SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.10.1.03
Vol.10 (22) (2014), 21–26

FINITELY BI-QUASIREGULAR RELATIONS

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Abstract. In this paper, analogously to the study of finitely regular (conjugative, dually normal and quasi-conjugative) relations, finitely bi-quasiregular relations are introduced and investigated. More concretely, after some preparations, an intrinsic characterization of such relations is established.

1. Introduction and preliminaries

In this article, since regular and finitely regular relations had important applications in lattice theory, following the concepts of finitely conjugative relations ([1], Jiang Guanghao and Xu Luoshan), finitely dual normal relations ([2], Jiang Guanghao and Xu Luoshan) and finitely quasi-conjugative relations ([5], D. A. Romano and M. Vinčić), we introduce and analyze the notions of finitely bi-quasiregular relations after citing some previous results of the second author on bi-quasiregular relations ([6]).

For a set X, we call α a binary relation on X if $\alpha \subseteq X^2$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X^2 : (\exists y \in X)((x, y) \in \alpha, (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $\mathcal{B}(X)$, with composition, is a monoid (semigroup with identity). Namely, $\triangle_X = \{(x, x) : x \in X\}$ is its identity element. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X^2 : (y, x) \in \alpha\}$ and $\alpha^c = X^2 \setminus \alpha$. Thus $(\alpha^c)^{-1} = (\alpha^{-1})^c$ holds.

Let A be a subset of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \ \alpha A = \{x \in X : (\exists b \in A)((x, b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

2010 *Mathematics Subject Classification.* 20M20, 03E02, 06A11.
Key words and phrases. Bi-quasiregular relations, finitely bi-quasiregular relations.

The fundamental works of K. A. Zareckii, B. M. Schein and others on regular relations motivated several mathematicians to investigate similar classes of relations, obtained by putting α^{-1}, α^c or $(\alpha^c)^{-1}$ in place of one or both α's on the right side of the regularity equation

$$\alpha = \alpha \circ \beta \circ \alpha$$

(where β is some relation). The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:

The relation $\alpha \in \mathcal{B}(X)$ is called:

– *dually normal* ([2]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^c)^{-1} \circ \beta \circ \alpha,$$

– *conjugative* ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha,$$

– *dually conjugative* ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1},$$

– *quasi-regular* ([4]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ \alpha.$$

Put $\alpha^1 = \alpha$. The previous definitions give equality

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, c\}$. We should investigate all other possibilities since some of the possibilities given in the previous equation have been investigated. (See, for example, articles [4], [5] and [6].)

Notions and notations which are not explicitly exposed but are used in this article, can be found in book [3] and articles [1], [2], [4], [5], [6] and [7].

The class of bi-quasiregular relations is described by the following definition:

Definition 1.1 ([6], Definition 4.1)**.** A relation $\alpha \in \mathcal{B}(X)$ is called a *bi-quasiregular* relation if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ \alpha^c.$$

Example 1.1. For example, the relation ∇_X on X, defined by $(x, y) \in \nabla_X \Longleftrightarrow x \neq y$, is a bi-quasiregular relation because the following equation

$$\nabla_X \;=\; \triangle_X \;\circ\; \nabla_X \circ \triangle_X = \nabla_X^c \circ \nabla_X \circ \nabla_X^c$$

holds. Besides, for a relation α on X such that α^c is a bijective function we have

$$\alpha = \triangle_X \circ \alpha \circ \triangle_X = (\alpha^c \circ (\alpha^c)^{-1}) \circ \alpha \circ ((\alpha^c)^{-1} \circ \alpha^c)$$
$$= \alpha^c \circ ((\alpha^c)^{-1} \circ \alpha \circ (\alpha^c)^{-1}) \circ \alpha^c = \alpha^c \circ \beta \circ \alpha^c.$$

So, such a relation α is be-quasiregular.

Theorem 1.1 ([6], Theorem 4.1). *For a relation $\alpha \in \mathcal{B}(X)$, the relation*

$$\alpha_* = ((\alpha^c)^{-1} \circ \alpha^c \circ (\alpha^c)^{-1})^C$$

is the maximal element in the family of all relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha^c \circ \beta \circ \alpha^c \subseteq \alpha.$$

A characterization of bi-quasiregular relation is given in the following theorem.

Theorem 1.2 ([6], Theorem 4.4). *For a relation $\alpha \in \mathcal{B}(X)$, the following conditions are equivalent:*

(1) *α is a bi-quasiregular,*
(2) *for all $x, y \in \alpha$ there exists $(u, v) \in X^2$ such that:*
 (a) $(x, u) \in \alpha^c, (v, y) \in \alpha^c$,
 (b) $(\forall s, t \in X)((s, u) \in \alpha^c, (v, t) \in \alpha^c \Longrightarrow (s, t) \in \alpha)$.
(3) $\alpha \subseteq \alpha^c \circ \alpha_* \circ \alpha^c$.

2. Finitely bi-quasiregular relations

In 2003. X.-Q. Xu and Y.-M.Lui, in [7], introduced a definition of finitely regular relations so that the relation α is finitely regular if and only if its finitely extension is regular. In this section we introduce the concept of finitely bi-quasiregular relations and give a characterization of that relations. For that we need the concept of finite extension of a relation. That notion and belonging notation we borrow from articles [1], [2] and [7]. For any set X, let

$$X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty}\}.$$

Definition 2.1. ([1], Definition 3.3; [2], Definition 3.4) Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the *finite extension* of α, by

$$(\forall F, G \in X^{(<\omega)})\,((F, G) \in \alpha^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha).$$

From Definition 2.1, we immediately obtain that

$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^c)^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^c)$

$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^{-1})^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^{-1} = \alpha F)$ and

$(\forall F, G \in X^{(<\omega)})((F, G) \in ((\alpha^{-1})^c)^{(<\omega)} \Longleftrightarrow G \subseteq F(\alpha^c)^{-1} = \alpha^c F)$.

Now, we can introduce a concept of *finitely bi-quasiregular relations.*

Definition 2.2. A relation $\alpha \in \mathcal{B}(X)$ is called *finitely bi-quasiregular* if there exists a relation β on $X^{(<\omega)}$ such that

$$\alpha^{(<\omega)} = (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}.$$

In accordance with [7], we call a relation α on X finitely bi-quasiregular if its finite extension to $X^{(<\omega)}$ is a bi-quasiregular relation:

$$\alpha^{(<\omega)} = (\alpha^{(<\omega)})^c \circ \beta \circ (\alpha^{(<\omega)})^c.$$

for some relation β. We will not use this option. That concept is different from our concept introduced in Definition 2.2.

Now we give an essential characterization of finitely bi-quasiregular relations.

Theorem 2.1. *A relation $\alpha \in \mathcal{B}(X)$ is a finitely bi-quasiregular if and only if for all $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$, there are $U, V \in X^{(<\omega)}$, such that*

(i) *$U \subseteq F\alpha^c$, $G \subseteq V\alpha^c$,*
(ii) *for all $S, T \in X^{(<\omega)}$, if $U \subseteq S\alpha^c$ and $T \subseteq V\alpha^C$, then $T \subseteq S\alpha$.*

Proof. (1) Suppose that α is a finitely bi-quasiregular. Then there is a relation β such that $(\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)} = \alpha^{(<\omega)}$. For all $(F, G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F, G) \in \alpha^{(<\omega)}$, then

$$(F, G) \in (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}.$$

Thus, there is $(U, V) \in (X^{(<\omega)})^2$ such that

$$(F, U) \in (\alpha^c)^{(<\omega)}, (U, V) \in \beta \text{ and } (V, G) \in (\alpha^c)^{(<\omega)},$$

i.e.,

$$U \subseteq F\alpha^c, G \subseteq V\alpha^c.$$

Hence we get condition (i).

Now we check (ii). For all $(S, T) \in (X^{(<\omega)})^2$, if $U \subseteq S\alpha^c$ and $T \subseteq V\alpha^c$, i.e., $(S, U) \in (\alpha^c)^{(<\omega)}$ and $(V, T) \in (\alpha^c)^{(<\omega)}$, then by $(U, V) \in \beta$, we have $(S, T) \in (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}$, i.e., $(S, T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$.

(2) Let α be a relation such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(F,G) \in \beta \Longleftrightarrow (\forall S, T \in X^{(<\omega)})((F \subseteq S\alpha^c \wedge T \cap G\alpha^c \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that (a) $(\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)} \subseteq \alpha^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H, W) \in (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}$, then there are $F, G \in X^{(<\omega)}$ with $(H, F) \in (\alpha^c)^{(<\omega)}$, $(F, G) \in \beta$ and $(G, W) \in (\alpha^c)^{(<\omega)}$. Then $F \subseteq H\alpha^c$ and $W \subseteq G\alpha^c$. For all $w \in W$, let $S = H$, $T = \{w\}$. Then $F \subseteq S\alpha^c$ and $G\alpha^c \cap T \neq \emptyset$ because $w \in T$ and $w \in T \subseteq W \subseteq G\alpha^c$. Since $(F, G) \in \beta$, we have that $F \subseteq S\alpha^c \wedge G\alpha^c \cap T \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H, W) = (S, W) \in \alpha^{(<\omega)}$. Therefore, we have $(\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)} \subseteq \alpha^{(<\omega)}$.

Second, check that (b) $\alpha^{(<\omega)} \subseteq (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H, W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), there are $A, B \in X^{(<\omega)}$ such that:
(i') $A \subseteq H\alpha^c$, $W \subseteq B\alpha^c$, and
(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq S\alpha^c$ and $T \subseteq B\alpha^c$, then $T \subseteq S\alpha$.

Now, we have to show that $(A, B) \in \beta$. Let be for all $(C, D) \in (X^{(<\omega)})^2$ the following $A \subseteq D\alpha^c$ and $D \cap B\alpha^c \neq \emptyset$ hold. From $D \cap B\alpha^c \neq \emptyset$ follows that there exists an element $d \in D \cap B\alpha^c (\neq \emptyset)$. So, $d \in D$ and $d \in B\alpha^c$. Put $S = C$ and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq S\alpha^c \wedge T = \{d\} \subseteq B\alpha^c) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

i.e. $\emptyset \neq \{d\} \cap S\alpha = T \cap S\alpha$. Therefore, $(A, B) \in \beta$ by definition of β. Finally, for $(H, A) \in (\alpha^c)^{(<\omega)}$, $(A, B) \in \beta$ and $(B, W) \in (\alpha^c)^{(<\omega)}$ follows that $(H, W) \in (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}$.

By assertion (a) and (b), finally we have $\alpha^{(<\omega)} = (\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)}$. □

Particularly, if we put $F = \{x\}$ and $G = \{y\}$ in the previous theorem, we conclude the following corollary.

Corollary 2.1. *A relation $\alpha \in \mathcal{B}(X)$ is a finitely bi-quasiregular if and only if for all elements $x, y \in X$ such that $(x, y) \in \alpha$ there are finite subsets $U, V \in X^{(<\omega)}$ such that*

(1^0) $(\forall u \in U)((x, u) \in \alpha^c)$ *and* $(\exists v \in V)((v, y) \in \alpha^c)$,

(2^0) *for all* $S \in X^{(<\omega)}$ *and* $t \in X$

holds

$$(U \subseteq S\alpha^c, (\exists v \in V)((v, t) \in \alpha^c)) \Longrightarrow (\exists s \in S)((s, t) \in \alpha).$$

Proof. Let α be a finitely bi-quasiregular relation and let x, y be elements of X such that $(x, y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem 2.1 then there exist finite U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold.

Conversely, assume now that for all elements $x, y \in X$ such that $(x, y) \in \alpha$ there are U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold. Define binary relation $\beta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(A, B) \in \beta \Longleftrightarrow (\forall S \in X^{(<\omega)})(\forall t \in X)((A \subseteq S\alpha^c,\ t \in B\alpha^c) \Longrightarrow t \in S\alpha).$$

The proof that the equality $(\alpha^c)^{(<\omega)} \circ \beta \circ (\alpha^c)^{(<\omega)} = \alpha^{(<\omega)}$ holds is analogous to the proof of Theorem 2.1. So, the relation α is finitely bi-quasiregular. $\square$

Acknowledgement. The authors are grateful to the referee for the helpful comments and suggestions which improved the paper.

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(Received: August 23, 2013)
(Revised: November 28, 2013)

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