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THE IDEAL-BASED ZERO-DIVISOR GRAPH OF COMMUTATIVE CHAINED RINGS

DAVID F. ANDERSON, S. EBRAHIMI ATANI, M. SHAJARI KOHAN, AND Z. EBRAHIMI SARVANDI

Abstract. Let I be a proper ideal of a commutative ring R with $1 \neq 0$. The ideal-based zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the (simple) graph with vertices $\{ x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I \}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, we study $\Gamma_I(R)$ for commutative rings R such that R/I is a chained ring.

1. Introduction

In the literature, there are many papers on assigning a graph to a ring (see, for example, [1] - [7], [9], and [11]). Among the most interesting graphs are zero-divisor graphs, because they involve both ring theory and graph theory. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. The concept of zero-divisor graph for a commutative ring R was introduced by I. Beck [7], where he was mainly interested in colorings. In his work, all elements of R were vertices of the graph, and distinct vertices x and y were adjacent if and only if $xy = 0$. This investigation of colorings of a commutative ring was then continued by D. D. Anderson and M. Naseer in [1]. Let $Z(R)$ be the set of zero-divisors of R. In [5], D. F. Anderson and P. S. Livingston associated a (simple) graph $\Gamma(R)$ to R, with vertices $Z(R)^* = Z(R)\setminus\{0\}$, the set of nonzero zero-divisors of R, and distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph $\Gamma(R)$ of R has been studied extensively; see the the survey articles [2] and [9].

Let R be a commutative ring with $1 \neq 0$, I a proper ideal of R, and $Z_I(R) = \{ x \in R \mid xy \in I$ for some $y \in R \setminus I \}$. In [11], S. P. Redmond introduced the *ideal-based zero-divisor graph* of R with respect to I, denoted by $\Gamma_I(R)$, with vertices $Z_I(R)^* = Z_I(R) \setminus I = \{ x \in R \setminus I \mid xy \in I$ for some

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$y \in R \setminus I\}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. Thus $\Gamma_{\{0\}}(R) = \Gamma(R)$ and $\Gamma_I(R)$ is the empty graph if and only if I is a prime ideal of R. In [11], he explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$ and showed, among other things, that $\Gamma_I(R)$ is connected with $\mathrm{diam}(\Gamma_I(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma_I(R)) \in \{3, 4, \infty\}$.

In [3], D. F. Anderson and A. Badawi studied $\Gamma(R)$ for several classes of rings which generalize valuation domains to the context of rings with zero-divisors. These rings include chained rings and rings R whose prime ideals contained in $Z(R)$ are linearly ordered. Recall that a ring R is a *chained ring* if the (principal) ideals of R are linearly ordered (by inclusion), equivalently, if either $x|y$ or $y|x$ for all $x, y \in R$. Examples of chained rings include valuation domains and factor rings of chained rings.

In this paper, we study $\Gamma_I(R)$ for commutative rings R such that R/I is a chained ring. Clearly, R/I is a chained ring when R is a chained ring; however, R/I may be a chained ring when R is not a chained ring. For example, let J be a proper ideal of a chained ring S (e.g., a valuation domain), $R = S[X]$ or $S[[X]]$, and $I = (J, X)$. Then R is not a chained ring, but $R/I \cong S/J$ is a chained ring. As another example, let R_1 and R_2 be chained rings and $R = R_1 \times R_2$ with ideals $I_1 = R_1 \times \{0\}$ and $I_2 = \{0\} \times R_2$. Then R is not a chained ring, but $R/I_1 \cong R_2$ and $R/I_2 \cong R_1$ are both chained rings.

In Section 2, we study the relationship between several natural subgraphs of $\Gamma_I(R)$. Then, in Section 3, we specialize to the case when R/I is a chained ring. We completely characterize the diameter and girth of the graph $\Gamma_I(R)$ for such rings in Theorem 3.8 and Theorem 3.9, respectively. Moreover, we extend several results in [3] to the more general ideal-based zero-divisor graph case. In fact, results in [3] for $\Gamma(R)$ when R is a chained ring are actually special cases of the results in this paper for $\Gamma_I(R)$ when R/I is a chained ring since if $I = \{0\}$, then R is a chained ring and $\Gamma(R) = \Gamma_I(R)$. We invite the interested reader to compare the results in [3] for $\Gamma(R)$ to the results in this paper for $\Gamma_I(R)$.

In order to make this paper easier to follow, we next recall various notions which will be used in the sequel. For a graph Γ, let $E(\Gamma)$ and $V(\Gamma)$ denote the sets of edges and vertices of Γ, respectively. By abuse of notation, we will often refer to a subgraph of $\Gamma_I(R)$ by its set of vertices; all such subgraphs will be induced subgraphs. We recall that a graph is *connected* if there exists a path connecting any two distinct vertices. At the other extreme, we say that a graph Γ is *totally disconnected* if no two vertices of Γ are adjacent. The distance between two distinct vertices a and b in Γ, denoted by $d(a, b)$, is the length of a shortest path connecting them ($d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path). The *diameter* of a graph Γ, denoted by diam(Γ), is $\sup\{\, d(a, b) \mid a, b \in V(\Gamma) \,\}$. A graph is *complete* if it is connected with diameter less than or equal to one. The *girth* of a graph Γ, denoted by gr(Γ), is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise, $\mathrm{gr}(\Gamma) = \infty$. Recall that a graph Γ is a *star graph* if it has a vertex that is adjacent to every other vertex and this is the only adjacency relation. Throughout this paper, all rings are assumed to be commutative with $1 \neq 0$. As usual, $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{Q}$ denote the rings of integers, integers modulo n, and rational numbers, respectively; for an ideal I of R, $\sqrt{I} = \{\, x \in R \mid x^n \in I$ for some integer $n \geq 1 \,\}$; and $\mathrm{nil}(R) = \sqrt{\{0\}}$. To avoid any trivialities when $\Gamma_I(R)$ is the empty graph, we will implicitly assume when necessary that I is not a prime ideal of R. For a ring theory reference, see [10]; for a graph theory reference, see [8].

2. Subgraphs of $\Gamma_I(R)$

Let I be a proper ideal of a commutative ring R. In this section, we investigate the relationship between several subgraphs of $\Gamma_I(R)$. It will be convenient to let $Z_I(R)^* = Z_I(R) \setminus I = \{\, x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I \,\}$. Note that $Z_{\{0\}}(R) = Z(R)$, $Z(R/I) = Z_I(R)/I$, and $V(\Gamma_I(R)) = Z_I(R)^*$. Moreover, $Z_I(R)^* = \emptyset$ (i.e., $Z_I(R) = I$) if and only if I is a prime ideal of R. Also, let $N_I(R) = \{\, x \in R \mid x^2 \in I \,\}$ and $N_I(R)^* = N_I(R) \setminus I$. Clearly, $I \subseteq N_I(R) \subseteq \sqrt{I}$, and $N_I(R)^* = \emptyset$ (i.e., $N_I(R) = I$) if and only if I is a radical ideal of R (i.e., $\sqrt{I} = I$).

Proposition 2.1. *Let I be a proper ideal of a commutative ring R, $\sqrt{I}^* = \sqrt{I} \setminus I$, $Z_I(R) = \{\, x \in R \mid xy \in I$ for some $y \in R \setminus I \,\}$, $Z_I(R)^* = Z_I(R) \setminus I$, and $N_I(R) = \{\, x \in R \mid x^2 \in I \,\}$. Then the following hold.*

(1) $\sqrt{I}^* \subseteq Z_I(R)^*$.
(2) $I \subseteq N_I(R) \subseteq \sqrt{I} \subseteq Z_I(R)$.
(3) *If $Z_I(R)$ is an ideal of R, then it is a prime ideal of R.*
(4) $N_I(R) = I$ *if and only if* $\sqrt{I} = I$.

Proof. (1) Let $x \in \sqrt{I}^* = \sqrt{I} \setminus I$. Let n ($n \geq 2$) be the least positive integer such that $x^n \in I$. As $x \notin I$, $x^{n-1} \notin I$, and $xx^{n-1} = x^n \in I$, we have $x \in Z_I(R) \setminus I = Z_I(R)^*$.

(2) This follows from part (1) and the above comments.

(3) Suppose that $Z_I(R)$ is an ideal of R, and let $x, y \in R$ such that $xy \in Z_I(R)$. Then there is a $z \in R \setminus I$ such that $(xy)z \in I$. If $yz \in I$, then $y \in Z_I(R)$. If $yz \notin I$, then $x \in Z_I(R)$. Thus $Z_I(R)$ is a prime ideal of R. (This also follows since $Z_I(R)$ is a union of prime ideals of R).

(4) This is clear. □

Theorem 2.2. *Let I be a proper ideal of a commutative ring R, $\sqrt{I}^* = \sqrt{I} \setminus I$, $Z_I(R) = \{ x \in R \mid xy \in I$ for some $y \in R \setminus I \}$, and $Z_I(R)^* = Z_I(R) \setminus I$. Then the following hold.*

(1) *If $x \in \sqrt{I}^*$ and $y \in Z_I(R)^*$, then $d(x, y) \leq 2$ in $\Gamma_I(R)$.*
(2) *The subgraph $Z_I(R) \setminus \sqrt{I}$ of $\Gamma_I(R)$ is totally disconnected if and only if $\sqrt{I}$ is a prime ideal of R.*

Proof. (1) We may assume that $x \neq y$ and $xy \notin I$. Since $y \in Z_I(R) \setminus I$ and $xy \notin I$, there is a $z \in Z_I(R) \setminus (I \cup \{x\})$ such that $zy \in I$. There is a least positive integer n such that $x^n z \in I$ since $x \in \sqrt{I}^*$. If $n = 1$, then $x - z - y$ is a path of length 2 from x to y. If $n \geq 2$, then $x - x^{n-1}z - y$ is a path of length 2 from x to y. Thus $d(x, y) \leq 2$ in $\Gamma_I(R)$.

(2) Assume that $\sqrt{I}$ is a prime ideal of R, and let x and y be distinct elements of $Z_I(R) \setminus \sqrt{I}$. If x and y are adjacent, then $xy \in I \subseteq \sqrt{I}$. Thus either x or y belongs to $\sqrt{I}$, a contradiction. Hence the subgraph $Z_I(R) \setminus \sqrt{I}$ is totally disconnected.

Conversely, assume that $\sqrt{I}$ is not a prime ideal of R. Then there are $x, y \in R \setminus \sqrt{I}$ with $xy \in \sqrt{I}$. Thus $x^n y^n = (xy)^n \in I$ for some positive integer n. If $x^n = y^n$, then $x^{2n} = x^n y^n \in I$; so $x \in \sqrt{I}$, a contradiction. Hence $x^n, y^n \in Z_I(R) \setminus \sqrt{I}$, $x^n \neq y^n$, and x^n and y^n are adjacent. Thus the subgraph $Z_I(R) \setminus \sqrt{I}$ is not totally disconnected. □

Proposition 2.3. *Let I be a proper ideal of a commutative ring R and $N_I(R) = \{ x \in R \mid x^2 \in I \}$. Then every vertex of the subgraph $\sqrt{I} \setminus N_I(R)$ of $\Gamma_I(R)$ is adjacent to a vertex of the subgraph $N_I(R)^* = N_I(R) \setminus I$ of $\Gamma_I(R)$.*

Proof. Let $x \in \sqrt{I} \setminus N_I(R)$, n $(n \geq 3)$ be the least positive integer such that $x^n \in I$, and $y = x^{n-1}$. Then $y = x^{n-1} \notin I$, $xy = xx^{n-1} = x^n \in I$, and $y^2 = (x^{n-1})^2 = x^{2n-2} \in I$ since $2n - 2 \geq n$ as $n \geq 3$. Thus $y \in N_I(R) \setminus I = N_I(R)^*$, $x \neq y$, and x is adjacent to y in $\Gamma_I(R)$ since $xy \in I$. □

Thus $\Gamma_I(R)$ is the union of three, possibly empty, disjoint subgraphs, $N_I(R)^* = N_I(R) \setminus I$, $\sqrt{I} \setminus N_I(R)$, and $Z_I(R) \setminus \sqrt{I}$. Suppose that the ideal I is not a prime ideal of R, but $\sqrt{I}$ is a prime ideal of R. Then $N_I(R)^*$ is nonempty by Proposition 2.1 (4) and $Z_I(R) \setminus \sqrt{I}$ is totally disconnected by Theorem 2.2 (2).

3. Chained rings

In this section, we investigate the ideal-based zero-divisor graph $\Gamma_I(R)$ with respect to a proper ideal I of a commutative ring R such that R/I is a chained ring. In particular, these results all hold when R is a chained ring.

Note that $\sqrt{I}$ is a prime ideal of R when R/I is a chained ring since radical ideals in chained rings are prime ideals.

We first show, among other things, that every vertex of the subgraph $Z_I(R) \setminus N_I(R)$ is adjacent to a vertex of the subgraph $N_I(R)^* = N_I(R) \setminus I$ and every two distinct vertices of $N_I(R)^*$ are adjacent (i.e., $N_I(R)^*$ is a complete subgraph of $\Gamma_I(R)$).

Proposition 3.1. *Let I be a proper ideal of a commutative ring R such that R/I is a chained ring, $Z_I(R) = \{ x \in R \mid xy \in I$ for some $y \in R \setminus I \}$, $Z_I(R)^* = Z_I(R) \setminus I$, $N_I(R) = \{ x \in R \mid x^2 \in I \}$, $N_I(R)^* = N_I(R) \setminus I$, and $x, y \in R$.*

(1) *If $xy \in I$, then either $x \in N_I(R)$ or $y \in N_I(R)$.*
(2) *If $x, y \in N_I(R)$, then $xy \in I$.*
(3) *If $x, y \in Z_I(R) \setminus N_I(R)$, then $xy \notin I$.*
(4) *If $x \in Z_I(R)^*$, then $xy \in I$ for some $y \in N_I(R)^*$.*
(5) *If $x_1, \ldots, x_n \in Z_I(R)^*$, then there is a $y \in N_I(R)^*$ such that $x_i y \in I$ for every integer i, $1 \leq i \leq n$.*
(6) *$N_I(R)$ is an ideal of R. Moreover, $N_I(R) = I$ if and only if I is a prime ideal of R.*
(7) *$N_I(R)$ is a prime ideal of R if and only if $N_I(R) = \sqrt{I}$.*
(8) *$Z_I(R)$ is a prime ideal of R.*

Proof. (1) Since R/I is a chained ring, we may assume that $(x + I)|(y + I)$ in R/I. Thus $y = ax + i$ for some $a \in R$ and $i \in I$. Hence $y^2 = (ax + i)y = axy + iy \in I$ since $xy \in I$; so $y \in N_I(R)$.

(2) Since R/I is a chained ring, we may assume that $(x + I)|(y + I)$ in R/I. Thus $y = ax + i$ for some $a \in R$ and $i \in I$. Hence $xy = x(ax + i) = ax^2 + xi \in I$ since $x^2 \in I$.

(3) This follows from part (1) above.

(4) If $x \in N_I(R)^*$, then let $y = x$. If $x \in Z_I(R) \setminus N_I(R)$, then there is a $y \in R \setminus I$ such that $xy \in I$. By part (3) above, we have $y \in N_I(R)^*$.

(5) Since R/I is a chained ring, there is an integer j, $1 \leq j \leq n$, such that $(x_j + I)|(x_i + I)$ for every integer i, $1 \leq i \leq n$. Thus $x_i = a_i x_j + b_i$ for some $a_i \in R$ and $b_i \in I$ for every integer i, $1 \leq i \leq n$. By part (4) above, there is a $y \in N_I(R)^*$ such that $x_j y \in I$. Hence $x_i y = (a_i x_j + b_i)y = a_i x_j y + b_i y \in I$ for every integer i, $1 \leq i \leq n$.

(6) Let $x, y \in N_I(R)$ and $r \in R$. Then $(rx)^2 = r^2 x^2 \in I$ since $x^2 \in I$; so $rx \in N_I(R)$. Thus we need only show that $x + y \in N_I(R)$. By assumption, $x^2, y^2 \in I$, and $xy \in I$ by part (2) above; so $(x + y)^2 = x^2 + 2xy + y^2 \in I$. Hence $N_I(R)$ is an ideal of R. The "moreover" statement follows from Proposition 2.1 (4) since I is a prime ideal of R if and only if I is a radical ideal of R as R/I is a chained ring.

(7) Suppose that $N_I(R)$ is a prime ideal of R. Then $N_I(R) = \sqrt{I}$ since $I \subseteq N_I(R) \subseteq \sqrt{I}$. Conversely, assume that $N_I(R) = \sqrt{I}$. Then $\sqrt{I}$ is a prime ideal of R since R/I is a chained ring.

(8) Since $Z_I(R)$ is closed under multiplication and by Proposition 2.1 (3), we need only show that $Z_I(R)$ is closed under addition. Let $x, y \in Z_I(R)$. Since R/I is a chained ring, we may assume that $(x+I)|(y+I)$ in R/I, and thus $y = ax + i$ for some $a \in R$ and $i \in I$. Let $z \in R \setminus I$ such that $xz \in I$. Then $(x + y)z = (x + ax + i)z = (1 + a)xz + iz \in I$; so $x + y \in Z_I(R)$. □

Remark 3.2. If R/I is not a chained ring, then $N_I(R)$ need not be an ideal of R. For example, let $R = \mathbb{Z}[X, Y]$ and $I = (X^2, Y^2)$. Then $X, Y \in N_I(R)$, but $X + Y \notin N_I(R)$. However, $N_I(R)$ is an ideal of R when $\text{char}(R) = 2$.

Let $R = \mathbb{Z}_2[X, Y]$ and $I = (X^4, Y^4)$. Then R/I is not a chained ring, $N_I(R)$ is an ideal of R since $\text{char}(R) = 2$, and $I \subsetneq (X^2, Y^2) = N_I(R) \subsetneq (X, Y) = \sqrt{I} = Z_I(R)$.

The next result improves Theorem 2.2 (2) when R/I is a chained ring.

Theorem 3.3. *Let I be a proper ideal ideal of a commutative ring R such that R/I is a chained ring, $Z_I(R) = \{ x \in R \mid xy \in I \text{ for some } y \in R \setminus I \}$, $N_I(R) = \{ x \in R \mid x^2 \in I \}$, and $N_I(R)^* = N_I(R) \setminus I$. Then $N_I(R)^*$ is a complete subgraph of $\Gamma_I(R)$ and the subgraph $Z_I(R) \setminus N_I(R)$ of $\Gamma_I(R)$ is totally disconnected. Moreover, $N_I(R)^*$ is nonempty if and only if $\Gamma_I(R)$ is nonempty.*

Proof. The first statement follows from parts (2) and (3) of Proposition 3.1, respectively. The "moreover" statement follows since $N_I(R)^* = \emptyset$ (i.e., $N_I(R) = I$) if and only if $\sqrt{I} = I$ by Proposition 2.1 (4), if and only if I is a prime ideal of R (since R/I is a chained ring), if and only if $Z_I(R)^* = \emptyset$ (i.e., $\Gamma_I(R)$ is the empty graph). □

Corollary 3.4. *Let I be a proper ideal ideal of a commutative ring R such that R/I is a chained ring. Then $\Gamma_I(R)$ is a complete graph if and only if $Z_I(R) = N_I(R)$. Moreover, if $\Gamma_I(R)$ is a complete graph, then $Z(R/I) = nil(R/I)$.*

Proof. We first show that $\Gamma_I(R)$ is complete if and only if $Z_I(R) = N_I(R)$. If $Z_I(R) = N_I(R)$, then $\Gamma_I(R)$ is complete by Theorem 3.3. Conversely, suppose that $N_I(R) \subsetneq Z_I(R)$. Let $x \in Z_I(R) \setminus N_I(R)$. Then $xy \in I$ for some $y \in N_I(R)^* = N_I(R) \setminus I$ by Proposition 3.1 (4), and thus $x + y \in Z_I(R)$ by Proposition 3.1 (8). Moreover, $x + y \notin N_I(R)$ since $y \in N_I(R)$, $x \notin N_I(R)$, and $N_I(R)$ is an ideal of R by Proposition 3.1 (6). Hence x and $x + y$ are distinct, nonadjacent vertices since $Z_I(R) \setminus N_I(R)$ is totally disconnected by Theorem 3.3. Hence $\Gamma_I(R)$ is not complete.

For the "moreover" statement, suppose that $\Gamma_I(R)$ is a complete graph. Then $Z_I(R) = N_I(R)$ by above, and thus $Z_I(R) = \sqrt{I}$ by Proposition 2.1 (2). Hence nil $(R/I) = \sqrt{I}/I = Z_I(R)/I = Z(R/I)$. □

Remark 3.5. (1) Corollary 3.4 also follows from [4, Theorem 4.6] and [6, Theorem 4.7]. Note that the converse of the "moreover" statement in Corollary 3.4 need not hold. For example, let $R = \mathbb{Z}_2 \times \mathbb{Z}_8$ and $I = \mathbb{Z}_2 \times \{0\}$. Then $R/I \cong \mathbb{Z}_8$ is a finite local ring; so $\mathrm{nil}(R/I) = Z(R/I)$. However, $\Gamma_I(R)$ is not complete (cf. Example 3.10). Also, Corollary 3.4 may fail if R/I is not a chained ring. For example, let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = \{(0,0)\}$. Then $\Gamma_I(R) = \Gamma(R)$ is the complete graph on two vertices, but $N_I(R) = I \subsetneq R \setminus \{(1,1)\} = Z_I(R)$.

(2) Let I be a proper ideal of a commutative ring R such that R/I is a chained ring. Note that if $\sqrt{I} \subsetneq Z_I(R)$, then R/I is infinite. This follows since if R/I is finite, then $\sqrt{I}/I$ is a prime, hence maximal, ideal of R/I contained in the prime ideal $Z_I(R)/I$; so $\sqrt{I} = Z_I(R)$. Moreover, if $\Gamma_I(R)$ is an infinite graph (i.e., I is not a prime ideal of R and either I is infinite or R/I is infinite), then the subgraph $Z_I(R) \setminus N_I(R)$ is infinite if it is nonempty. This is clear if $N_I(R)$ is finite. If $N_I(R)$ is infinite, it follows since $x + N_I(R) \subseteq Z_I(R) \setminus N_I(R)$ for $x \in Z_I(R) \setminus N_I(R)$.

(3) Let I be a proper ideal of a commutative ring R such that R/I is a chained ring. Then there are eight possibilities for equals or strict inclusion in the chain of ideals $I \subseteq N_I(R) \subseteq \sqrt{I} \subseteq Z_I(R)$ (i.e., for the subgraphs $N_I(R) \setminus I$, $\sqrt{I} \setminus N_I(R)$, and $Z_I(R) \setminus \sqrt{I}$ of $\Gamma_I(R)$ being empty or nonempty). If $N_I(R) = I$, then I is a prime ideal of R by Proposition 3.1 (6); so in this case, all four ideals are equal and $\Gamma_I(R)$ is the empty graph. Easy examples show that the other four cases are all possible. For example, let $R = \mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$ and $I = (X^2)$. Then R is a valuation domain; so R/I is a chained ring. Note that $I \subsetneq (X) = N_I(R) \subsetneq X\mathbb{Q}[[X]] = \sqrt{I} \subsetneq (2) = Z_I(R)$.

When R/I is a chained ring, the graph $\Gamma_I(R)$ is easy to describe. It is the union of two disjoint subgraphs, $N_I(R)^* = N_I(R) \setminus I$ (nonempty when $\Gamma_I(R)$ is nonempty) and $Z_I(R) \setminus N_I(R)$ (possibly empty), where $N_I(R)^*$ is complete and $Z_I(R) \setminus N_I(R)$ is totally disconnected by Theorem 3.3, and every vertex of $Z_I(R) \setminus N_I(R)$ is adjacent to some vertex of $N_I(R)$ by Proposition 3.1(4).

Recall that $\mathrm{diam}(\Gamma_I(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma_I(R)) \in \{3, 4, \infty\}$ for every proper ideal I of a commutative ring R. Stronger results hold for the diameter and girth of $\Gamma_I(R)$ when R/I is a chained ring.

Theorem 3.6. *Let I be a proper ideal of a commutative ring R such that R/I is a chained ring. Then diam$(\Gamma_I(R)) \in \{0, 1, 2\}$.*

Proof. Let $Z_I(R)^* = Z_I(R) \setminus I = V(\Gamma_I(R))$ and $N_I(R) = \{\, x \in R \mid x^2 \in I \,\}$. If $|Z_I(R)^*| \leq 1$, then $\mathrm{diam}(\Gamma_I(R)) = 0$. So we may assume that $|Z_I(R)^*| \geq 2$.

Let $x, y \in Z_I(R)^*$ with $x \neq y$. If $x, y \in N_I(R)$, then $xy \in I$ by Proposition 3.1 (2), and thus $d(x, y) = 1$. If $x \in N_I(R)$ and $y \notin N_I(R)$, then $yz \in I$ for some $z \in N_I(R)^* \subseteq Z_I(R)^*$ by Proposition 3.1 (4) and $xz \in I$ by Proposition 3.1 (2). If $x = z$, then $d(x, y) = 1$. Otherwise, $x - z - y$ is a path of length 2 from x to y, and hence $d(x, y) \leq 2$. Finally, let $x, y \notin N_I(R)$. Then $xz, yz \in I$ for some $z \in N_I(R)^* \subseteq Z_I(R)^*$ by Proposition 3.1 (5). Thus $x - z - y$ is a path of length 2 from x to y, and hence $d(x, y) \leq 2$ (actually, $d(x, y) = 2$ since $xy \notin I$ by Proposition 3.1(3)). Thus $\mathrm{diam}(\Gamma_I(R)) \in \{0, 1, 2\}$. □

Remark 3.7. $\mathrm{diam}(\Gamma_I(R)) = 0$ (i.e., $|Z_I(R)^*| \leq 1$) if and only if either $\Gamma_I(R)$ is the empty graph (i.e., I is a prime ideal of R) or $I = \{0\}$ (i.e., $\Gamma_I(R) = \Gamma(R)$) and $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, both of which are chained rings.

Next, we explicitly determine when the diameter of $\Gamma_I(R)$ is either 0, 1, or 2.

Theorem 3.8. *Let I be a proper ideal of a commutative ring R such that R/I is a chained ring, $Z_I(R) = \{\, x \in R \mid xy \in I$ for some $y \in R \setminus I \,\}$, $Z_I(R)^* = Z_I(R) \setminus I$, and $N_I(R) = \{\, x \in R \mid x^2 \in I \,\}$. Then exactly one of the following three cases must occur.*

(1) $|Z_I(R)^*| \leq 1$. *In this case, diam*$(\Gamma_I(R)) = 0$.
(2) $|Z_I(R)^*| \geq 2$ *and* $N_I(R) = Z_I(R)$. *In this case, diam*$(\Gamma_I(R)) = 1$.
(3) $|Z_I(R)^*| \geq 2$ *and* $N_I(R) \subsetneq Z_I(R)$. *In this case, diam*$(\Gamma_I(R)) = 2$.

Proof. This follows directly from Proposition 3.1 and the proof of Theorem 3.6. □

We next show that $\mathrm{gr}(\Gamma_I(R)) \in \{3, \infty\}$ when R/I is a chained ring.

Theorem 3.9. *Let I be a proper ideal of a commutative ring R such that R/I is a chained ring, $Z_I(R) = \{\, x \in R \mid xy \in I$ for some $y \in R \setminus I \,\}$, $N_I(R) = \{\, x \in R \mid x^2 \in I \,\}$, and $N_I(R)^* = N_I(R) \setminus I$. Then exactly one of the following four cases must occur.*

(1) $|N_I(R)^*| \leq 1$. *In this case, gr*$(\Gamma_I(R)) = \infty$.
(2) $|N_I(R)^*| = 2$ *and* $N_I(R) = Z_I(R)$. *In this case, gr*$(\Gamma_I(R)) = \infty$.
(3) $|N_I(R)^*| = 2$ *and* $N_I(R) \subsetneq Z_I(R)$. *In this case, gr*$(\Gamma_I(R)) = 3$.
(4) $|N_I(R)^*| \geq 3$. *In this case, gr*$(\Gamma_I(R)) = 3$.

Proof. (1) We may assume that $N_I(R)^* \neq \emptyset$ by the "moreover" statement in Theorem 3.3. Let $N_I(R)^* = \{x\}$. If $N_I(R)^* = Z_I(R)^*$, then $\mathrm{gr}(\Gamma_I(R)) = \infty$. If $N_I(R)^* \subsetneq Z_I(R)^*$, then $\Gamma_I(R)$ is a star graph with center x by parts (3) and (4) of Proposition 3.1. Thus $\mathrm{gr}(\Gamma_I(R)) = \infty$.

(2) By hypothesis, $|Z_I(R)^*| = 2$; hence $\mathrm{gr}(\Gamma_I(R)) = \infty$.

(3) Let $N_I(R)^* = \{x, y\}$. Then $xy \in I$ by Proposition 3.1 (2) and $x+y \in N_I(R)$ by Proposition 3.1 (6). If $x + y \in N_I(R) \setminus I = N_I(R)^*$, then either $x + y = x$ or $x + y = y$. Thus either $y = 0$ or $x = 0$, a contradiction. Hence $x + y \in I$. Let $z \in Z_I(R) \setminus N_I(R)^*$. Then either $xz \in I$ or $yz \in I$ by Proposition 3.1 (4). However, in either case, $xz, yz \in I$ since $x + y \in I$. Thus $x - y - z - x$ is a triangle in $\Gamma_I(R)$; so $\mathrm{gr}(\Gamma_I(R)) = 3$.

(4) If $|N_I(R)^*| \geq 3$, then $\mathrm{gr}(\Gamma_I(R)) = 3$ by Proposition 3.1 (2). □

The final example illustrates the above results. In particular, it shows that all possible values may be realized for $\mathrm{diam}(\Gamma_I(R))$ and $\mathrm{gr}(\Gamma_I(R))$ when R/I is a chained ring and I is a nonzero ideal of R. For the $\mathrm{diam}(\Gamma_I(R)) = 0$ case, see Remark 3.7.

Example 3.10. Note that $\mathbb{Z}_n$ is a chained ring if and only if n is a prime power. Let p be a prime number, and for every positive integer n, let $R_n = \mathbb{Z}_2 \times \mathbb{Z}_{p^n}$ and $I_n = \mathbb{Z}_2 \times \{0\}$. Then $R_n/I_n \cong \mathbb{Z}_{p^n}$ is a chained ring. It is easily verified (cf. Theorem 3.9) that $\Gamma_{I_1}(R_1)$ is the empty graph, $\mathrm{gr}(\Gamma_{I_2}(R_2)) = \infty$ if $p = 2$, $\mathrm{gr}(\Gamma_{I_2}(R_2)) = 3$ if $p \neq 2$, and $\mathrm{gr}(\Gamma_{I_n}(R_n)) = 3$ for $n \geq 3$ since $(0, p) - (1, p^{n-1}) - (0, p^{n-1}) - (0, p)$ is a triangle. It is also easily verified that $\mathrm{diam}(\Gamma_{I_2}(R_2)) = 1$ and $\mathrm{diam}(\Gamma_{I_n}(R_n)) = 2$ for $n \geq 3$ (cf. Theorem 3.8).

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David F. Anderson
Department of Mathematics
The University of Tennessee
Knoxville, TN 37996
U.S.A.
anderson@math.utk.edu

S. Ebrahimi Atani, M. Shajari Kohan,
Z. Ebrahimi Sarvandi
Department of Mathematics
University of Guilan
P.O. Box 1914, Rasht
Iran
ebrahimiatani@gmail.com