SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.10.1.04
Vol.10 (22) (2014), 27–35

MULTIVARIATE FRACTIONAL REPRESENTATION FORMULA AND OSTROWSKI TYPE INEQUALITY

GEORGE A. ANASTASSIOU

Abstract. Here we derive a multivariate fractional representation formula involving ordinary partial derivatives of first order. Then we prove a related multivariate fractional Ostrowski type inequality with respect to uniform norm.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [3]:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P_1(x, t) f'(t)\, dt, \tag{1}$$

where $P_1(x, t)$ is the Peano kernel

$$P_1(x, t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x, \\ \frac{t-b}{b-a}, & x < t \leq b, \end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a, b]. \tag{4}$$

Properties of the above operator can be found in [4].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

2010 *Mathematics Subject Classification.* 26A33, 26D10, 26D15.

Key words and phrases. Multivariate fractional integral, representation formula, multivariate Ostrowski inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha}\,\Gamma(\alpha) \cdot \left\{ \frac{J_a^{\alpha} f(b)}{b-a} - J_a^{\alpha-1}\left(P_1(x,b) f(b)\right) + J_a^{\alpha}\left(P_1(x,b) f'(b)\right) \right\}. \quad (5)$$

When $\alpha = 1$ the last (5) *reduces to classic Montgomery identity* (1).

We may rewrite (5) as follows

$$f(x) = (b-x)^{1-\alpha}\left[\frac{1}{b-a}\int_a^b (b-t)^{\alpha-1} f(t)\,dt - (\alpha-1)\int_a^b (b-t)^{\alpha-2} P_1(x,t) f(t)\,dt + \int_a^b (b-t)^{\alpha-1} P_1(x,t) f'(t)\,dt\right]. \quad (6)$$

In this article based on (5), we establish a multivariate fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i,b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate fractional Ostrowski type inequality.

2. Main results

We make

Assumption 2. *Let $f \in C^1\left(\prod_{i=1}^m [a_i,b_i]\right)$.*

Assumption 3. *Let $f : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$, and it is x_j-integrable for all $j = 1,\ldots,m$. Furthermore $\frac{\partial f}{\partial x_i}(t_1,\ldots,t_i,x_{i+1},\ldots,x_m)$ it is integrable on $\prod_{j=1}^i [a_j,b_j]$, for all $i = 1,\ldots,m$, for any $(x_{i+1},\ldots,x_m) \in \prod_{j=i+1}^m [a_j,b_j]$.*

Convention 4. *We set*

$$\prod_{j=1}^{0} \cdot = 1. \quad (7)$$

Notation 5. *Here $x = \overrightarrow{x} = (x_1,\ldots,x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. Likewise $t = \overrightarrow{t} = (t_1,\ldots,t_m)$, and $d\overrightarrow{t} = dt_1 dt_2 \ldots dt_m$. We denote the kernel*

$$P_1(x_i,t_i) = \begin{cases} \frac{t_i-a_i}{b_i-a_i}, & a_i \leq t_i \leq x_i, \\ \frac{t_i-b_i}{b_i-a_i}, & x_i < t_i \leq b_i, \end{cases} \quad (8)$$

We need

Definition 6. [see [2]] Let $\prod_{i=1}^{m}[a_i,b_i] \subset \mathbb{R}^m$, $m \in \mathbb{N}-\{1\}$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$. Let $\alpha > 0$, $f \in L_1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$. We define the left mixed Riemann-Liouville fractional multiple integral of order α:

$$\left(I_{a+}^{\alpha}f\right)(x) := \frac{1}{(\Gamma(\alpha))^m}\int_{a_1}^{x_1}\dots\int_{a_m}^{x_m}\left(\prod_{i=1}^{m}(x_i-t_i)\right)^{\alpha-1} f(t_1,\dots,t_m)\,dt_1\dots dt_m, \tag{9}$$

where $x_i \in [a_i,b_i]$, $i=1,\dots,m$, and $x=(x_1,\dots,x_m)$, $a=(a_1,\dots,a_m)$, $b=(b_1,\dots,b_m)$.

We present the following multivariate fractional representation formula

Theorem 7. *Let f as in Assumption 2 or Assumption 3, $\alpha \geq 1$, $x_i \in [a_i,b_i)$, $i=1,\dots,m$. Then*

$$f(x_1,\dots,x_m) = \frac{\left(\prod_{i=1}^{m}(b_i-x_i)\right)^{1-\alpha}(\Gamma(\alpha))^m}{\prod_{i=1}^{m}(b_i-a_i)}\left(I_{a+}^{\alpha}f\right)(b) + \sum_{i=1}^{m}A_i(x_1,\dots,x_m) + \sum_{i=1}^{m}B_i(x_1,\dots,x_m), \tag{10}$$

where for $i=1,\dots,m$:

$$A_i(x_1,\dots,x_m) := \frac{-(\alpha-1)\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j-a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i-1}(b_j-t_j)\right)^{\alpha-1} \cdot (b_i-t_i)^{\alpha-2}P_1(x_i,t_i)\,f(t_1,\dots,t_i,x_{i+1},\dots,x_m)\,dt_1\dots dt_i, \tag{11}$$

and

$$B_i(x_1,\dots,x_m) := \frac{\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j-a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i}(b_j-t_j)\right)^{\alpha-1} \cdot P_1(x_i,t_i)\frac{\partial f}{\partial x_i}(t_1,\dots,t_i,x_{i+1},\dots,x_m)\,dt_1dt_2\dots dt_i. \tag{12}$$

Proof. By (6) we have

$$f(x_1,\dots,x_m) = (b_1-x_1)^{1-\alpha}\left[\frac{1}{b_1-a_1}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}f(t_1,x_2,\dots,x_m)\,dt_1 - (\alpha-1)\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)\,f(t_1,x_2,\dots,x_m)\,dt_1 + \int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,\dots,x_m)\,dt_1\right], \tag{13}$$

and

$$f(t_1, x_2, \dots, x_m) =$$
$$= (b_2 - x_2)^{1-\alpha} \left[\frac{1}{b_2 - a_2} \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-1} f(t_1, t_2, x_3, \dots, x_m) \, dt_2 \right.$$
$$- (\alpha - 1) \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-2} P_1(x_2, t_2) f(t_1, t_2, x_3, \dots, x_m) \, dt_2$$
$$\left. + \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-1} P_1(x_2, t_2) \frac{\partial f}{\partial x_2}(t_1, t_2, x_3, \dots, x_m) \, dt_2 \right]. \quad (14)$$

We plug in (14) into (13). Hence

$$f(x_1, \dots, x_m) = (b_1 - x_1)^{1-\alpha} \left[\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} (b_2 - x_2)^{1-\alpha} \cdot \right.$$
$$\left[\frac{1}{b_2 - a_2} \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-1} f(t_1, t_2, x_3, \dots, x_m) \, dt_2 \right.$$
$$- (\alpha - 1) \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-2} P_1(x_2, t_2) f(t_1, t_2, x_3, \dots, x_m) \, dt_2$$
$$\left. + \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-1} P_1(x_2, t_2) \frac{\partial f}{\partial x_2}(t_1, t_2, x_3, \dots, x_m) \, dt_2 \right] dt_1$$
$$- (\alpha - 1) \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-2} P_1(x_1, t_1) f(t_1, x_2, \dots, x_m) \, dt_1$$
$$\left. + \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} P_1(x_1, t_1) \frac{\partial f}{\partial x_1}(t_1, x_2, \dots, x_m) \, dt_1 \right]. \quad (15)$$

That is we have so far

$$f(x_1, \dots, x_m) = \frac{(b_1 - x_1)^{1-\alpha} (b_2 - x_2)^{1-\alpha}}{(b_1 - a_1)(b_2 - a_2)}$$
$$\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha-1} (b_2 - t_2)^{\alpha-1} f(t_1, t_2, x_3, \dots, x_m) \, dt_1 dt_2$$
$$- \frac{(\alpha - 1)(b_1 - x_1)^{1-\alpha} (b_2 - x_2)^{1-\alpha}}{(b_1 - a_1)}$$
$$\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha-1} (b_2 - t_2)^{\alpha-1} P_1(x_2, t_2) f(t_1, t_2, x_3, \dots, x_m) \, dt_1 dt_2$$
$$+ \frac{(b_1 - x_1)^{1-\alpha} (b_2 - x_2)^{1-\alpha}}{(b_1 - a_1)}$$

$$\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha-1} (b_2 - t_2)^{\alpha-1} P_1 (x_2, t_2) \frac{\partial f}{\partial x_2} (t_1, t_2, x_3, \dots, x_m) \, dt_1 dt_2$$

$$- (\alpha - 1) (b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-2} P_1 (x_1, t_1) f (t_1, x_2, \dots, x_m) \, dt_1$$

$$+ (b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} P_1 (x_1, t_1) \frac{\partial f}{\partial x_1} (t_1, x_2, \dots, x_m) \, dt_1. \quad (16)$$

Call

$$A_1 (x_1, \dots, x_m) := - (\alpha - 1) (b_1 - x_1)^{1-\alpha} \cdot \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-2} P_1 (x_1, t_1) f (t_1, x_2, \dots, x_m) \, dt_1, \quad (17)$$

$$B_1 (x_1, \dots, x_m) := (b_1 - x_1)^{1-\alpha} \cdot \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} P_1 (x_1, t_1) \frac{\partial f}{\partial x_1} (t_1, x_2, \dots, x_m) \, dt_1, \quad (18)$$

$$A_2 (x_1, x_2, \dots, x_m) := - \frac{(\alpha - 1) (b_1 - x_1)^{1-\alpha} (b_2 - x_2)^{1-\alpha}}{(b_1 - a_1)}$$
$$\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha-1} (b_2 - t_2)^{\alpha-1} P_1 (x_2, t_2) f (t_1, t_2, x_3, \dots, x_m) \, dt_1 dt_2, \quad (19)$$

and

$$B_2 (x_1, x_2, \dots, x_m) := \frac{(b_1 - x_1)^{1-\alpha} (b_2 - x_2)^{1-\alpha}}{(b_1 - a_1)}$$
$$\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha-1} (b_2 - t_2)^{\alpha-1} P_1 (x_2, t_2) \frac{\partial f}{\partial x_2} (t_1, t_2, x_3, \dots, x_m) \, dt_1 dt_2. \quad (20)$$

We rewrite (16) as follows

$$f (x_1, \dots, x_m) = \frac{((b_1 - x_1) (b_2 - x_2))^{1-\alpha}}{(b_1 - a_1) (b_2 - a_2)}$$
$$\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} ((b_1 - t_1) (b_2 - t_2))^{\alpha-1} f (t_1, t_2, x_3, \dots, x_m) \, dt_1 dt_2$$
$$+ A_2 (x_1, \dots, x_m) + B_2 (x_1, \dots, x_m) + A_1 (x_1, \dots, x_m) + B_1 (x_1, \dots, x_m) . \quad (21)$$

We continue with

$$f(t_1,t_2,x_3,\dots,x_m) \overset{(6)}{=}$$
$$\overset{(6)}{=} \frac{(b_3-x_3)^{1-\alpha}}{b_3-a_3}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1} f(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_3$$
$$-(\alpha-1)(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-2}P_1(x_3,t_3)\,f(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_3$$
$$+(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}P_1(x_3,t_3)\,\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_3. \tag{22}$$

Next plug (22) into (21). Hence it holds

$$f(x_1,\dots,x_m) = \frac{((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)(b_3-a_3)}$$
$$\cdot \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2)(b_3-t_3))^{\alpha-1}$$
$$\cdot f(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_1dt_2dt_3 - \frac{(\alpha-1)((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}$$
$$\cdot \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2))^{\alpha-1}(b_3-t_3)^{\alpha-2}P_1(x_3,t_3)$$
$$\cdot f(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_1dt_2dt_3 + \frac{((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}$$
$$\cdot \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2)(b_3-t_3))^{\alpha-1}P_1(x_3,t_3)$$
$$\cdot \frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_1dt_2dt_3$$
$$+A_1(x_1,\dots,x_m)+A_2(x_1,\dots,x_m)+B_1(x_1,\dots,x_m)+B_2(x_1,\dots,x_m). \tag{23}$$

Call

$$A_3(x_1,\dots,x_m) := -\frac{(\alpha-1)((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}$$
$$\cdot \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2))^{\alpha-1}(b_3-t_3)^{\alpha-2}P_1(x_3,t_3)$$
$$\cdot f(t_1,t_2,t_3,x_4,\dots,x_m)\,dt_1dt_2dt_3, \tag{24}$$

and

$$B_3(x_1,\dots,x_m) := \frac{((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)} \cdot \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} ((b_1-t_1)(b_2-t_2)(b_3-t_3))^{\alpha-1} P_1(x_3,t_3) \cdot \frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,\dots,x_m)\, dt_1 dt_2 dt_3. \quad (25)$$

Thus we have proved

$$f(x_1,\dots,x_m) = \frac{\left(\prod_{i=1}^{3}(b_i-x_i)\right)^{1-\alpha}}{\prod_{i=1}^{3}(b_i-a_i)} \cdot \int_{\prod_{i=1}^{3}[a_i,b_i]} \left(\prod_{i=1}^{3}(b_i-t_i)\right)^{\alpha-1} f(t_1,t_2,t_3,x_4,\dots,x_m)\, dt_1 dt_2 dt_3 + \sum_{i=1}^{3} A_i(x_1,\dots,x_m) + \sum_{i=1}^{3} B_i(x_1,\dots,x_m). \quad (26)$$

Working similarly we finally obtain the fractional representation formula

$$f(x_1,\dots,x_m) = \frac{\left(\prod_{i=1}^{m}(b_i-x_i)\right)^{1-\alpha}}{\prod_{i=1}^{m}(b_i-a_i)} \int_{\prod_{i=1}^{m}[a_i,b_i]} \left(\prod_{i=1}^{m}(b_i-t_i)\right)^{\alpha-1} f\left(\overrightarrow{t}\right) d\overrightarrow{t} + \sum_{i=1}^{m} A_i(x_1,\dots,x_m) + \sum_{i=1}^{m} B_i(x_1,\dots,x_m). \quad (27)$$

The proof of the theorem is now completed. □

We make

Remark 8. Let $f \in C^1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i,b_i)$, $i = 1,\dots,m$. Denote by

$$\|f\|_{\infty}^{\sup} := \sup_{x\in\prod_{i=1}^{m}[a_i,b_i]} |f(x)|. \quad (28)$$

We observe that

$$|B_i(x_1,\dots,x_m)| \overset{(12)}{\leq} \frac{\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j-a_j)} \int_{\prod_{j=1}^{i}[a_j,b_j]} \left(\prod_{j=1}^{i}(b_j-t_j)\right)^{\alpha-1} \cdot |P_1(x_i,t_i)|\, dt_1\dots dt_i) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup} \quad (29)$$

$$= \frac{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}}{\prod_{j=1}^{i-1}(b_j - a_j)}\left(\prod_{j=1}^{i-1}\int_{a_j}^{b_j}(b_j - t_j)^{\alpha-1}dt_j\right)$$

$$\cdot\left(\int_{a_i}^{b_i}(b_i - t_i)^{\alpha-1}|P_1(x_i, t_i)|\,dt_i\right) \quad (30)$$

$$= \frac{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}}{(b_i - a_i)\prod_{j=1}^{i-1}(b_j - a_j)}\frac{\left(\prod_{j=1}^{i-1}(b_j - a_j)\right)^{\alpha}}{\alpha^{i-1}}$$

$$\cdot\left[\int_{a_i}^{x_i}(b_i - t_i)^{\alpha-1}(t_i - a_i)\,dt_i + \int_{x_i}^{b_i}(b_i - t_i)^{\alpha-1}(b_i - t_i)\,dt_i\right]$$

$$= \frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j - a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i - a_i)} \quad (31)$$

$$\cdot\left[\int_{a_i}^{x_i}(b_i - t_i)^{\alpha-1}\left[(b_i - a_i) - (b_i - t_i)\right]dt_i + \frac{(b_i - x_i)^{\alpha+1}}{\alpha+1}\right]$$

$$= \frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j - a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i - a_i)} \quad (32)$$

$$\cdot\left[(b_i - a_i)\left[\frac{(b_i - a_i)^{\alpha}}{\alpha} - \frac{(b_i - x_i)^{\alpha}}{\alpha}\right] - \frac{(b_i - a_i)^{\alpha+1}}{\alpha+1} + \frac{2(b_i - x_i)^{\alpha+1}}{\alpha+1}\right]$$

$$= \frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j - a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i - a_i)}$$

$$\cdot\left[\frac{(b_i - a_i)^{\alpha+1}}{\alpha(\alpha+1)} + \frac{2(b_i - x_i)^{\alpha+1}}{\alpha+1} - (b_i - a_i)\frac{(b_i - x_i)^{\alpha}}{\alpha}\right]. \quad (33)$$

We have proved for $i = 1, \ldots, m$, that

$$|B_i(x_1, \ldots, x_m)| \leq \frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j - a_j)\right)^{\alpha-1}}{\alpha^{i-1}}$$

$$\cdot\left[\frac{(b_i - a_i)^{\alpha}}{\alpha(\alpha+1)} + \frac{2(b_i - x_i)^{\alpha+1}}{(\alpha+1)(b_i - a_i)} - \frac{(b_i - x_i)^{\alpha}}{\alpha}\right]. \quad (34)$$

We have established the following multivariate fractional Ostrowski type inequality.

Theorem 9. *Let* $f \in C^1\left(\prod_{i=1}^m [a_i, b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, \ldots, m$. *Then*

$$\left| f(x_1, \ldots, x_m) - \frac{\left(\prod_{i=1}^m (b_i - x_i)\right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^\alpha f\right)(b)}{\prod_{i=1}^m (b_i - a_i)} - \sum_{i=1}^m A_i(x_1, \ldots, x_m) \right|$$
$$\leq \sum_{i=1}^m \left\{ \frac{\left\| \frac{\partial f}{\partial x_i} \right\|_\infty^{\sup} \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\prod_{j=1}^{i-1} (b_j - a_j)\right)^{\alpha-1}}{\alpha^{i-1}} \cdot \left[\frac{(b_i - a_i)^\alpha}{\alpha(\alpha+1)} + \frac{2(b_i - x_i)^{\alpha+1}}{(\alpha+1)(b_i - a_i)} - \frac{(b_i - x_i)^\alpha}{\alpha} \right] \right\}. \quad (35)$$

References

[1] G. Anastassiou, M. Hooshmandasl, A. Ghasemi and F. Moftakharzadeh, *Montgomery identities for fractional integrals and related fractional inequalities*, J. Inequal. Pure Appl. Math., 10 (4) (2009), Article 97, 6 pages.

[2] T. Mamatov and S. Samko, *Mixed fractional integration operators in mixed weighted Hölder spaces*, Fract. Calc. Appl. Anal., 13 (3) (2010), 245–259.

[3] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, Netherlands, (1994).

[4] S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA, (1993).

(Received: October 24, 2013)

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
U.S.A.
ganastss@memphis.edu