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SUPERADDITIVITY OF FUNCTIONALS RELATED TO GAUSS' TYPE INEQUALITES

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Abstract. In this paper we prove superadditivity of some functionals associated with the Gauss-Winckler and the Gauss-Pólya inequalities.

1. Introduction

In [2] C. F. Gauss mentioned the following inequality between the second and the fourth absolute moments.

If f is a non-negative and decreasing function, then

$$\left(\int_0^\infty x^2 f(x)\,dx\right)^2 \le \frac{5}{9}\int_0^\infty f(x)\,dx \int_0^\infty x^4 f(x)\,dx. \tag{1.1}$$

Until now, there are a lot of generalizations, sharpenings and improvements of inequality (1.1). One of major lines of generalization is due to A. Winckler and the other springing from a pair of results of G. Pólya.

A. Winckler, [7], gave the following result which is known as the Gauss-Winckler inequality in the recent literature. More about it and its history one can find in [1].

Theorem 1.1. *If f is a non-negative, continuous and non-increasing function on $[0,\infty)$ such that $\int_0^\infty f(x)dx = 1$, then for $m \le r$*

$$\left((m+1)\int_0^\infty x^m f(x)\,dx\right)^{\frac{1}{m}} \le \left((r+1)\int_0^\infty x^r f(x)\,dx\right)^{\frac{1}{r}}. \tag{1.2}$$

Another generalization was done by G. Pólya and today those type of inequalities are called the Gauss-Pólya inequalites. Namely, in the book "Problems and Theorems in Analysis " (see [5, Vol I, p. 83, Vol II, p. 129] one can find the following results.

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Theorem 1.2.
(i) *Let* $f : [0,\infty) \to \mathbf{R}$ *be a non-negative and decreasing function. If* a *and* b *are non-negative real numbers, then*

$$\left(\int_0^\infty x^{a+b} f(x)dx\right)^2 \le \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^\infty x^{2a}f(x)dx \int_0^\infty x^{2b}f(x)dx$$

if all the integrals exist.
(ii) *Let* $f : [0,1] \to \mathbf{R}$ *be a non-negative and increasing function. If* a *and* b *are non-negative real numbers, then*

$$\left(\int_0^1 x^{a+b} f(x)dx\right)^2 \ge \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^1 x^{2a}f(x)dx \int_0^1 x^{2b}f(x)\,dx.$$

J. Pečarić and S. Varošanec treated the above mentioned inequalities in a unified way and proved the following generalizations, [4], [6].

Theorem 1.3. *Let* $g : [a,b] \to \mathbf{R}$ *be a non-negative increasing differentiable function and let* $f : [a,b] \to \mathbf{R}$, *be a non-negative function such that* $x \mapsto \frac{f(x)}{g'(x)}$ *is a non-decreasing function. Let* $p_i\,(i=1,\dots,n)$ *be positive real numbers such that* $\sum_{i=1}^n \frac{1}{p_i} = 1$. *If* $a_i\,(i=1,\dots,n)$ *are real numbers such that* $a_i > -\frac{1}{p_i}$, *then*

$$\int_a^b g(x)^{a_1+\dots+a_n} f(x)\,dx \ge \frac{\prod_{i=1}^n (a_i p_i+1)^{\frac{1}{p_i}}}{1+\sum_{i=1}^n a_i}\prod_{i=1}^n\left(\int_a^b g(x)^{a_i p_i} f(x)\,dx\right)^{\frac{1}{p_i}}. \tag{1.3}$$

If $g(a)=0$ *and if the quotient function* $\frac{f}{g'}$ *is non-increasing, then the reverse inequality in (1.3) holds.*

As a consequence of the above results we conclude that if f and g satisfy the assumptions of Theorem 1.3, then the function

$$Q(r) = (r+1)\int_a^b g^r(x)f(x)\,dx$$

is log-concave when $\frac{f}{g'}$ is a non-decreasing function and the function Q is log-convex when $g(a)=0$ and $\frac{f}{g'}$ is non-increasing.

Using that property, the following generalization of the Gauss-Winckler inequality was proved in [6]:

Theorem 1.4. *Let* f *and* g *be defined as in Theorem* 1.3, $\frac{f}{g'}$ *be a non-decreasing function and* $p,\ q,\ r,\ s$ *be real numbers from the domain of definition of the function* Q.

If $p \le q$, $r \le s$ and $p > r$, $q > s$, then

$$\left(\frac{(p+1) \int_a^b g^p(x) f(x) dx}{(r+1) \int_a^b g^r(x) f(x) dx} \right)^{\frac{1}{p-r}} \ge \left(\frac{(q+1) \int_a^b g^q(x) f(x) dx}{(s+1) \int_a^b g^s(x) f(x) dx} \right)^{\frac{1}{q-s}} . \tag{1.4}$$

If $g(a) = 0$ and $\frac{f}{g'}$ is non-increasing, then the reverse inequality holds.

Remark 1.5. In [6] the authors considered the case when $g(x) = x$, f is non-increasing and $a = 0$. In that case inequalities (1.3) and (1.4) hold with $b = \infty$ and then we get results for moments.

In the next section we investigate properties of mappings which arise from Gauss-Pólya's inequalities, while in the third section we research functional related to the Gauss-Winckler inequality (1.4). The main tool of this investigation is the Hölder type inequality which we give in the following form, [3]:

Proposition 1.6. *Let $a_i, b_i, p_i, (i = 1, \dots, n)$ be non-negative real numbers such that $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then*

$$a_1^{\frac{1}{p_1}} \cdots a_n^{\frac{1}{p_n}} + b_1^{\frac{1}{p_1}} \cdots b_n^{\frac{1}{p_n}} \le \prod_{i=1}^n (a_i + b_i)^{\frac{1}{p_i}} . \tag{1.5}$$

It is a simple consequence of weighted AM-GM inequality

$$\frac{a_1^{\frac{1}{p_1}} \cdots a_n^{\frac{1}{p_n}}}{(a_1+b_1)^{\frac{1}{p_1}} \cdots (a_n+b_n)^{\frac{1}{p_n}}} + \frac{b_1^{\frac{1}{p_1}} \cdots b_n^{\frac{1}{p_n}}}{(a_1+b_1)^{\frac{1}{p_1}} \cdots (a_n+b_n)^{\frac{1}{p_n}}}$$

$$\le \frac{a_1}{p_1(a_1+b_1)} + \cdots + \frac{a_n}{p_n(a_n+b_n)} + \frac{b_1}{p_1(a_1+b_1)} + \cdots + \frac{b_n}{p_n(a_n+b_n)} = 1.$$

2. Functionals related to the Gauss-Pólya inequalites

Throughout this section functions $f, g : [a, b] \to \mathbf{R}$ are non-negative, g is increasing differentiable, numbers $p_i\,(i = 1, \dots, n)$ are positive reals such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and $a_i\ (i = 1, \dots, n)$ are real numbers such that $a_i > -\frac{1}{p_i}$.
Let us consider the functional G defined as

$$G(f) = \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_i p_i} f(x)\, dx \right)^{\frac{1}{p_i}} - (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1 + \cdots + a_n} f(x)\, dx.$$

It is obvious that $f \mapsto G(f)$ is positive homogeneous, i.e. $G(\lambda f) = \lambda G(f)$ for any $\lambda \geq 0$. As a consequence of Theorem 1.3, if f/g' is a non-decreasing function, then $G(f) \leq 0$, while if f/g' is non-increasing and $g(a) = 0$, then $G(f) \geq 0$.

The following theorem gives superadditivity property of the functional G.

Theorem 2.1. *Let $f_1, f_2, g : [a,b] \to \mathbf{R}$ be non-negative functions, g increasing differentiable, numbers $p_i\,(i = 1, \ldots, n)$ be positive reals such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and a_i $(i = 1, \ldots, n)$ be real numbers such that $a_i > -\frac{1}{p_i}$. Then*

$$G(f_1 + f_2) \geq G(f_1) + G(f_2),$$

i.e. G is a superadditive functional.

Furthermore, if $f_1 \geq f_2$ such that $\frac{f_1 - f_2}{g'}$ is non-increasing, $g(a) = 0$, then

$$G(f_1) \geq G(f_2),$$

i.e. G is non-decreasing.

Proof. Let us consider a difference $G(f_1 + f_2) - G(f_1) - G(f_2)$.

$$\begin{aligned}
G(f_1{+}f_2){-}G(f_1){-}G(f_2) ={}& \prod_{i=1}^n (a_ip_i{+}1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_ip_i} (f_1 + f_2)(x)\, dx \right)^{\frac{1}{p_i}} \\
& - (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1 + \cdots + a_n} (f_1 + f_2)(x)\, dx \\
& - \prod_{i=1}^n (a_ip_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_ip_i} f_1(x)\, dx \right)^{\frac{1}{p_i}} \\
& + (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1 + \cdots + a_n} f_1(x)\, dx \\
& - \prod_{i=1}^n (a_ip_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_ip_i} f_2(x) dx \right)^{\frac{1}{p_i}} \\
& + (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1 + \cdots + a_n} f_2(x)\, dx \\
={}& \prod_{i=1}^n (a_ip_i + 1)^{\frac{1}{p_i}} \left[\prod_{i=1}^n \left(\int_a^b g(x)^{a_ip_i} (f_1 + f_2)(x)\, dx \right)^{\frac{1}{p_i}} \right. \\
& \left. - \prod_{i=1}^n \left(\int_a^b g(x)^{a_ip_i} f_1(x)\, dx \right)^{\frac{1}{p_i}} - \prod_{i=1}^n \left(\int_a^b g(x)^{a_ip_i} f_2(x)\, dx \right)^{\frac{1}{p_i}} \right].
\end{aligned}$$

Setting in (1.5):

$$a_i = \int_a^b g(x)^{a_i p_i} f_1(x)\,dx, \quad b_i = \int_a^b g(x)^{a_i p_i} f_2(x)\,dx, \quad i = 1, 2, \dots, n$$

and using the Hölder inequality we have that $G(f_1+f_2)-G(f_1)-G(f_2) \geq 0$, so G is superadditive.

If $f_1 \geq f_2$, $\frac{f_1-f_2}{g'}$ is non-increasing and $g(a) = 0$, then $G(f_1 - f_2) \geq 0$, so, we have

$$\begin{aligned} G(f_1) &= G(f_2 + (f_1 - f_2)) \\ &\geq G(f_2) + G(f_1 - f_2) \geq G(f_2). \end{aligned}$$

□

Corollary 2.2. *Let f_1, f_2, g be non-negative functions on $[a, b]$, g increasing differentiable, $g(a) = 0$, numbers p_i $(i = 1, \dots, n)$ be positive reals such that $\sum_{i=1}^n \frac{1}{p_i} = 1$, a_i $(i = 1, \dots, n)$ be real numbers such that $a_i > -\frac{1}{p_i}$ and $c, C \in \mathbf{R}$ such that $Cf_2 - f_1$, $f_1 - cf_2$ are non-negative and $\frac{Cf_2-f_1}{g'}$, $\frac{f_1-cf_2}{g'}$ are non-negative non-increasing functions. Then*

$$\begin{aligned} &C\Bigg\{ \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_i p_i} f_2(x)\,dx \right)^{\frac{1}{p_i}} \\ &\qquad\qquad - (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1+\dots+a_n} f_2(x)\,dx \Bigg\} \\ &\geq \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_i p_i} f_1(x)\,dx \right)^{\frac{1}{p_i}} \\ &\qquad\qquad - (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1+\dots+a_n} f_1(x)\,dx \\ &\geq c\Bigg\{ \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left(\int_a^b g(x)^{a_i p_i} f_2(x)\,dx \right)^{\frac{1}{p_i}} \\ &\qquad\qquad - (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1+\dots+a_n} f_2(x)\,dx \Bigg\}. \end{aligned}$$

Proof. Using previous results we have

$$CG(f_2) = G(Cf_2) = G((Cf_2 - f_1) + f_1) \geq G(Cf_2 - f_1) + G(f_1) \geq G(f_1)$$

and

$$G(f_1) = G((f_1 - cf_2) + cf_2) \geq G(f_1 - cf_2) + G(cf_2) \geq G(cf_2) = cG(f_2)$$

from which the conclusion of the corollary is established. □

The following theorem contains a result about concavity of function $G \circ \phi$ where ϕ is concave.

Theorem 2.3. *Let $\phi : [0,\infty) \to [0,\infty)$ be a concave function, f_1, f_2, g be non-negative functions on $[a,b]$ such that $(\phi \circ (\alpha f_1 + (1-\alpha) f_2) - [\alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2)])/g'$ is non-increasing for some $\alpha \in [0,1]$, $g(a) = 0$. Then*

$$G \circ \phi \circ (\alpha f_1 + (1-\alpha) f_2) \geq \alpha (G \circ \phi \circ f_1) + (1-\alpha)(G \circ \phi \circ f_2).$$

Proof. For any $x \in [a,b]$ we have

$$\begin{aligned}(\phi \circ (\alpha f_1 + (1-\alpha) f_2))(x) &= \phi(\alpha f_1(x) + (1-\alpha) f_2(x)) \\ &\geq \alpha \phi(f_1(x)) + (1-\alpha)\phi(f_2(x)) \\ &= (\alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2))(x),\end{aligned}$$

where a concavity of function ϕ is used. So, we have $\phi \circ (\alpha f_1 + (1-\alpha) f_2) \geq \alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2)$. Using properties of G and the above-proved inequality we have

$$\begin{aligned}&G(\phi \circ (\alpha f_1 + (1-\alpha) f_2)) \geq G(\alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2)) \\ &\quad \geq G(\alpha(\phi \circ f_1)) + G((1-\alpha)(\phi \circ f_2)) = \alpha G(\phi \circ f_1) + (1-\alpha) G(\phi \circ f_2)\end{aligned}$$

and the proof is established. □

Remark 2.4. Let us consider a case when $g(x) = x$, $a = 0$, $b = \infty$ and f is non-increasing as it is mentioned in Remark 1.5. Let us denote by $\mu_r(f)$ a moment of the order r i.e.

$$\mu_r(f) = \int_0^\infty x^r f(x)\, dx.$$

Then the functional G has a form

$$G(f) = \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \mu_{a_i p_i}^{\frac{1}{p_i}}(f) - (1 + \sum_{i=1}^n a_i)\mu_{a_1 + \cdots + a_n}(f)$$

and G is superadditive. Also, if $f_1 \geq f_2$ such that $f_1 - f_2$ is non-increasing, then $G(f_1) \geq G(f_2)$.

3. Functionals related to the Gauss-Winckler inequality

Putting in (1.4) $r = s = 0$ we get the Gauss-Winckler inequality for f/g' non-decreasing function:

$$\left(\frac{(p+1)\int_a^b g^p(x) f(x)\, dx}{\int_a^b f(x)\, dx} \right)^{\frac{1}{p}} \geq \left(\frac{(q+1)\int_a^b g^q(x) f(x)\, dx}{\int_a^b f(x)\, dx} \right)^{\frac{1}{q}}$$

where $0 < p \leq q$. If f/g' is non-increasing and $g(a) = 0$, then the reversed inequality holds.

Let us consider a functional W defined as

$$W(f) = \left(\int_a^b f(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f(x)\,dx\right)^{\frac{p}{q}} - (p+1)\int_a^b g^p(x)f(x)\,dx.$$

The following theorem gives superadditivity and monotonicity of the functional W.

Theorem 3.1. *Let $f_1, f_2, g : [a,b] \to \mathbf{R}$ be non-negative functions, g increasing differentiable, numbers p, q be positive real such that $p \leq q$. Then*

$$W(f_1 + f_2) \geq W(f_1) + W(f_2).$$

Additionally, if $f_1 \geq f_2$ such that $\frac{f_1 - f_2}{g'}$ is non-increasing, $g(a) = 0$, then

$$W(f_1) \geq W(f_2).$$

Proof. Let us transform $W(f_1 + f_2) - W(f_1) - W(f_2)$.

$$\begin{aligned}
&W(f_1+f_2) - W(f_1) - W(f_2)\\
&= \left(\int_a^b (f_1+f_2)(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)(f_1+f_2)(x)\,dx\right)^{\frac{p}{q}}\\
&- (p+1)\int_a^b g^p(x)(f_1+f_2)(x)\,dx\\
&- \left(\int_a^b f_1(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_1(x)\,dx\right)^{\frac{p}{q}}\\
&+ (p+1)\int_a^b g^p(x)f_1(x)dx - \left(\int_a^b f_2(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_2(x)dx\right)^{\frac{p}{q}}\\
&+ (p+1)\int_a^b g^p(x)f_2(x)\,dx\\
&= \left(\int_a^b (f_1+f_2)(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)(f_1+f_2)(x)\,dx\right)^{\frac{p}{q}}\\
&- \left(\int_a^b f_1(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_1(x)\,dx\right)^{\frac{p}{q}}\\
&- \left(\int_a^b f_2(x)\,dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_2(x)\,dx\right)^{\frac{p}{q}} \geq 0
\end{aligned}$$

where in the last inequality we use the Hölder inequality with

$$n=2,\ \ \frac{1}{p_1}=1-\frac{p}{q}>0,\ \ \frac{1}{p_2}=\frac{p}{q}>0,\ \ a_1=\int_a^b f_1(x)\,dx,\ \ b_1=\int_a^b f_2(x)\,dx,$$

$$a_2=(q+1)\int_a^b g^q(x)f_1(x)\,dx,\ \ b_2=(q+1)\int_a^b g^q(x)f_2(x)\,dx.$$

So, superadditivity of the functional W is established.

If $\frac{f_1-f_2}{g'}$ is non-increasing, $g(a)=0$, then from Theorem 1.4 we obtain $W(f_1-f_2)\geq 0$ and

$$W(f_1)=W(f_2+(f_1-f_2))\geq W(f_2)+W(f_1-f_2)\geq W(f_2).$$

□

Remark 3.2. Let us consider a case when $g(x)=x$, $a=0$, $b=\infty$ and f is non-increasing as it is mentioned in Remark 1.5. Now the functional W has the form

$$W(f)=(q+1)^{\frac{p}{q}}\left(\mu_0(f)\right)^{1-\frac{p}{q}}\mu_q^{\frac{p}{q}}(f)-(p+1)\mu_p(f)$$

and W is superadditive. Also, if $f_1\geq f_2$ such that f_1-f_2 is non-increasing, then $W(f_1)\geq W(f_2)$.

The following result is an interesting inequality for the Beta function.

Corollary 3.3. *Let* $0<p\leq q$, $y_1,y_2>-1$. *Then*

$$\left(\frac{1}{y_1+1}+\frac{1}{y_2+1}\right)^{1-\frac{p}{q}}\left[\beta(q+1,y_1+1)+\beta(q+1,y_2+1)\right]^{\frac{p}{q}}$$

$$\geq\left(\frac{1}{y_1+1}\right)^{1-\frac{p}{q}}\beta^{\frac{p}{q}}(q+1,y_1+1)+\left(\frac{1}{y_2+1}\right)^{1-\frac{p}{q}}\beta^{\frac{p}{q}}(q+1,y_2+1)$$

where β *is the Beta function defined as* $\beta(x+1,y+1)=\int_0^1 t^x(1-t)^y dt$.

Proof. It is a consequence of the previous theorem with $[a,b]=[0,1]$, $f_i(t)=(1-t)^{y_i}$, $i=1,2$, $g(x)=x$. □

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