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A FAST CONVERGENT APPROXIMATION METHOD FOR THE SOLUTION OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The Riccati equation method is used to obtain a fast convergent approximation method for the solution of second order linear ordinary differential equations. By using examples it is shown how fast the proposed method can converge.

1. INTRODUCTION

Let p be a real-valued continuous function on $[T_0,T]$. Consider the second order linear ordinary differential equation

$$\phi'' + p(t)\phi = 0, \;\; t \in [T_0,T]. \tag{1.1}$$

In practice, the problem of finding the values of solutions of differential equations (in particular, of Eq. (1.1)) arises very often. This problem is solvable in the case when the solutions of a differential equation (in particular in the case of Eq. (1.1)) are representable in a closed form trough the known data of the equation. However this occurs in very rare cases. To solve this problem, many numerical methods have been developed for solving differential equations (in particular for solving Eq. (1.1)), and many works are devoted to them (see [1-10]) and cited works therein). Among them notice [4] in which an impressive fast convergent numerical method for solving second order linear ordinary differential equations is developed. Unfortunately, the fast convergence of this method has been demonstrated practically in some examples, but has not yet been proved mathematically, which is why it is unclear to which equations it can be effectively applied.

In this paper we propose a new approximation method for the solution of Eq. (1.1), based on the Riccati equation method. We show with examples how fast this method can converge.

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2. Auxiliary propositions

. Let $[\alpha,\beta]$ be a subset of $[T_0,T]$. Consider the equation

$$\theta'' + p(t)\theta = 0, \quad t \in [\alpha,\beta] \tag{2.1}$$

and associated with it the Riccati equation

$$u' = u^2 + p(t), \quad t \in [\alpha,\beta]. \tag{2.2}$$

All solutions u of (2.2), existing on $[\alpha,\beta]$, are connected with solutions $\theta(t)$ of Eq. (2.1) by the relations (see [11], p. 332)

$$\theta(t) = \theta(t_0)\exp\left\{-\int\limits_{t_0}^{t} u(\tau)d\tau\right\}, \quad \theta(t_0) \neq 0, \quad t_0, t \in [\alpha,\beta]. \tag{2.3}$$

For any $x \in \mathbb{C}([\alpha,\beta])$ denote by $\|x\|_{[\alpha,\beta]} = \|x\|$ the norm of x in $\mathbb{C}([\alpha,\beta])$. Set

$$p_1(t) \equiv \int\limits_{\alpha}^{t} p(\tau)d\tau, \quad p_{n+1}(t) \equiv \mathbb{P}_n(t)\int\limits_{\alpha}^{\tau} \frac{p(\tau) - p_n^2(\tau)}{\mathbb{P}_n(\tau)}d\tau,$$

where $\mathbb{P}_n(t) \equiv \exp\left\{2\int\limits_{\alpha}^{t} p_n(\tau)d\tau\right\}, \quad t \in [\alpha,\beta], \quad n = 1,2,\ldots.$

Theorem 2.1. *Let the following conditions be satisfied:*
1) $(\beta-\alpha)(1+\|p\|) \leq 1$;
2) $(\beta-\alpha)(c^2+\|p\|)e^{(\beta-\alpha)c} \leq c$ *for some* $c \in (0,1]$.

Then the following assertions are valid:

I) *the solution* u_* *of Eq.* (2.2) *with* $u_*(\alpha) = 0$ *exists on* $[\alpha,\beta]$*;*

II) *the sequence* $\{p_n(t)\}_{n=1}^{+\infty}$ *converges to* u_* *in* $\mathbb{C}^1([\alpha,\beta])$ *and*

$$\|u_* - p_n\| \leq \frac{e^{-2(\beta-\alpha)c}}{\beta-\alpha}E_n(\rho), \quad n = 1,2,... \tag{2.4}$$

$$\|u_*' - p_n'\| \leq \frac{2ce^{-2(\beta-\alpha)c}}{\beta-\alpha}E_n(\rho) + \frac{e^{-4(\beta-\alpha)c}}{(\beta-\alpha)^2}E_{n-1}^2(\rho), \quad n = 2,3,... \tag{2.5}$$

where $\rho \equiv (\beta-\alpha)e^{(\beta-\alpha)c}\min\{\|p\|,c\}, \quad E_1(\rho) \equiv \rho^2, \quad E_2(\rho) \equiv \frac{\rho^4}{3}$,

$$E_n(\rho) \equiv \frac{\rho^{2^n}}{(2^1-1)^{2^{n-1}}(2^2-1)^{2^{n-2}}...(2^n-1)}, \quad n = 3,4,....$$

Proof. Set $M \equiv \max\limits_{t\in[\alpha,\beta],\ 0\leq|u|\leq\gamma} |u^2 + p(t)|, \quad h \equiv \min\{\beta-\alpha, \frac{\gamma}{M}\}, \quad \gamma > 0$. Since the function $f(t;u) \equiv u^2 + p(t)$ is continuous on the domain $\{(t;u) : t \in [\alpha,\beta], \ 0 \leq u \leq \gamma\}$, by Peano's theorem (see [11] p. 10) Eq. (2.2) has a solution u_* on $[\alpha,\beta]$. Therefore the assertion I) will be proved if we show that it is always possible to take $h = \beta-\alpha$. If $\|P\| = 0$, then for $\gamma = \frac{1}{\beta-\alpha}$, we have $h = \beta-\alpha$ (since in this case

$\frac{\gamma}{M}=\beta-\alpha$). If $\|p\|\neq 0$, then taking $\gamma=\|p\|$, we obtain $M\leq\|p\|^2+\|p\|$. By the condition 1), from here it follows that $\frac{\gamma}{M}\geq\frac{\|p\|}{\|p\|^2+\|p\|}\geq\frac{1}{\|p\|+1}\geq\beta-\alpha$. Therefore in this case, we have also $h=\beta-\alpha$. The assertion I) is proved. Now we prove II). By (2.2), we have

$$u_*(t)=u_1(t)+p_1(t),\ \ t\in[\alpha,\beta], \tag{2.6}$$

where $u_1(t)\equiv\int\limits_\alpha^t u_*^2(\tau)d\tau,\ \ t\in[\alpha,\beta]$. Using (2.2), from here, we obtain

$$u_*'(t)-2p_1(t)u_*(t)=u_1^2(t)-p_1^2(t)+p(t),\ \ t\in[\alpha,\beta]. \tag{2.7}$$

Let $\mathfrak{M}_1$ be an integral operator, acting on $\mathbb{C}([\alpha,\beta])$ by the rule

$$(\mathfrak{M}_1u)(t)=\mathbb{P}_1(t)\int\limits_\alpha^t\frac{u(\tau)}{\mathbb{P}_1(\tau)}d\tau,\ \ u\in\mathbb{C}([\alpha,\beta]).$$

Acting on both sides of (2.7) by $\mathfrak{M}_1$ and taking into account that $u_*(\alpha)=0$, we obtain

$$u_*(t)=u_2(t)+p_2(t),\ \ t\in[\alpha,\beta] \tag{2.8}$$

where $u_2(t)\equiv\mathbb{P}_1(t)\int\limits_\alpha^t\frac{u_1^2(\tau)}{\mathbb{P}_1(\tau)}d\tau, t\in[\alpha,\beta]$. Using again (2.2), by the analogy of (2.7), from here we obtain

$$u_*'(t)-2p_2(t)u_*(t)=u_2^2(t)-p_2^2(t)+p(t),\ \ t\in[\alpha,\beta]. \tag{2.9}$$

Let $\mathfrak{M}_2$ be an integral operator, acting on $\mathbb{C}([\alpha,\beta])$ by the rule

$$(\mathfrak{M}_2u)(t)=\mathbb{P}_2(t)\int\limits_\alpha^t\frac{u(\tau)}{\mathbb{P}_2(\tau)}d\tau,\ \ u\in\mathbb{C}([\alpha,\beta]).$$

Acting on both sides of (2.9) by $\mathfrak{M}_2$ and taking into account that $u_*(\alpha)=0$, we get

$$u_*(t)=u_3(t)+p_3(t),\ \ t\in[\alpha,\beta],$$

where $u_3(t)\equiv\mathbb{P}_2(t)\int\limits_\alpha^t\frac{u_2^2(\tau)}{\mathbb{P}_2(\tau)}d\tau, t\in[\alpha,\beta]$, and so on. Continuing this process of recursive determination of $u_1(t),\ u_2(t),\ u_3(t),...$ for the general case of n (taking into account (2.6)-(2.9)) we obtain the recursive formulae

$$u_*(t)=u_n(t)+p_n(t),\ \ t\in[\alpha,\beta], \tag{2.10}$$

where $u_n(t)\equiv\mathbb{P}_n(t)\int\limits_\alpha^t\frac{u_{n-1}^2(\tau)}{\mathbb{P}_n(\tau)}d\tau,\ \ t\in[\alpha,\beta],\ \ n=2,3,\ldots$. Let us estimate the norms $\|u_*\|,\ \ \|p_n\|,\ \ n=1,2,\ldots$. Now we show that

$$\|p_n\|\leq c,\ \ n=1,2,\ldots. \tag{2.11}$$

By 2) for $n = 1$, we have

$$\|p_1\| \le (\beta-\alpha)\|p\| \le (\beta-\alpha)(c^2+\|p\|)e^{2(\beta-\alpha)c} \le c.$$

Therefore (2.11) is valid for $n = 1$. Suppose (2.11) is valid for some $n = k$. We show that it is valid also for $n = k+1$. Since $\|p_k\| \le c$, we have

$$\|p_{k+1}\| = \max_{t\in[\alpha,\beta]}\Big|\int\limits_\alpha^t \exp\Big\{2\int\limits_\tau^t p_k(s)ds\Big\}\big(p(\tau)-p_k^2(\tau)\big)d\tau\Big| \le (\beta-\alpha)e^{2(\beta-\alpha)c}[\|p\|+c^2].$$

This together with 2) implies (2.11) for $n = k+1$. Therefore (2.11) is valid for all $n = 1,2,\dots$. Obviously

$$|u_1(t)| = \Big|\int\limits_\alpha^t u_*^2(\tau)d\tau\Big| \le (t-\alpha)\|u_*\|^2, \ \ t \in [\alpha,\beta].$$

From here and from (2.11) we get

$$|u_2(t)| \le e^{2(\beta-\alpha)c}\frac{(t-\alpha)^3}{3}\|u_*\|^{2^2}, \ \ t\in[\alpha,\beta],$$

which together with (2.11) implies

$$|u_3(t)| \le e^{(2+2^2)(\beta-\alpha)c}\frac{(t-\alpha)^7}{3^2 7^1}\|u_*\|^{2^3}, \ \ t\in[\alpha,\beta],$$

and so on. Continuing this process of successive estimations in the general case of n we obtain

$$|u_n(t)| \le e^{(2+2^2+\dots+2^{n-1})(\beta-\alpha)c}\frac{(t-\alpha)^{2^n-1}}{(2^1-1)^{2^{n-1}}(2^2-1)^{2^{n-2}}\dots(2^n-1)}\|u_*\|^{2^n}, \ \ t\in[\alpha,\beta],$$

$n = 1,2,\dots$. From here it follows

$$\|u_n\| \le \frac{e^{-2(\beta-\alpha)c}}{\beta-\alpha}\frac{\big[(\beta-\alpha)e^{(\beta-\alpha)c}\|u_*\|\big]^{2^n}}{(2^1-1)^{2^{n-1}}(2^2-1)^{2^{n-2}}\dots(2^n-1)}, \ \ n = 1,2,\dots. \tag{2.12}$$

By the Peano's Theorem we have $\|u_*\| \le \|p\|$. This together with 2) implies

$$(\beta-\alpha)e^{(\beta-\alpha)c}\|u_*\| \le (\beta-\alpha)e^{2(\beta-\alpha)c}\|p\| \le \frac{c\|p\|}{c_2+\|p\|} \le \frac{\|p\|}{c^2+\|p\|} < 1.$$

From here, from (2.10) and (2.12) it follows that the sequence of functions $\{p_n(t)\}_{n=1}^{+\infty}$ converges to $u_*(t)$ in $\mathbb{C}([\alpha,\beta])$. Then since $\|u_*\| \le \|p\|$and by (2.11) $\|p_n\| \le c, n = 1,2,\dots$ we have

$$\|u_*\| \le \min\{\|p\|, c\}. \tag{2.13}$$

This together with (2.6), (2.10) and (2.12) implies (2.4). As far as $E_n(\rho) \to 0$ for $n \to +\infty$ then to complete the proof of the theorem it is enough to prove (2.5). It is not difficult to verify that

$$u'_*(t) - p'_n(t) = u^2_{n-1}(t) + 2p_n(t)u_n(t), \;\; n = 2,3,\ldots.$$

This together with (2.4), (2.10) and (2.11) implies (2.5). The theorem is proved. □

For any matrix $A \equiv (a_{ij})^2_{i,j=1}$ $(a_{ij} \in \mathbb{R},\; i,j = 1,2)$ denote by $\|A\|$ the norm $\max\limits_{j=1,2} \sum\limits_{i=1}^{2} |a_{ij}|$ of A. Then for any matrix $B \equiv (b_{i,j})^2_{i,j=1}$ $(b_{ij} \in \mathbb{R},\; i,j = 1,2)$ the following relations are valid.

$$\|\lambda A + \mu B\| \le |\lambda|\|A\| + |\mu|\|B\|, \;\; \lambda,\mu \in \mathbb{R}, \;\; \|AB\| \le \|A\|\,\|B\|. \tag{2.14}$$

By (2.3) under the conditions of Theorem 2.1 we have a solution of Eq. (2.1) of the form

$$\theta_0(t) \equiv \exp\left\{-\int_\alpha^t u_*(\tau)d\tau\right\}, \;\; t \in [\alpha,\beta].$$

Another solution of Eq. (2.1), linearly independent of $\theta_0(t)$, can be given by the formula (see [11], p. 327)

$$\theta_1(t) \equiv \theta_0(t)\int_\alpha^t \frac{d\tau}{\theta_0^2(\tau)}, \;\; t \in [\alpha,\beta]. \tag{2.15}$$

Set

$$\theta_{n,0}(t) \equiv \exp\left\{-\int_\alpha^t p_n(\tau)d\tau\right\}, \;\; \theta_{n,1}(t) \equiv \theta_{n,0}(t)\int_\alpha^t \frac{d\tau}{\theta_{n,0}^2(\tau)},$$

$$\Theta(t) \equiv \begin{pmatrix} \theta_0(t) & \theta_1(t) \\ \theta'_0(t) & \theta'_1(t) \end{pmatrix}, \;\; \Theta_n(t) \equiv \begin{pmatrix} \theta_{n,0}(t) & \theta_{n,1}(t) \\ \theta'_{n,0}(t) & \theta'_{n,1}(t) \end{pmatrix}, \;\; t \in [\alpha,\beta], \;\; n = 1,2,\ldots.$$

Corollary 2.1. *Let the conditions of Theorem* 2.1 *be satisfied. Then the sequences* $\{\theta_{n,0}(t)\}_{n=1}^{+\infty}$ *and* $\{\theta_{n,1}(t)\}_{n=1}^{+\infty}$ *converge respectively to* $\theta_0(t)$ *and* $\theta_1(t)$ *in* $\mathbb{C}^2([\alpha,\beta])$ *and*

$$\|\theta_0 - \theta_{n,0}\| \le e^{-(\beta-\alpha)c}E_n(\rho), \; n = 1,2,\ldots, \tag{2.16}$$

$$\|\theta'_0 - \theta'_{n,0}\| \le \left[\frac{1}{\beta-\alpha} + c\right]e^{-(\beta-\alpha)c}E_n(\rho), \; n = 1,2,\ldots, \tag{2.17}$$

$$\|\theta''_0 - \theta''_{n,0}\| \le \|p\|e^{-(\beta-\alpha)c}E_n(\rho) + \tfrac{e^{-4(\beta-\alpha)c}}{(\beta-\alpha)^2}\left(E_{n-1}(\rho) + E_n(\rho)\right)^2, \; n = 2,3,\ldots \tag{2.18}$$

$$\|\theta_1 - \theta_{n,1}\| \le e^{-(\beta-\alpha)c}E_n(\rho), \; n = 1,2,\ldots, \tag{2.19}$$

$$\|\theta'_1 - \theta'_{n,1}\| \le [2 + c]e^{-(\beta-\alpha)c}E_n(\rho), \; n = 1,2,\ldots, \tag{2.20}$$

$$\|\theta_1'' - \theta_{n,1}''\| \le \|p\| e^{-(\beta-\alpha)c} E_n(\rho) + \tfrac{e^{-3(\beta-\alpha)c}}{\beta-\alpha}\big(E_{n-1}(\rho) + E_n(\rho)\big)^2,\ n = 2,3,\dots \tag{2.21}$$

$$\|\Theta(t) - \Theta_n(t)\| \le S_0 e^{-(\beta-\alpha)c} E_n(\rho),\ \ t \in [\alpha,\beta],\ m = 1,2,\dots, \tag{2.22}$$

where $S_0 \equiv \max\{1 + \frac{1}{\beta-\alpha} + c, 3 + c\}$.

Proof. The inequality (2.22) we can obtain easily from (2.16), (2.17), (2.19) and (2.20) by using (2.14). The convergence of the sequences $\{\theta_{n,0}(t)\}_{n=1}^{+\infty}$ and $\{\theta_{n,1}(t)\}_{n=1}^{+\infty}$ respectively to $\theta_0(t)$ and $\theta_1(t)$ in $\mathbb{C}^2([\alpha,\beta])$ follows immediately from (2.16)-(2.21). Therefore to complete the proof of the corollary it is enough to prove (2.16)-(2.21). We have

$$|\theta_0(t) - \theta_{n,0}(t)| = \left|\exp\left\{-\int_\alpha^t u_*(\tau)d\tau\right\} - \exp\left\{-\int_\alpha^t p_n(\tau)d\tau\right\}\right| \le$$

$$\le \left|\int_\alpha^t (u_*(\tau) - p_n(\tau))d\tau\right| \exp\left\{\left|\int_\alpha^t u_*(\tau)d\tau\right|,\ \left|\int_\alpha^t p_n(\tau)d\tau\right|\right\},\ t \in [\alpha,\beta].$$

This together with (2.4), (2.11) and (2.13) implies (2.16). Obviously by (2.11) and (2.13) we have

$$\|\theta_0\| \le e^{(\beta-\alpha)c},\ \ \|\theta_{n,0}\| \le e^{(\beta-\alpha)c}. \tag{2.23}$$

From here and from (2.16) it follows: $\|\theta_0' - \theta_{n,0}'\| = \|-u_*\theta_0 + p_n\theta_{n,0}\| \le \|\theta_0\|\,\|u_* - p_n\| + \|p_n\|\,\|\theta_0 - \theta_{n,0}\| \le e^{(\beta-\alpha)c}\|u_* - p_n\| + \|p_n\| e^{-(\beta-\alpha)c} E_n(\rho),\ \ n = 1,2,\dots.$ This together with (2.4) and (2.11) implies (2.17). We prove (2.18). Using the easily verifiable equalities

$$p_n'(t) = 2p_{n-1}(t)p_n(t) + p(t) - p_{n-1}^2(t),\ \ t \in [\alpha,\beta],\ \ n = 2,3,\dots,$$

we obtain

$$\theta_{n,0}''(t) = ([p_n(t) - p_{n-1}(t)]^2 - p(t))\theta_{n,0},\ \ t \in [\alpha,\beta],\ \ n = 2,3,\dots$$

Then

$$\theta_0''(t) - \theta_{n,0}''(t) = p(t)(\theta_{n,0}(t) - \theta_0(t)) - [p_n(t) - p_{n-1}(t)]^2\theta_{n,0}(t),\ \ t \in [\alpha,\beta].$$

From here it follows

$$\|\theta_0'' - \theta_{n,0}''\| \le \|[p_n - p_{n-1}]^2\|\,\|\theta_{n,0}\| + \|p\|\,\|\theta_{n,0} - \theta_0\| \le$$

$$\le (\|u_* - p_n\| + \|u_* - p_{n-1}\|)^2\,\|\theta_{n,0}\| + \|p\|\,\|\theta_{n,0} - \theta_0\|.$$

This together with (2.4), (2.16) and (2.23) implies (2.18). It is not difficult to verify that

$$\|\theta_1\| \le (\beta - \alpha)e^{(\beta-\alpha)c},\ \ \|\theta_{n,1}\| \le (\beta-\alpha)e^{(\beta-\alpha)c},\ \ n = 1,2,3,\dots \tag{2.24}$$

We have

$$|\theta_1(t)-\theta_{n,1}(t)| =$$

$$= \left|\int_\alpha^t \exp\left\{-\int_\tau^t u_*(s)ds + \int_\alpha^\tau u_*(s)ds\right\}d\tau - \int_\alpha^t \exp\left\{-\int_\tau^t u_*(s)ds + \int_\alpha^\tau u_*(s)ds\right\}d\tau\right|$$

$$\leq \left|\int_\alpha^t \left(\int_\alpha^\tau [u_*(s)-p_n(s)]ds - \int_\tau^t [u_*(s)-p_n(s)]ds\right)\times\right.$$

$$\times \exp\left\{\max\left\{\left|-\int_\tau^t u_*(s)ds + \int_\alpha^\tau u_*(s)ds\right|, \left|-\int_\tau^t p_n(s)ds + \int_\alpha^\tau p_n(s)ds\right|\right\}d\tau,\ t\in[\alpha,\beta],\right.$$

$$n = 1,2,3,\dots\ .$$

This together with (2.4), (2.11) and (2.13) implies (2.19). Since

$$\theta_1'(t) = -u_*(t)\theta_1(t) = \frac{1}{\theta_0(t)},\ \theta_{n,1}'(t) = -p_n(t)\theta_{n,1}(t) = \frac{1}{\theta_{n,0}(t)},\ t\in[\alpha,\beta],\ n=1,2,\dots$$

we have

$$\|\theta_1' - \theta_{n,1}'\| \leq \|u_*\|\|\theta_1 - \theta_{n,1}\| + \|\theta_{n,1}\|\|u_* -' p_n\| + \left\|\frac{1}{\theta_0} - \frac{1}{\theta_{n,0}}\right\|,\ n=1,2,\dots. \quad (2.25)$$

We have also

$$\left|\frac{1}{\theta_0(t)} - \frac{1}{\theta_{n,0}(t)}\right| = \left|\exp\left\{\int_\alpha^t u_*(\tau)d\tau\right\} - \exp\left\{\int_\alpha^t p_n(\tau)d\tau\right\}\right| \leq$$

$$\leq \left|\int_\alpha^t [u_*(\tau)-p_n(\tau)]\,d\tau\right| \exp\left\{\max\left\{\left|\int_\alpha^t u_*(\tau)d\tau\right|, \left|\int_\alpha^t p_n(\tau)d\tau\right|\right\}\right\}, t\in[\alpha,\beta], n=1,2,\dots.$$

This together with (2.4), (2.11) and (2.13) implies

$$\left\|\frac{1}{\theta_0} - \frac{1}{\theta_{n,0}}\right\| \leq e^{-(\beta-\alpha)c}E_n(\rho),\ \ n=1,2,\dots.$$

From here, from (2.4), (2.13), (2.24) and (2.25) we obtain it follows (2.20). It is not difficult to verify that

$$\theta_{n,1}''(t) = [(p_n(t)-p_{n-1}(t))^2 - p(t)]\theta_{n,1}(t),\ \ t\in[\alpha,\beta].\ \ n=2,3,\dots.$$

Then since $\theta_1''(t) = -p(t)\theta_1(t),\ \ t\in[\alpha,\beta]$, we have

$$\theta_1''(t) - \theta_{n,1}''(t) = \theta_{n,1}(t)(p_n(t)-p_{n-1}(t))^2 - p(t)(\theta_1(t)-\theta_{n,1}(t)), t\in[\alpha,\beta], n=2,3,\dots.$$

From here it follows

$$\|\theta_1'' - \theta_{n,1}\| \leq \|p\|\,\|\theta_1 - \theta_{n,1}\| + \|\theta_{n,1}\|\,(\|p_n - u_*\| + \|p_{n-1} - u_*\|)^2.$$

This together with (2.4), (2.19) and (2.24) implies (2.21).
The corollary is proved. □

3. Fast convergent approximation method

From the conditions of Theorem 2.1 is seen that they are satisfied if $\beta-\alpha$ is small enough. This suggests how to to use Theorem 2.1 to construct approximate solutions for Eq. (1.1) on arbitrarily large intervals $[T_0,T]$. Obviously to do this it is enough to partition the interval $[T_0,T]$ in a sum (union) of small intervals so that for each of them Theorem 2.1 holds, and after that to construct an approximate solution on each of the partitions and then "glue" them properly. Next we show how we realize this idea.

Let $T_0=t_0<t_1<\ldots>t_{2^N}=T$ be the partition of the interval $[T_0,T]$ so that for each $[t_k,t_{k+1}]=[\alpha,\beta]$ $(k=\overline{0,2^N-1})$ the conditions of Theorem 2.1 are satisfied. Then according to Theorem 2.1 for every $k=\overline{0,2^N-1}$ the equation

$$y'=y^2+p(t),\ \ t\in[t_k,t_{k+1}]$$

has a solution $y^*_k(t)$ on $[t_k,t_{k+1}]$ with $y^*_k(t_k)=0$. Set

$$\phi_{0,k}(t)\equiv\exp\left\{-\int\limits_{t_k}^{t}y^*_k(\tau)d\tau\right\},\ \ \phi_{1,k}(t)\equiv\phi_{0,k}(t)\int\limits_{t_k}^{t}\frac{d\tau}{\phi^2_{0,k}(t)},$$

$$p_{1,k}(t)\equiv\int\limits_{t_k}^{t}p(\tau)d\tau,\ \ p_{n,k}(t)\equiv\mathbb{P}_{n-1,k}(t)\int\limits_{t_l}^{t}\frac{p(\tau)-p^2_{n-1,k}(\tau)}{\mathbb{P}_{n-1,k}(\tau)}d\tau,$$

where

$$\mathbb{P}_{n-1,k}(t)\equiv\exp\left\{2\int\limits_{t_k}^{t}p_{n-1,k}(\tau)d\tau\right\},\ \ t\in[t_k,t_{k+1}],\ \ k=\overline{0,2^N-1},\ \ n=2,3,\ldots,$$

$$\phi_{n,0,k}(t)\equiv\exp\left\{-\int\limits_{t_k}^{t}p_{n,k}(\tau)d\tau\right\},\ \ \phi_{n,1,k}(t)\equiv\phi_{n,0,k}(t)\int\limits_{t_k}^{t}\frac{d\tau}{\phi^2_{n,0,k}(t)},$$

$$\Phi_k(t)\equiv\begin{pmatrix}\phi_{0,k}(t) & \phi_{1,k}(t)\\ \phi'_{0,k}(t) & \phi'_{1,k}(t)\end{pmatrix},\ \ \Phi_{n,k}(t)\equiv\begin{pmatrix}\phi_{n,0,k}(t) & \phi_{n,1,k}(t)\\ \phi'_{n,0,k}(t) & \phi'_{n,1,k}(t)\end{pmatrix},\ t\in[t_k,t_{k+1}],$$

$k=\overline{0,2^N-1}$, $n=1,2,\ldots$. It is not difficult to verify that

$$\Phi_k(t_k)=\Phi_{n,k}(t_k)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \ k=\overline{0,2^N-1},\ \ n=1,2,\ldots. \tag{3.1}$$

By induction on m define: $t_{1,k}\equiv t_{2k}$, $k=\overline{0,2^{N-1}-1}$, $t_{m+1,k}\equiv t_{m,2k}$, $k=\overline{0,2^{N-m}-1}$, $m=\overline{1,N}$,

$$\Phi^0_{1,k}(t)\equiv\begin{cases}\Phi_{2k}(t),\ \ t\in[t_{2k},t_{2k+1}],\\ \Phi_{2k+1}(t)\Phi_{2k}(t_{2k+1}),\ \ t\in[t_{2k+1},t_{2k+2}],\end{cases}$$

$$\Phi^0_{1,n,k}(t)\equiv\begin{cases}\Phi_{n,2k}(t),\ \ t\in[t_{2k},t_{2k+1}],\\ \Phi_{n,2k+1}(t)\Phi_{2k}(t_{2k+1}),\ \ t\in[t_{2k+1},t_{2k+2}].\end{cases}$$

The matrix functions $\Phi_k(t)$, $\Phi_{n,k}(t)$ and the intervals $[t_k, t_{k+1}]$, $k = \overline{0, 2^N - 1}$, $n = 1, 2, \ldots$ we will call the matrix functions and the intervals of level 0 respectively, and the matrix functions $\Phi^0_{1,k}(t)$, $\Phi^0_{1,n,k}(t)$ and the intervals $[t_{1,k}, t_{1,k+1}]$, $k = \overline{0, 2^{N-1} - 1}$, $n = 1, 2, \ldots$ we will call the matrix functions and the intervals of level 1 respectively. Let $\phi^0_{m,k}(t)$ and $\Phi^0_{m,n,k}(t)$, $k = \overline{0, 2^{N-m} - 1}$, $n = 1, 2, \ldots$ be matrix functions of level m on the intervals $[t_{m,k}, t_{m,k+1}]$, $k = \overline{0, 2^{N-m} - 1}$ of level m. Define by induction on m the matrix functions $\phi^0_{m,k}(t)$ and $\Phi^0_{m,n,k}(t)$, $k = \overline{0, 2^{N-m-1} - 1}$, $n = 1, 2, \ldots$ on the intervals $[t_{m+1,k}, t_{m+1,k+1}]$, $k = \overline{0, 2^{N-m-1} - 1}$ of level m+1 respectively as follows:

$$\Phi^0_{m+1,k}(t) \equiv \begin{cases} \Phi^0_{m,k}(t), \ t \in [t_{m,k}, t_{m,k+1}], \\ \Phi^0_{m,k+1}(t)\Phi^0_{m,k}(t_{m,k+1}), \ t \in [t_{m,k+1}, t_{m,k+2}], \end{cases}$$

$$\Phi^0_{m+1,n,k}(t) \equiv \begin{cases} \Phi^0_{m,n,k}(t), \ t \in [t_{m,k}, t_{m,k+1}], \\ \Phi^0_{m,n,k+1}(t)\Phi^0_{m,n,k}(t_{2k+1}), \ t \in [t_{m,k+1}, t_{m,k+2}], \end{cases}$$

$$k = \overline{0, 2^{N-m-1} - 1}, \ m = \overline{1, N-1}, \ n = 1, 2, \ldots.$$

Set: $\Phi_*(t) \equiv \Phi^0_{N,0}(t), \Phi_{*,n}(t) \equiv \Phi^0_{N,n,0}(t)$, $t \in [t_{N,0}, t_{N,1}] = [T_0, T]$, $n = 1, 2, \ldots$.

Since $\Phi^0_*(T_0) = \Phi_0(T_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by the uniqueness theorem and (2.3), (2.15) the matrix function $\Phi^0_*(t)$ is a fundamental matrix of Eq. (1.1) on $[T_0, T]$. Next our goal is to estimate

$$\max_{t \in [T_0, T]} \|\Phi_*(t) - \Phi_{*,n}(t)\|.$$

Let c_k be a constant for which the conditions of Theorem 2.1 with $[\alpha, \beta] = [t_k, t_{k+1}]$ are satisfied ($k = \overline{0, 2^N - 1}$). Set:

$$d_k \equiv \|p\|_{[t_k, t_{k+1}]}, \ \rho_k \equiv (t_{k+1} - t_k)e^{(t_{k+1}-t_k)c_k} \min\{d_k, c_k\}, \ \rho \equiv \max\{\rho_k, \ k = \overline{0, 2^N - 1}\},$$

$$S_{k+1} \equiv \max\left\{1 + \frac{1}{t_{k+1} - t_k} + c_k, 3 + c_k\right\}, \ k = \overline{0, 2^N - 1}, \ S \equiv \max\{S_k, \ k = \overline{1, 2^N}\}.$$

Then by Corollary 2.1 (see (2.22)) we have

$$\|\Phi_k(t) - \Phi_{n,k}(t)\| \le SE_n(\rho), \ t \in [t_k, t_{k+1}], \ k = \overline{0, 2^N - 1}, \ n = 1, 2, \ldots \tag{3.2}$$

Set

$$\Delta_{0,n} \equiv \max_{k = \overline{0, 2^N - 1}} \max_{t \in [t_k, t_{k+1}]} \|\Phi_k(t) - \Phi_{n,k}(t)\|,$$

$$\Delta_{m,n} \equiv \max_{k = \overline{0, 2^{N-m} - 1}} \max_{t \in [t_{m,k}, t_{m,k+1}]} \|\Phi^0_{m,k}(t) - \Phi^0_{m,n,k}(t)\|, \ m = \overline{1, N}, \ n = 1, 2, \ldots.$$

By (2.4) we have

$\|\Phi^0_{1,k}(t) - \Phi^0_{1,n,k}(t)\| = \|\Phi_{2k+1}(t)\Phi_{2k}(t_{2k+1}) - \Phi_{n,2k+1}(t)\Phi_{n,2k}(t_{2k+1})\| \le \|\Phi_{2k+1}(t) - \Phi_{n,2k+1}(t)\| \, \|\Phi_{2k}(t_{2k+1})\| + \|\Phi_{2k}(t_{2k+1})\| \, \|\Phi_{2k}(t_{2k+1}) - \Phi_{n,2k}(t_{2k+1})\|$, $t \in [t_{2k+1}, t_{2k+2}]$.

Hence,

$$\|\Phi^0_{1,k}(t)-\Phi^0_{1,n,k}(t)\| \le \|\Phi_{2k}(t_{2k+1})\|\|\Phi_{2k+1}(t)-\Phi_{n,2k+1}(t)\|+$$
$$+|\Phi_{2k+1}(t)-\Phi_{n,2k+1}(t)\||\Phi_{2k+1}(t_{2k+1})-\Phi_{n,2k+1}(t_{2k+1})\|+$$
$$+\|\Phi_{2k+1}(t)\|\|\Phi_{2k}(t_{2k+1})-\Phi_{n,2k}(t_{2k+1})\|,\ t\in[t_{2k+1},t_{2k+2}],\ k=\overline{0,2^{N_1}-1}. \quad (3.3)$$

For any $\zeta \in [T_0,T]$ denote by $\Phi(\zeta;t)$ the fundamental matrix of Eq. (1.1) with $\Phi(\zeta;\zeta)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Set $M \equiv \max\limits_{\zeta\in[T_0,T]}\max\limits_{t\in[\zeta,T]}\|\Phi(\zeta;t)\|$. Then from (3.3) it follows

$$\|\Phi^0_{1,k}(t)-\Phi^0_{1,n,k}(t)\| \le 2M\Delta_{0,n}+\Delta^2_{0,n},\ \ t\in[t_{2k+1},t_{2k+2}].$$

By obvious inequality $M\ge 1$ from here it follows

$$\|\Phi^0_{1,k}(t)-\Phi^0_{1,n,k}(t)\| \le 2M\Delta_{0,n}+\Delta^2_{0,n},\ \ t\in[t_{1,k},t_{1,k+1}].$$

Hence

$$\Delta_{1,n}\le 2M\Delta_{0,n}+\Delta^2_{0,n},$$

and in general for any $m=0,1,\dots,N-1$ it can be shown that

$$\Delta_{m+1,n}\le 2M\Delta_{m,n}+\Delta^2_{m,n}. \quad (3.4)$$

From here we obtain

$$\Delta_{m+2,n}\le(2M)^2\Delta_{m,n}+(2M+(2M)^2)\Delta^2_{m,n}+4M\Delta^3_{m,n}+\Delta^4_{m,n}, m=\overline{0,2^{N-2}}, n=1,2,\dots,$$
$$\Delta_{m+3.n}\le(2M)^3\Delta_{m,n}+[(2M)^2+(2M)^3+(2M)^4]\Delta^2_{m,n}+16M^2\Delta^3_{m,n}+$$
$$[4M^2+48M^3+16M^4]\Delta^4_{m,n}+[24M^2+32M^3]\Delta^5_{m,n}+[4M+8M^2]\Delta^6_{m,n}+8M\Delta^7_{m,n}+\Delta^8_{m,n}.$$

and finally

$$\max_{t\in[T_0,T]}\|\Phi_*(t)-\Phi_{*,n}(t)\|=\Delta_{N,n}\le(2M)^N\Delta_{0,n}+\Delta^2_{0,n}Q_N(\Delta_{0,n}),\ \ n=1,2,...,$$

where $Q_N(t)$ is a polynomial of degree 2^N-2, with positive coefficients (depending only on M) such that $Q_N(0)\ne 0$. From here and from (3.2) we obtain the following immediately

Theorem 3.1. *The sequence* $\{\Phi_{*,n}(t)\}_{n=1}^{+\infty}$ *converges to the fundamental matrix* $\Phi_*(t)$ *of Eq. (1.1)* $\left(\Phi_*(T_0)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$ *on* $[T_0,T]$ *by the norm of matrices uniformly in t and the following estimates are valid*

$$\max\|\Phi_*(t)-\Phi_{*,n}(t)\|\le(2M)^N SE_n(\rho)+(SE_n(\rho))^2Q_N(SE_n(\rho)),\ \ n=1,2,\dots.$$

4. Examples

In this section we show how fast the proposed approximation can converge. Consider the Mathieu equation (see [12], [13], p. 111)

$$\phi''+(1-\varepsilon+\delta\cos 2t)\phi=0,\ \ t\in[T_0,T]. \quad (4.1)$$

In the case $\varepsilon = \delta = 0$ this equation becomes an equation with constant coefficients, that is:

$$\phi'' + \phi = 0, \ \ t \in [T_0, T].$$

Obviously for this equation the matrix function

$$\Phi_0(t;\zeta) \equiv \begin{pmatrix} \cos(t-\zeta) & \sin(t-\zeta \\ -\sin(t-\zeta) & \cos(t-\zeta) \end{pmatrix}, (T_0 \le \zeta \le t \le T)$$

is its fundamental matrix with $\Phi_0(\zeta;\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $\zeta \in [T_0, T]$. It is also obvious that $\|\Phi_0(t;\zeta)\| \le \sqrt{2}, \ T_0 \le \zeta \le t \le T$. Due to this we will assume that the parameters ε and δ are so small, that

$$\|\Phi(t;\zeta)\| \le 2, \ \ T_0 \le \zeta \le t \le T, \tag{4.2}$$

where $\Phi(t;\zeta)$ is the fundamental matrix for Eq. (4.1) with $\Phi(\zeta;\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $\zeta \in [T_0, T]$.

Example 4.1. *Let $n = 2, \ [T_0, T] = [0, 1]$. Take $t_k = \frac{k}{8}, \ k = \overline{0, 8}$. For this case we have $\|p\| \le 1, \ N = 3$ and by (4.2) $M = 2$. Then it is not difficult to check that the conditions of Theorem 2.1 with $[\alpha, \beta] = [t_k, t_{k+1}] \ c_k = \frac{1}{7} \ (k = \overline{0, t})$ are satisfied for Eq. (4.1). It is not difficult to verify also, that for this case $\rho = \max\limits_{k=\overline{0,8}} \rho_k = \frac{1}{56} e^{\frac{1}{56}} < \frac{1}{55}$. Then $S = 1 + 8 + \frac{1}{7}$, and $SE_2(\rho) \le \frac{64}{21} \left(\frac{1}{55}\right)^4$. Then applying (3.4) three times for successive estimation of $\Delta_{1,2}, \ \Delta_{2,2}, \ \Delta_{3,2}$ via $SE_2(\rho)$ ($\Delta_{1,2}$ via $\Delta_{0,2} = SE_2(\rho)$, $\Delta_{2,2}$ via $\Delta_{1,2}$ and $\Delta_{3,2}$ via $\Delta_{2,2}$) from here we obtain*

$$\|\Phi_*(t) - \Phi_{*,2}(t)\| \le 0.00003, \ \ t \in [0, 1].$$

Example 4.2. *Let $n = 2, \ [T_0, T] = [0, 8]$. Take $t_k = \frac{k}{16}, \ k = \overline{0, 128}$. For this case we have $N = 7, \ M = 2$. Then it is not difficult to verify that for $c_k = \frac{1}{15} (\ k = \overline{0, 127})$ the conditions of Theorem 2.1 with $[\alpha, \beta] = [t_k, t_{k+1}]$ for Eq. (4.1) are satisfied. It is also not difficult to verify that for this case $\rho = \max\limits_{k=\overline{0,128}} \rho_k = \frac{1}{240} e^{\frac{1}{240}} < \frac{1}{238}$. Hence, since for this case $S = \frac{256}{15}$, we have*

$$\Delta_{0,2} = SE_2(\rho) < \frac{256}{135} \left(\frac{1}{238}\right)^4.$$

Then applying (3.4) for successive estimations of $\Delta_{1,2}, \ldots, \Delta_{7,2}$ via $\Delta_{0,2}$ from here we obtain

$$\|\Phi_*(t) - \Phi_{*,2}(t)\| \le 0.000001, \ \ t \in [0, 8].$$

Example 4.3. *Let $n = 3, \ [T_0, T] = [0, 128]$. Take $t_k = \frac{k}{4}, \ k = \overline{0, 512}$. For this case we have $N = 9, \ M = 2$. Then it is not difficult to verify that for $c_k = \frac{1}{3} (\ k = \overline{0, 511})$*

the conditions of Theorem 2.1 with $[\alpha,\beta]=[t_k,t_{k+1}]$ *for Eq. (4.1) are satisfied. For this case we have* $\rho=\max\limits_{k=\overline{0,128}}\rho_k=\frac{1}{12}e^{\frac{1}{12}}<\frac{1}{11}$. *Hence, since for this case* $S=\frac{16}{3}$, *we have*

$$\Delta_{0,2}=SE_2(\rho)<\frac{16}{189}\left(\frac{1}{11}\right)^8.$$

Then applying (3.4) for successive estimations of $\Delta_{1,2},\dots,\Delta_{9,2}$ *via* $\Delta_{0,2}$ *from here we obtain*

$$\|\Phi_*(t)-\Phi_{*,3}(t)\|\le 0.00004,\ \ t\in[0,128].$$

Example 4.4. *Let* $n=4,\ \ [T_0,T]=[0,1048576]$. *Take* $t_k=\frac{k}{4},\ \ k=\overline{0,512}$. *For this case we have* $N=22,\ \ M=2$,

$$SE_4(\rho)<\frac{16}{3}\frac{1}{3^4 7^2 15}\left(\frac{1}{11}\right)^{16}$$

and, finally, the estimate

$$\|\Phi_*(t)-\Phi_{*,4}(t)\|\le 0.0000001,\ \ t\in[0,1048576].$$

Remark 4.1. The values of the elements of the matrices $\Phi_{*,n}(t),\ \ n=1,2$ can be effectively calculated in the particular case when $p(t)=\sum\limits_{j=1}^{N}\left[a_j t^j+\frac{b_j}{t+c_j}+\frac{e_j}{t^2+f_j}\right]$, $t+c_j\ne\ 0,\ \ f_j\ge 0, j=\overline{1,N},\ \ t\ge t_0$.

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