SUFFICIENT CONDITIONS FOR TWO SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

BASEM AREF FRASIN

ABSTRACT. In the present paper, we determine necessary and sufficient condition for

$$\Phi_q^m(z) := z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n$$

whose coefficients are probabilities of Pascal distribution to be in the class $\mathcal{H}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ of analytic functions with negative coefficients defined in the open unit disk. Further, we give necessary and sufficient condition for the integral operator $\mathcal{G}_q^m f(z) = \int_0^z \frac{\Phi_q^m(t)}{t}dt$ to be in the class $\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ z \in \mathbb{U}. \tag{1.2}$$

For some $0 \le \alpha < 1$ and $\beta_j \geqslant 0,\ j = 1,2,\cdots,k$,and functions of the form (1.1), we let $\mathcal{H}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ be the subclass of $\mathcal{A}$ satisfying the analytic criteria

$$\Re\left\{ \frac{f(z)}{z} + \beta_1 z(\frac{f(z)}{z})' + \beta_2 z^2(\frac{f(z)}{z})'' + \cdots + \beta_k z^k(\frac{f(z)}{z})^{(k)} \right\} > \alpha\ (z \in \mathbb{U}),$$

and also, let $\mathcal{G}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ be the subclass of $\mathcal{A}$ satisfying the analytic criteria

$$\Re\left\{ f'(z) + \beta_1 z f''(z) + \beta_2 z^2 f'''(z) + \cdots + \beta_k z^k f^{(k+1)}(z) \right\} > \alpha\ (z \in \mathbb{U}).$$

2020 *Mathematics Subject Classification.* Primary 30C45; Secondary 33C10, 33C20.

Key words and phrases. Analytic functions, univalent functions, Hadamard product, Pascal distribution series.

Also denote

$$\mathcal{H}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)=\mathcal{H}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)\cap\mathcal{T}$$

and

$$\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)=\mathcal{G}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)\cap\mathcal{T}.$$

The classes $\mathcal{H}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ and $\mathcal{G}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$ were introduced by Frasin [9] (see also, [10]). In particular, the class $\mathcal{H}(0,0,\ldots,0;\alpha)=\mathcal{B}(\alpha)$ was studied by Chen [3,4] and Goal [14], and the class $\mathcal{G}(0,0,\ldots,0;\alpha)=\mathcal{C}(\alpha)$ was studied by Sarangi and Uralegaddi [25], Owa and Uralegaddi [24], and Srivastava and Owa [27] (see also, [8]).

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$,$\tau\in\mathbb{C}\backslash\{0\}$, $-1\leq B<A\leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\ z\in\mathbb{U}.$$

This class was introduced by Dixit and Pal [7].

A variable X is said to be a Pascal distribution if it takes the values $0,1,2,3,\ldots$ with probabilities

$$(1-q)^m,\ \frac{qm(1-q)^m}{1!},\ \frac{q^2m(m+1)(1-q)^m}{2!},\ \frac{q^3m(m+1)(m+2)(1-q)^m}{3!},\ldots,$$

respectively, where q and m are called the parameters, and thus

$$P(X=r)=\binom{r+m-1}{m-1}q^r(1-q)^m,\ r=0,1,2,3,\ldots.$$

Very recently, El-Deeb et al. [6] (see also, [2,18]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi_q^m(z):=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\ z\in\mathbb{U},$$

where $m\geq 1$, $0\leq q\leq 1$, and we note that, by the ratio test the radius of convergence of the above series is infinity. We also define the series

$$\Phi_q^m(z):=2z-\Psi_q^m(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\ z\in\mathbb{U}. \tag{1.3}$$

Let us consider the linear operator $I_q^m:\mathcal{A}\to\mathcal{A}$ defined by the convolution or Hadamard product

$$I_q^m f(z):=\Psi_q^m(z)*f(z)=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m a_n z^n,\ z\in\mathbb{U},$$

where $m\geq 1$ and $0\leq q\leq 1$.

Inspired by earlier results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example, [5,

13, 16, 26, 28]) and by using various distributions such as Yule-Simon distribution, Logarithmic distribution, Poisson distribution, Binomial distribution, Beta-Binomial distribution, Zeta distribution, Geometric distribution and Bernoulli distribution (see for example, [1, 11, 12], [17]- [23]), we determine the necessary and sufficient condition for Φ_q^m to be in the class $\mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$.
Furthermore, we give a sufficient condition for $I_q^m(\mathcal{R}^{\tau}(A,B)) \subset \mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ and finally, we give a necessary and sufficient condition for the function f such that its image by the integral operator $\mathcal{G}_q^m f(z) = \int_0^z \frac{\Phi_q^m(t)}{t} dt$ belongs to the class $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$.

To establish our main results, we need the following Lemmas.

Lemma 1.1. *[9] A function $f \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k;$ $\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} [1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]|a_n| \leq 1-\alpha\,. \quad (1.4)$$

The result (1.4) is sharp.

Lemma 1.2. *[9] A function $f \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k;$ $\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]|a_n| \leq 1-\alpha. \quad (1.5)$$

The result (1.5) is sharp.

Lemma 1.3. *[7] If $f \in \mathcal{R}^{\tau}(A,B)$ is of the form (1.1) , then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \; n \in \mathbb{N}-\{1\}.$$

The result is sharp for the function

$$f(z) = \int_0^z (1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt, \; (z \in \mathbb{U}; n \in \mathbb{N}-\{1\}).$$

2. NECESSARY AND SUFFICIENT CONDITION FOR $\Phi_q^m \in \mathcal{H}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$

In order to prove our main results, we will use the following notation, for $m \geq 1$ and $0 \leq q < 1$:

$$\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n = \frac{1}{(1-q)^m}, \quad \sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n = \frac{1}{(1-q)^{m-1}},$$
$$\sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{1}{(1-q)^{m+1}}, \quad \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n = \frac{1}{(1-q)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\begin{aligned}\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1} &= \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^{n}-1\\ &= \frac{1}{(1-q)^{m}}-1, \qquad (2.1)\\ \sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1} &= qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^{n}\\ &= \frac{qm}{(1-q)^{m+1}}, \qquad (2.2)\\ \sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}&\\ = q^{2}m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^{n} &= \frac{q^{2}m(m+1)}{(1-q)^{m+2}} = 2q^{2}\frac{\binom{m+1}{m-1}}{(1-q)^{m+2}}. \qquad (2.3)\end{aligned}$$

and in general, we have

$$\begin{aligned}&\sum_{n=k+1}^{\infty}(n-1)(n-2)\cdots(n-k)\binom{n+m-2}{m-1}q^{n-1}\\ &= q^{k}m(m+1)\cdots(m+k-1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^{n}\\ &= \frac{q^{k}m(m+1)\cdots(m+k-1)}{(1-q)^{m+k}} = k!q^{k}\frac{\binom{m+k-1}{m-1}}{(1-q)^{m+k}}. \qquad (2.4)\end{aligned}$$

Unless otherwise mentioned, we shall assume in this paper that $0 \leq \alpha < 1$ and $\beta_j \geqslant 0,\ j = 1, 2, \cdots, k$, while $m \geq 1$ and $0 \leq q < 1$.

Firstly, we obtain the necessary and sufficient condition for Φ_q^m to be in the class $\mathcal{H}_{\mathcal{I}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$.

Theorem 2.1. *The series* $\Phi_q^m \in \mathcal{H}_{\mathcal{I}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$, *if and only if*

$$\sum_{j=1}^{k} j!q^{j}\beta_{j}\frac{\binom{m+j-1}{m-1}}{(1-q)^{j}} \leq (1-q)^{m}-\alpha. \qquad (2.5)$$

Proof. Since Φ_q^m is defined by (1.3), in view of Lemma 1.1 it is sufficient to show that

$$\begin{aligned}&\sum_{n=2}^{\infty}[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\\ &\times\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m} \leq 1-\alpha.\end{aligned}$$

Using (2.1)-(2.4), we get

$$\begin{aligned}&\sum_{n=2}^{\infty}[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\\&\times\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\&=\beta_1\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\&+\beta_2\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\&+\cdots+\beta_k\sum_{n=k+1}^{\infty}(n-1)(n-2)\cdots(n-k)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\&+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\&=\beta_1\frac{qm}{(1-q)}+\beta_2\frac{q^2m(m+1)}{(1-q)^2}+\cdots+\beta_k\frac{q^km(m+1)\cdots(m+k-1)}{(1-q)^k}+(1-(1-q)^m)\end{aligned}$$

but this last expression is upper bounded by $1-\alpha$ if and only if (2.5) hold. □

3. SUFFICIENT CONDITION FOR $I_q^m(\mathcal{R}^\tau(A,B))\subset\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$

Making use of Lemma 1.3, we will study the action of the Pascal distribution series on the class $\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$.

Theorem 3.1. *If $f\in\mathcal{R}^\tau(A,B)$ and the inequality*

$$(A-B)|\tau|\left[\sum_{j=1}^{k}j!q^j\beta_j\frac{\binom{m+j-1}{m-1}}{(1-q)^j}+(1-(1-q)^m)\right]\leq 1-\alpha$$

is satisfied, then $I_q^m f\in\mathcal{G}_{\mathcal{T}}(\beta_1,\beta_1,\ldots,\beta_k;\alpha)$.

Proof. According to Lemma 1.2 it is sufficient to show that

$$\begin{aligned}&\sum_{n=2}^{\infty}n[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\\&\times\binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n|\leq 1-\alpha.\end{aligned}$$

Since $f\in\mathcal{R}^\tau(A,B)$, using Lemma 1.3 we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n},\ n\in\mathbb{N}\setminus\{1\},$$

therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\\
&\times \binom{n+m-2}{m-1} q^{n-1}(1-q)^m\,|a_n|\\
&\leq (A-B)\,|\tau| \left[\sum_{n=2}^{\infty} [1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots \right.\\
&\left. +\beta_k(n-1)(n-2)\cdots(n-k)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \right].
\end{aligned}$$

The remaining part of the proof is similar to that of Theorem 2.1, and so we omit the details. □

4. INTEGRAL OPERATOR

Theorem 4.1. *If the function* $\mathcal{G}_q^m$ *is given by*

$$\mathcal{G}_q^m(z) := \int_0^z \frac{\Phi_q^m(t)}{t}dt,\ z \in \mathbb{U},$$

then $\mathcal{G}_q^m \in \mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ *if and only if the inequality (2.5) holds.*

Proof. According to (1.3) it follows that

$$\mathcal{G}_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n},\ z \in \mathbb{U}.$$

Using Lemma 1.1, the function $\mathcal{G}_q^m(z)$ belongs to $\mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ if and only if

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[1+\beta_1(n-1)+\beta_2(n-1)(n-2)+\cdots+\beta_k(n-1)(n-2)\cdots(n-k)]\\
&\times \frac{1}{n}\binom{n+m-2}{m-1} q^{n-1}(1-q)^m \leq 1-\alpha.
\end{aligned}$$

By a similar proof like that of Theorem 2.1 we get that $\mathcal{G}_q^m f \in \mathcal{G}_{\mathcal{T}}(\beta_1, \beta_1, \ldots, \beta_k; \alpha)$ if and only if (2.5) holds. □

5. ACKNOWLEDGEMENTS

The author would like to thank the referees for their helpful comments and suggestions.

REFERENCES

[1] R. M. El-Ashwah and W. Y Kota, *Some condition on a Poisson distribution series to be in subclasses of univalent functions*, Acta Univ. Apulensis Math. Inform., No. 51/2017, pp. 89-103.

[2] S.Çakmak, S.Yalçın, and Ş. Altınkaya, *Some connections between various classes of analytic functions associated with the power series distribution*, Sakarya Univ. J. Sci., 23(5)(2019), 982–985.

[3] M.-P. Chen, *On the regular functions satisfying* $Re\{f(z)/z\} > \alpha$, Bull. Inst. Math. Acad. Sinica 3 (1975), 65-70.

[4] M. -P. Chen, *On functions satisfying* $Re\{f(z)/z\} > \alpha$, Tamkang J. Math. 5 (1974) 231-234.

[5] N. E. Cho, S. Y. Woo and S. Owa, *Uniform convexity properties for hypergeometric functions*, Fract. Cal. Appl. Anal., 5(3) (2002), 303–313.

[6] S. M. El-Deeb, T. Bulboacă and J. Dziok, *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Math. J., 59(2019), 301–314.

[7] K.K. Dixit and S.K. Pal, *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math., 26(1995), no. 9, 889-896.

[8] B.A. Frasin, *Some properties of certain analytic and univalent functions*, Tamsui Oxford J. Math. 23(1)(2007),67-77.

[9] B.A. Frasin, *An application of generalized Bessel functions on two subclasses of analytic functions*, Theory Appl. Math. Comput. Sci, 10 (2) (2020) 39-45.

[10] B.A. Frasin,*Two subclasses of analytic functions associated with Poisson distribution*, Turkish J. Ineq., 4 (1) (2020), Pages 25 –30.

[11] B.A. Frasin, *On certain subclasses of analytic functions associated with Poisson distribution series*, Acta Univ. Sapientiae, Mathematica, 11, 1 (2019) 78–86.

[12] B. A. Frasin and Ibtisam Aldawish, *On subclasses of uniformly spirallike functions associated with generalized Bessel functions*, J. Funct. Spaces, Volume 2019, Article ID 1329462, 6 pages.

[13] B.A. Frasin, Tariq Al-Hawary and Feras Yousef, *Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions*, Afr. Mat., Volume 30, Issue 1–2, 2019, pp. 223–230.

[14] R. M. Goel, *On functions satisfying* $Re\{f(z)/z\} > \alpha$, Pub. Math. Debrecen 18 (1971), 111-117.

[15] T. H. Macgregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Sot. 104 (1962), 532-537.

[16] E. Merkes and B. T. Scott, *Starlike hypergeometric functions*, Proc. Amer. Math. Soc., 12 (1961), 885-888.

[17] G. Murugusundaramoorthy, *Subclasses of starlike and convex functions involving Poisson distribution series* , Afr. Mat. (2017) 28:1357-1366.

[18] G. Murugusundaramoorthy, B. A. Frasin and Tariq Al-Hawary, *Uniformly convex spiral functions and uniformly spirallike function associated with Pascal distribution series*, Math. Bohem., in press.

[19] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series*, Hacet. J. Math. Stat. 45(2016), no. 4, 1101-1107.

[20] W. Nazeer, Q. Mehmood, S.M. Kang, and A. Ul Haq, *An application of Binomial distribution series on certain analytic functions*, J. Comput. Anal. Appl., Volume 26,No.1(2019),11-17.

[21] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., (2014), Art. ID 984135, 1–3.

[22] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat. 27, 1021–1027 (2016). https://doi.org/10.1007/s13370-016-0398-z
[23] A. T. Oladipo, *Bounds for probability of the generalized distribution defined by generalized polylogarithm*, Punjab Univ. J. Math, Vol 51(7) (2019), 19-26.
[24] S. Owa and B. A. Uralegaddi, *An application of the fractional calculus*, J. Karnatak Unia. Sci. 30 (1985).
[25] S. M. Sarangi and B. A. Uralegaddi, *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients*, I, Alti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nurur. (8) 65 (1978), 38-42.
[26] H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl. 172 (1993), 574–581.
[27] H. M. Srivastava and Owa, *Certain classes of analytic functions with varying arguments*, J. Math. Anal. Appl. 136, 217-228 (1988).
[28] H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integr. Transf. Spec. Func. 18 (2007), 511–520.

(Received: May 03, 2020)
(Revised: May 05, 2021)

Basem Aref Frasin
AL al-Bayt University
Faculty of Science
Department of Mathematics
e-mail: *bafrasin@yahoo.com*