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RESULTS ON DIFFERENCE-DIFFERENTIAL POLYNOMIAL OF ENTIRE FUNCTIONS

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ABSTRACT. In this paper, we deal with the uniqueness results of derivatives of difference-differential polynomials of entire functions sharing a value with CM(IM). The results of this paper extend the results of K.Liu, X.L.Liu and T.B.Cao[8,9] and K.Zhang and H.Yi[17] and many others.

1. INTRODUCTION

In this paper a meromorphic function f means that f is meromorphic for the whole complex plane C. We shall use the standard notations of value distribution theory such as $T(r,f), m(r,f), N(r,f), S(r,f)$,...etc (see [5]). Two meromorphic functions f and g share a point $a(\in$ C)CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities and similarly we can say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. The function which is denoted by $N_{(k}(r,\frac{1}{f-a})$, counts the number of zeros of $f-a$ with multiplicity greater or equal to k and $\overline{N}_{(k}(r,\frac{1}{f-a})$ is the corresponding one for which multiplicity is not counted.
Now we denote, $N_k(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}_{(2}(r,\frac{1}{f-a}) + ... + \overline{N}_{(k}(r,\frac{1}{f-a})$, where a is any finite complex number and k is a constant. Further, we consider $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, a polynomial in z of degree m, where $a_0(\neq 0), a_1, ..., a_{m-1}, a_m(\neq 0)$ are complex constants.

In this paper, we establish theorem 2.2 and theorem 2.3, which improve theorem 1.1 (Liu et al. [8]) and theorem 1.2 (Liu et al. [8]); those theorems i.e.,theorem 2.2 and theorem 2.3, also improve theorem 1.3 (Liu et al. [9]) and theorem 1.4 (Liu et al. [9]) as per as the polynomial functions is concerned. Moreover, our intention in this chapter is to generalize the polynomial functions of theorem 1.5 (Zhang et al. [17]) and the theorem 1.6 (Zhang et al. [17]). We are quite successful in doing so and in this direction, we prove theorem 2.4 and theorem 2.5 . In the latter theorem, we have also taken into account the sharing properties with IM (i.e., while disregarding multiplicities) which the authors in [17] did not.

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We now begin by stating the theorems due to K. Liu, X. L. Liu and T. B. Cao [8] in 2012:

Theorem 1.1. [8] *Let f and g be transcendental entire functions of finite order, $n \geq 2k+6$. If $[f(z)^n f(z+c)]^{(k)}$ and $[g(z)^n g(z+c)]^{(k)}$ share the value 1 CM, then either, $f(z) = c_1 e^{Cz}$, $f(z) = c_2 e^{-Cz}$ where c_1, c_2 and C are constants satisfying $(-1)^k (c_1 c_2)^{n+1} [(n+1)C]^{2k} = 1$ or $f = tg$ where $t^{n+1} = 1$.*

Theorem 1.2. [8] *Let f and g be transcendental entire functions of finite order, $n \geq 5k+12$. If $[f(z)^n f(z+c)]^{(k)}$ and $[g(z)^n g(z+c)]^{(k)}$ share the value 1 IM, then either, $f(z) = c_1 e^{Cz}$, $f(z) = c_2 e^{-Cz}$ where c_1, c_2 and C are constants satisfying $(-1)^k (c_1 c_2)^{n+1} [(n+1)C]^{2k} = 1$ or $f = tg$ where $t^{n+1} = 1$.*

Meanwhile, in 2011, the same author [9] obtained the following results:

Theorem 1.3. [9] *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) < 1$, $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$. ($\rho_2(f)$ is the hyper order of f)*

Theorem 1.4. [9] *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) < 1$, $n \geq 5k+4m+12$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value 1 IM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

Subequently, Zhang and Yi [17] looked into the zeros of a certain kind of difference differential polynomial in 2014 and came up with the following theorems:

Theorem 1.5. [17] *Let f and g be two transcendental entire functions of finite order,$\alpha(z) \neq 0$ be a common small function with respect to f and g, c_j(j=1.2,....,d) be distinct finite complex numbers, and n,m,d and v_j(j=1,2,....,d) be non negative integers. If $n \geq 2k+m+\sigma+5$ and the differential-difference polynomials $[f^n(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$ and $[g^n(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ share $\alpha(z)$ CM, then $f = tg$ where $t^m = t^{n+\sigma} = 1$ where $\sigma = v_1 + ... + v_d$.*

Theorem 1.6. [17] *Let f and g be two transcendental entire functions of finite order,$\alpha(z) \neq 0$ be a common small function with respect to f and g, c_j(j=1.2,....,d) be distinct finite complex numbers, and n,m,d and v_j(j=1,2,....,d) be non negative integers. If $n \geq 2k+m+\sigma+5$ and the differential-difference polynomials $[f^n(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$ and $[g^n(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ share $\alpha(z)$ CM, then $f = tg$ where $t^m = t^{n+\sigma} = 1$ where $\sigma = v_1 + ... + v_d$.*

Now the following questions naturally arise:
What would happen if we replace

(i) the polynomial function of theorem 1.3 and of theorem 1.4 by the polynomial function $f^n P(f) f(z+c)$? and

(ii) the polynomial function of theorem 1.5 and of theorem 1.6 by the polynomial function $f^nP(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$?

Furthermore, our investigation is concerned about the sharing properties of IM . As mentioned earlier, in this paper, we are trying to answer those questions which in turn also generalize the aforementioned theorems.

2. MAIN RESULTS

In this paper we are trying to figure out those questions which in turn generalize the theorem (1.1 - 1.6) and many more theorems. Our theorems are the following:

Theorem 2.1. *Let f be a transcendental entire function of finite order. Then $[f^nP(f)f(z+c)]^{(k)}-1$ has infinitely many zeros for $n\geq k+2$.*

Theorem 2.2. *Let f and g be transcendental entire functions of finite order and c be a complex constant and n, m and k be non negative integers such that $n\geq 2k+m+6$. If $[f^nP(f)f(z+c)]^{(k)}$ and $[g^nP(g)g(z+c)]^{(k)}$ share 1 CM, then :*

(i) *$f=tg$ where $t^d=1$ and d = g.c.d. of $(n+m,n+m-1,\dots,n)$*
or

(ii) *$f(z)$ and $g(z)$ satisfy the algebraic equation $Q(f,g)=0$ where,*
$Q(w_1,w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+\ldots+a_0)w_1(z+c)-w_2^n(a_mw_2^m$
$+a_{m-1}w_2^{m-1}+\ldots+a_0)w_2(z+c)$.

Theorem 2.3. *Let f and g be transcendental entire functions of finite order and c be a complex constant and n, m and k be non negative integers such that $n\geq 5k+4m+12$. If $[f^nP(f)f(z+c)]^{(k)}$ and $[g^nP(g)g(z+c)]^{(k)}$ share 1 IM, then :*

(i) *$f=tg$ where $t^d=1$ and d = g.c.d. of $(n+m,n+m-1,\dots,n)$*
or

(ii) *$f(z)$ and $g(z)$ satisfy the algebraic equation $Q(f,g)=0$ where,*
$Q(w_1,w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+\ldots+a_0)w_1(z+c)-w_2^n(a_mw_2^m$
$+a_{m-1}w_2^{m-1}+\ldots+a_0)w_2(z+c)$.

Theorem 2.4. *Let f and g be transcendental entire functions of finite order and c_j(j=1,.....,d) be distinct finite complex numbers and n,m,d and v_j(j=1,....,d) be non negative integers such that $n\geq 2k+m+\sigma+5$ where $\sigma=v_1+.....+v_d$. If $[f^nP(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$ and $[g^nP(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ share 1 CM, then*

(i) *$f=tg$ where $t^d=1$ and d=g.c.d. of (n+m,n+m-1,...,n)*
or

(ii) *$f(z)$ and $g(z)$ satisfy the algebraic equation $Q(f,g)=0$ where -*
$Q(w_1,w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+\ldots+a_0)w_1(z+c)-w_2^n(a_mw_2^m$
$+a_{m-1}w_2^{m-1}+\ldots+a_0)w_2(z+c)$.

Theorem 2.5. *Let f and g be transcendental entire functions of finite order and c_j(j=1,.....,d) be distinct finite complex numbers and n,m,d and v_j)(j=1,....,d) be* *non negative integers such that* $n \geq 5k+4m+4\sigma+8$ *where* $\sigma = v_1 + ... + v_d$. *If* $[f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$ *and* $[g^n P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ *share 1 IM, then*

(i) $f = tg$ *where* $t^d = 1$ *and d=g.c.d. of (n+m,n+m-1,...,n)*
or

(ii) $f(z)$ *and* $g(z)$ *satisfy the algebraic equation* $Q(f,g) = 0$ *where -*
$Q(w_1,w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \ldots + a_0)w_1(z+c) - w_2^n(a_m w_2^m$
$+ a_{m-1}w_2^{m-1} + \ldots + a_0)w_2(z+c)$.

3. LEMMAS

Before going to the details of the proof of the theorems, we need to mention some results in the form of lemmas.

Lemma 3.1. *[13] Let f be a non constant meromorphic function and p,k be positive integers. Then*

$$T(r,f^{(k)}) \leq T(r,f) + k\overline{N}(r,f) + S(r,f)$$
$$N_p(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f)$$
$$N_p(r,\frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f).$$

Lemma 3.2. *[11] Let* $f(z)$ *and* $g(z)$ *be two non constant meromorphic functions. If* $f(z)$ *and* $g(z)$ *share the value* 1 *CM, then one of the following three cases holds :*

(i) $T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$*; the same inequality holds for* $T(r,g)$*;*

(ii) $fg = 1$*;*

(iii) $f \equiv g$.

Lemma 3.3. *[14] Let f and g be two non constant entire functions. If f and g share* 1 *IM, then one of the following cases holds :*

(i) $T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + 2\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + S(r,f) + S(r,g)$ *the same inequality holds for* $T(r,g)$*,*

(ii) $f = g$

(iii) $fg \equiv 1$.

Lemma 3.4. *[13] Let* $f_1(z)$ *and* $f_2(z)$ *be two non constant meromorphic functions. If* $c_1f_1 + c_2f_2 = c_3$ *where,* c_1, c_2 *and* c_3 *are non zero constants, then*

$$T(r,f_1) \leq \overline{N}(r,f_1) + \overline{N}(r,\frac{1}{f_1}) + \overline{N}(r,\frac{1}{f_2}) + S(r,f_1).$$

Lemma 3.5. *Let f and g be two transcendental entire functions of finite order and c be a non zero constant and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots . + a_1 z + a_0$ where $a_0(\neq 0), a_1, a_2, \ldots, a_m(\neq 0)$ are complex constants.*
If $[f^n P(f) f(z+c)]^{(k)} = [g^n P(g) g(z+c)]^{(k)}$, then:

(i) *$f = tg$ for a constant t such that $t^d = 1$ where $d = g.c.d.$ of $(n+m, n+m-1, \ldots, n)$.*
or,

(ii) *$f(z)$ and $g(z)$ satisfy the algebraic equation $Q(f,g) = 0$, where -*
$Q(u,v) = u^n(a_m u^m + a_{m-1} u^{m-1} + \ldots .. + a_0) u(z+c) - v^n(a_m v^m + a_{m-1} v^{m-1} + \ldots . + a_0) v(z+c)$.

Proof: From the given condition, we can say, $f^n P(f) f(z+c) = g^n P(g) g(z+c) + R(z)$ where $R(z)$ is a polynomial of degree at most $(k-1)$.
Now if $R(z) \neq 0$, then we have, $\frac{f^n P(f) f(z+c)}{R(z)} = \frac{g^n P(g) g(z+c)}{R(z)} + 1$.
Using Lemma 3.4, we have -

$$
\begin{aligned}
(n+m+1)T(r,f) &= T(r, \frac{f^n P(f) f(z+c)}{R(z)}) + S(r,f) \\
&\leq \overline{N}(r, \frac{f^n P(f) f(z+c)}{R(z)}) + \overline{N}(r, \frac{R(z)}{f^n P(f) f(z+c)}) \\
&\quad + \overline{N}(r, \frac{Q(z)}{g^n P(g) g(z+c)}) + S(r,f) \\
&\leq \overline{N}(r, \frac{1}{f^n P(f)}) + \overline{N}(r, \frac{1}{f(z+c)}) \\
&\quad + \overline{N}(r, \frac{1}{g^n P(g)}) + \overline{N}(r, \frac{1}{g(z+c)}) + S(r,f) \\
&\leq (m+2)T(r,f) + (m+2)T(r,g) + S(r,f) + S(r,g).
\end{aligned}
$$

Similarly, we have,
$(n+m+1)T(r,g) \leq (m+2)T(r,g) + (m+2)T(r,f) + S(r,f) + S(r,g)$.

Adding these two, we get
$(n+m+1)[T(r,f) + T(r,g)] \leq 2(m+2)[T(r,f) + T(r,g)] + S(r,f) + S(r,g)$.

As $n \geq m+4$, it is a contradiction.

Hence $R(z) = 0$, which means

$$
f^n P(f) f(z+c) = g^n P(g) g(z+c). \tag{3.1}
$$

Suppose $h = \frac{f}{g}$ and h is a constant. Then $f = gh$. Substitute this in (3.1) we have $(a_m g^m h^m + a_{m-1} g^{m-1} h^{m-1} + \ldots + a_0 h^n h(z+c)) = (a_m g^m + a_{m-1} g^{m-1} + \ldots + a_0)$
i.e., $a_m g^m (h^{n+m} h(z+c) - 1) + a_{m-1} g^{m-1} (h^{n+m-1} h(z+c) - 1) + \ldots$
$+ a_0 (h^n h(z+c) - 1) = 0.$

Now since g is transcendental, from the last equation we have, $h^d = 1$ where d is highest common divisor of (n+m,n+m-1,...,n).

Thus $f = tg$ for a constant t such that $t^d = 1$.
Now let $h \neq$ a constant, then from (3.1) we can conclude that $f(z)$ and $g(z)$ satisfy an algebraic equation $Q(f,g) = 0$ where -
$Q(u,v) = u^n P(u)u(z+c) - v^n P(v)v(z+c)$.

Lemma 3.6. *Let f and g be two transcendental entire functions of finite order and $c_j (j = 1,\ldots,d)$ be non zero complex constant and $P(z) = a_m z^m + a_{m-1}z^{m-1} + \ldots. + a_1 z + a_0$ where $a_0(\neq 0), a_1, a_2, \ldots, a_m(\neq 0)$ are complex constant. If $[f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)} = [g^n P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ and $n \geq m+\sigma+3$, where $n, m, v_j (j = 1,\ldots,d)$ are non negetive integers and $\sigma = v_1 + \ldots + v_d$ then:*

(i) *$f = tg$ for a constant t such that $t^d = 1$ where $d = g.c.d.of(n+m, n+m-1, \ldots, n)$,*
or,
(ii) *$f(z)$ and $g(z)$ satisfy the algebraic equation $Q(f,g) = 0$,where -*
$Q(p,q) = p^n P(p)\prod_{j=1}^{d} p(z+c_j)^{v_j} - q^n P(q)\prod_{j=1}^{d} q(z+c_j)^{v_j}$.

Proof. From the given condition, we can say,
$f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} = g^n P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} + Q(z)$, where $Q(z)$ is a polynomial of degree at most $(k-1)$. Now if $Q(z) \neq 0$, then we have,
$\frac{f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}}{Q(z)} = \frac{g^n P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}}{Q(z)} + 1$.
Using Lemma 3.4, we have -

$$\begin{aligned}
(n+m+\sigma)T(r,f) &= T\left(r, \frac{f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}}{Q(z)}\right) + S(r,f)\\
&\leq \overline{N}\left(r, \frac{f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}}{Q(z)}\right) + \overline{N}\left(r, \frac{Q(z)}{f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}}\right)\\
&\quad + \overline{N}\left(r, \frac{Q(z)}{g^n P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}}\right) + S(r,f)\\
&\leq \overline{N}\left(r, \frac{1}{f^n P(f)}\right) + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{v_j}}\right) + \overline{N}\left(r, \frac{1}{g^n P(g)}\right)\\
&\quad + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{v_j}}\right) + S(r,f)\\
&\leq (m+\sigma+1)[T(r,f) + T(r,g)] + S(r,f) + S(r,g).
\end{aligned}$$

Similarly, we get-
$(n+m+\sigma)T(r,g) \leq (m+\sigma+1)[T(r,f) + T(r,g)] + S(r,f) + S(r,g)$.
Adding these two ineuality, we get-
$(n+m+\sigma)[T(r,f) + T(r,g)] \leq 2(m+\sigma+1)[T(r,f) + T(r,g)] + S(r,f) + S(r,g)$.

This contradicts the fact that $n \geq m+\sigma+3$. Hence $R(z)=0$, which means

$$f^nP(f)\prod_{j=1}^{d}f(z+c_j)^{\nu_j}=g^nP(g)\prod_{j=1}^{d}g(z+c_j)^{\nu_j}. \tag{3.2}$$

Now suppose $h=\frac{f}{g}$ and h is a constant. Then $f=gh$. Substitute this in(3.2), we have

$$h^n[a_mg^mh^m+a_{m-1}g^{m-1}h^{m-1}+\ldots+a_0]\prod_{j=1}^{d}h(z+c_j)^{\nu_j}=[a_mg^m+a_{m-1}g^{m-1}+\ldots.+a_0]$$

$$i.e., \quad a_mg^m[h^{n+m}\prod_{j=1}^{d}h(z+c_j)^{\nu_j}-1]+a_{m-1}g^{m-1}[h^{n+m-1}\prod_{j=1}^{d}h(z+c_j)^{\nu_j}-1]+\ldots$$
$$+a_0[h^n\prod_{j=1}^{d}h(z+c_j)^{\nu_j}-1]=0.$$

As g is transcendental, we have, $h^d=1$, where d is g.c.d. of (n+m,n+m-1,...,n)
Thus $f=tg$ for a constant t such that $t^d=1$.
Now let $h\neq$ a constant, then from (3.2) we can say that $f(z)$ and $g(z)$ satisfy an algebraic equation $Q(f,g)=0$ where
$Q(p,q)=p^nP(p)\prod_{j=1}^{d}p(z+c_j)^{\nu_j}-q^nP(q)\prod_{j=1}^{d}q(z+c_j)^{\nu_j}$. □

Lemma 3.7. *Let f be an entire function of finite order and $F=f^nP(f)f(z+c)$. Then $T(r,F)=(n+m+1)T(r,f)+S(r,f)$.*

4. Proof of the Main Results

Proof of Theorem 2.1. :
Let $F(z)=f^nP(f)f(z+c)$. Then $F(z)$ is not a constant. Now let us consider that $F^{(k)}-1$ has only finitely many zeros. Then

$$T(r,F^{(k)})\leq\overline{N}(r,F^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-1})+S(r,F^{(k)}).$$

Since F is a transcendental entire function, by the Lemma 3.1, we have

$$\begin{aligned}T(r,F^{(k)}) &\leq N_1(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-1})+S(r,F^{(k)})\\ &\leq T(r,F^{(k)})-T(r,F)+N_{k+1}(r,\frac{1}{F})+S(r,F^{(k)}).\end{aligned}$$

So,

$$\begin{aligned}(n+m+1)T(r,f) &\leq (k+1)T(r,f)+mT(r,f)+T(r,f)+S(r,f)\\ &=(k+m+2)T(r,f)+S(r,f)\end{aligned}$$

which contradicts our assumption that $n\geq k+2$. Hence the conclusion of the theorem is proved.

Proof of Theorem 2.2. :
Let, $F = f^n P(f) f(z+c)$, $G = g^n P(g) g(z+c)$
and $F_1 = [f^n P(f) f(z+c)]^{(k)} = F^{(k)}$, $G_1 = [g^n P(g) g(z+c)]^{(k)} = G^{(k)}$.
Then F_1 and G_1 are also entire functions. Now by the given condition and the lemma 3.2, we have,

Case I:
$T(r, F_1) \leq N_2(r, \frac{1}{F_1}) + N_2(r, \frac{1}{G_1}) + S(r, f) + S(r, g)$
i.e., $T(r, F^{(k)}) \leq N_2(r, \frac{1}{F^{(k)}}) + N_2(r, \frac{1}{G^{(k)}}) + S(r, f) + S(r, g)$.
By Lemma 3.1, we have

$$\begin{aligned} T(r, F^{(k)}) \leq\ & T(r, F^{(k)}) - T(r, F) + N_{k+2}(r, \frac{1}{F}) + T(r, G^{(k)}) - T(r, G) \\ & + N_{k+2}(r, \frac{1}{G}) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned} i.e., \quad T(r, F) + T(r, G) &\leq T(r, G^{(k)}) + N_{k+2}(r, \frac{1}{F}) + N_{k+2}(r, \frac{1}{G}) + S(r, f) + S(r, g) \\ &\leq T(r, G) + k\overline{N}(r, G) + N_{k+2}(r, \frac{1}{F}) + N_{k+2}(r, \frac{1}{G}) + S(r, f) + S(r, g). \end{aligned}$$

$$So, T(r, F) \leq N_{k+2}(r, \frac{1}{F}) + N_{k+2}(r, \frac{1}{G}) + S(r, f) + S(r, g). \tag{4.1}$$

Similarly we have,

$$T(r, G) \leq N_{k+2}(r, \frac{1}{G}) + N_{k+2}(r, \frac{1}{F}) + S(r, f) + S(r, g). \tag{4.2}$$

Adding (3.3) and (3.4), we get,
$(n+m+1)[T(r, f) + T(r, g)] \leq 2(k+2+m+1)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$.
This contradicts $n \geq 2k + m + 6$.

Case II:

$$\begin{aligned} F_1 G_1 &= 1 \\ i.e., \quad [f^n P(f) f(z+c)]^{(k)} [g^n P(g) g(z+c)]^{(k)} &= 1. \end{aligned} \tag{4.3}$$

As $n \geq 2k + m + 6$, from (3.5) we can say that $f(z)$ has no zeros. Therefore we may assume $f(z) = e^{\alpha(z)}$.
Then,
$[f^n P(f) f(z+c)]^{(k)} = [e^{n\alpha(z)}(a_m e^{m\alpha(z)} + a_{m-1} e^{(m-1)\alpha(z)} + \ldots + a_1 e^{\alpha(z)} + a_0) e^{\alpha(z+c)}]^{(k)}$.
Set, $\alpha(z+c) = \beta(z)$, we have
$= [a_m e^{(n+m)\alpha(z)+\beta(z)} + \ldots + a_1 e^{(n+1)\alpha(z)+\beta(z)} + a_0 e^{n\alpha(z)+\beta(z)}]^{(k)}$
$= e^{(n+m)\alpha(z)+\beta(z)} P_m(\alpha^{(1)}, \beta^{(1)}, \ldots, \alpha^{(k)}, \beta^{(k)}) + \ldots + e^{(n+1)\alpha(z)+\beta(z)} P_1(\alpha^{(1)}, \beta^{(1)}, \ldots,$
$\alpha^{(k)}, \beta^{(k)}) + e^{n\alpha(z)+\beta(z)} P_0(\alpha^{(1)}, \beta^{(1)}, \ldots, \alpha^{(k)}, \beta^{(k)})$
$= e^{n\alpha(z)+\beta(z)}[P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)} + P_0]$.
Now, $T(r, P_i) = S(r, f)$ for $i = 0, 1, \ldots, m$ and $T(r, \frac{1}{P_m e^{m\alpha} + \ldots + P_0}) = S(r, f)$.

Therefore from the second fundamental theorem,

$$\begin{aligned} mT(r,f) &= T(r,P_me^{m\alpha}+\ldots+P_1e^{\alpha})+S(r,f)\\ &\le \overline{N}\Big(r,\frac{1}{P_me^{m\alpha}+\ldots+P_1e^{\alpha}}\Big)+\overline{N}\Big(r,\frac{1}{P_me^{m\alpha}+\ldots+P_0}\Big)\\ &+\overline{N}\Big(r,P_me^{m\alpha}+\ldots+P_1e^{\alpha}\Big)+S(r,f)\\ &\le \overline{N}\Big(r,\frac{1}{P_me^{(m-1)\alpha}+\ldots+P_0}\Big)+S(r,f)\\ &\le (m-1)T(r,f)+S(r,f). \end{aligned}$$

Thus we arrive at a contradiction.

Case III: $F_1=G_1$.

By Lemma 3.5., we get the desired results.

Proof of Theorem 2.3. :

The functions F,G,F_1 and G_1 are defined the same as in the proof of theorem 2.2. As F_1 and G_1 share 1 IM, then by the lemma 3.3, we have

Case I:

$T(r,F_1)\le N_2(r,\frac{1}{F_1})+N_2(r,\frac{1}{G_1})+2\overline{N}(r,\frac{1}{F_1})+\overline{N}(r,\frac{1}{G_1})+S(r,f)+S(r,g)$

i.e., $T(r,F^{(k)})\le N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+2\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S(r,f)+S(r,g)$.

Applying the lemma 3.1, we have

$$\begin{aligned} T(r,F^{(k)}) &\le T(r,F^{(k)})-T(r,F)+N_{k+2}(r,\frac{1}{F})+T(r,G^{(k)})-T(r,G)\\ &\quad+N_{k+2}(r,\frac{1}{G})+2\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S(r,f)+S(r,g)\\ i.e., T(r,F)+T(r,G) &\le T(r,G)+k\overline{N}(r,G)+N_{k+2}(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{G})\\ &\quad+2\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S(r,f)+S(r,g)\\ i.e., T(r,F) &\le N_{k+2}(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{G})+2\overline{N}(r,\frac{1}{F^{(k)}})\\ &\quad+\overline{N}(r,\frac{1}{G^{(k)}})+S(r,f)+S(r,g)\\ &\le N_{k+2}(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{G})+2N_1(r,\frac{1}{F^{(k)}})+N_1(r,\frac{1}{G^{(k)}})\\ &\quad+S(r,f)+S(r,g)\\ &\le N_{k+2}(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{G})+2N_{k+1}(r,\frac{1}{F})+2k\overline{N}(r,F)\\ &\quad+N_{k+1}(r,\frac{1}{G})+k\overline{N}(r,G)+S(r,f)+S(r,g). \end{aligned}$$

So, $(n+m+1)T(r,f) \leq (3k+3m+7)T(r,f) + (2k+2m+5)T(r,g) + S(r,f) + S(r,g)$.
Similarly, $(n+m+1)T(r,g) \leq (3k+3m+7)T(r,g) + (2k+2m+5)T(r,f) + S(r,f) + S(r,g)$.
Adding these two, we have
$(n+m+1)[T(r,f)+T(r,g)] \leq (5k+5m+12)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)$.
As $n \geq 5k+4m+12$, the above inequality is not valid.
Proof of the remaining part of the theorem is in the same line as the previous theorem.

Proof of Theorem 2.4. :
Let $F = f^n P(f) \prod_{j=1}^{d} f(z+c_j)^{v_j}, G = g^n P(g) \prod_{j=1}^{d} g(z+c_j)^{v_j}$ and
$F_1 = [f^n P(f) \prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)} = [F]^{(k)}, G = [g^n P(g) \prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)} = [G]^{(k)}$.
Then F_1 and G_1 are also entire functions. Now by Lemma 3.2, we have -

Case I:
Following the same procedure as in the proof of theorem 2.2 (CaseI), we have -
$T(r,F) \leq N_{k+2}(r, \frac{1}{F}) + N_{k+2}(r, \frac{1}{G}) + S(r,f) + S(r,g)$.
So, $(n+m+\sigma)T(r,f) = T(r,F) \leq (k+m+\sigma+2)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)$.
Similarly, we get-
$(n+m+\sigma)T(r,g) = T(r,G) \leq (k+m+\sigma+2)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)$.
Adding these two inequalities, we have -
$(n+m+\sigma)[T(r,f)+T(r,g)] \leq 2(k+m+\sigma+2)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)$.
As $n \geq 2k+m+\sigma+5$, the inequality in not valid.

Case II:

$$F_1 G_1 = 1$$
$$i.e., [f^n P(f) \prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)} [g^n P(g) \prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)} = 1. \tag{4.4}$$

As $n \geq 2k+m+\sigma+5$, from (3.6),we see that $f(z)$ has no zeros. So we may assume $f(z) = e^{\alpha(z)}$. Now, $[f^n P(f) \prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$.
Set, $\alpha(z+c_j) = \beta_j(z)$
$= [e^{n\alpha(z)}(a_m e^{m\alpha(z)} + a_{m-1} e^{(m-1)\alpha(z)} + \ldots + a_1 e^{\alpha(z)} + a_0) \prod_{j=1}^{d} (e^{\beta_j(z)})^{v_j}]^{(k)}$
$= [a_m e^{(n+m)\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)} + \ldots. + a_1 e^{(n+1)\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)} + a_0 e^{n\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)}]^{(k)}$
$= e^{(n+m)\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)} P_m(\alpha^{(1)}, \beta_1^{(1)}, ..., \beta_d^{(1)}, \ldots, \alpha^{(k)}, \beta_1^{(k)}, ..., \beta_d^{(k)}) + \ldots$
$+ e^{(n+1)\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)} P_1(\alpha^{(1)}, \beta_1^{(1)}, ..., \beta_d^{(1)}, \ldots, \alpha^{(k)}, \beta_1^{(k)}, ..., \beta_d^{(k)})$
$+ e^{n\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)} P_0(\alpha^{(1)}, \beta_1^{(1)}, ..., \beta_d^{(1)}, \ldots, \alpha^{(k)}, \beta_1^{(k)}, ..., \beta_d^{(k)})$
$= e^{n\alpha(z) + \sum_{j=1}^{d} v_j \beta_j(z)} [P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)} + P_0]$.

Now, $T(r, P_i) = S(r,f)$ for $i = 0, 1, \ldots, m$ and $T(r, \frac{1}{P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)} + P_0}) = S(r,f)$.

Therefore from the second fundamental theorem,

$$\begin{aligned} mT(r,f) &= T(r, P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)}) + s(r,f) \\ &\le \overline{N}\Big(r, \frac{1}{P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)}}\Big) + \overline{N}\Big(r, \frac{1}{P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)} + P_0}\Big) \\ &\quad + \overline{N}\Big(r, P_m e^{m\alpha(z)} + \ldots + P_1 e^{\alpha(z)}\Big) + S(r,f) \\ &\le \overline{N}\Big(r, \frac{1}{P_m e^{(m-1)\alpha + \ldots + P_0}}\Big) + S(r,f) \\ &\le (m-1)T(r,f) + S(r,f). \end{aligned}$$

Thus we arrive at a contradiction.

Case III:

$F_1 G_1 = 1$.

By Lemma 3.6., we get the desired results.

Proof of Theorem 1.5. :

The functions F, G, F_1 and G_1 are defined the same as in the proof of theorem 2.4. Now by Lemma 3.3 and CaseI of the proof of theorem 2.3, we can easily deduce that -

$$\begin{aligned} T(r,F) \le N_{k+2}(r, \frac{1}{F}) + N_{k+2}(r, \frac{1}{G}) + 2N_{k+1}(r, \frac{1}{F}) + 2k\overline{N}(r,F) \\ + N_{k+1}(r, \frac{1}{G}) + k\overline{N}(r,G) + S(r,f) + S(r,g). \end{aligned}$$

So,

$(n+m+\sigma)T(r,f) = T(r,F) \le (3k+3m+3\sigma+4)T(r,f) + (2k+2m+2\sigma+3)T(r,g) + S(r,f) + S(r,g)$.

Similarly,

$(n+m+\sigma)T(r,g) = T(r,G) \le (3k+3m+3\sigma+4)T(r,g) + (2k+2m+2\sigma+3)T(r,f) + S(r,f) + S(r,g)$.

Adding these two, we get -

$(n+m+\sigma)[T(r,f)+T(r,g)] \le (5k+5m+5\sigma+7)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)$

This contradicts

$n \ge 5k+4m+4\sigma+8$.

Proof of the remaining part of the theorem is in the same line as the proof of theorem 2.4.

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