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A CLASS OF DIFFERENTIAL EQUATIONS OF COMBINED HADAMARD AND RIEMANN-LIOUVILLE OPERATORS

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ABSTRACT. This paper deals with the study of a class of nonlinear coupled equations of Riemann-Liouville fractional differential equations with Hadamard fractional integral nonlocal conditions. The existence of solutions is established by the Schauder fixed point theorem. Another result is proved using Leray-Schauder's degree. Finally, some illustrative examples are discussed.

1. INTRODUCTION

In this paper, we are concerned with the investigation of the existence of solutions for a new class of nonlinear Riemann-Liouville fractional differential equations combined with nonlocal Hadamard fractional integral boundary conditions of the following form

$$\begin{cases} {}_{RL}D^{p}u(t) = f\left(t, u(t), v(t),\ {}_{RL}D^{p-1}v(t)\right),\ t \in [0,1], 1 < p \leq 2, \\ {}_{RL}D^{q}v(t) = g\left(t, u(t), v(t),\ {}_{RL}D^{q-1}u(t)\right),\ t \in [0,1], 1 < q \leq 2, \\ u(0) = 0,\ u(1) = A\ {}_{H}I^{\rho}v(\xi),\ \rho > 0,\ 0 < \xi < 1, \\ v(0) = 0,\ v(1) = B\ {}_{H}I^{\sigma}u(\xi),\ \sigma > 0,\ 0 < \xi < 1, \end{cases} \tag{1.1}$$

where ${}_{RL}D^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, $\alpha \in \{p, q, p-1, q-1\}$, $A, B \in \mathbb{R}$, and ${}_{H}I^{\rho}$, ${}_{H}I^{\sigma}$ are the Hadamard fractional integrals of orders ρ and σ respectively, $f, g \in C\left([0,1] \times \mathbb{R}^3, \mathbb{R}\right)$.

Several interesting results dealing with the existence and uniqueness of solutions, stabilities and properties of solutions, analytic and numerical methods of solutions for fractional differential equations can be found in the following research papers [1–4, 6, 8–10, 16, 18–21, 23]. One can also see the paper [22] where the authors have concentrated on the study of existence and uniqueness of solutions for coupled systems of Riemann-Liouville and Hadamard fractional derivatives of Langevin equation with fractional integral conditions.
We cite also the paper of B. Ahmad et al. [5], where the authors have investigated the existence of solutions for nonlinear Langevin equations and inclusions

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involving Hadamard and Caputo type fractional derivatives equipped with nonlocal fractional integral conditions.
Very recently, the authors of the paper [7] have studied a new coupled differential system with nonlinearities involving two unknown functions and their derivatives, subject to new kinds of multi-point and multi-strip boundary value conditions. Some existence results have been established.
However, it has been observed that most of the papers have involved either the standard derivatives, or Riemann-Liouville and Caputo type fractional derivatives. Besides these two derivatives, there is the Hadamard one. It is different kind of fractional derivatives that has been introduced by Hadamard in 1892 [15]. This derivative differs from the other ones in the sense that its kernel contains a logarithmic function of arbitrary exponent, see [11, 12, 15–17].
The importance of studying the present paper is that, the problem (1.1) contains both of Riemann-Liouville and Hadamard operators. To the best of our knowledge, this is the first time that a such problem was introduced.
The paper is organized as follows: In section 2, we recall some preliminaries and lemmas that will be needed later. In Section 3, we present our main results on the existence of one solution for the problem (1.1). Some illustrative examples are presented in the last section.

2. PRELIMINARIES

In this section, we present some useful definitions, notations and lemmas of fractional calculus [13].

Definition 2.1. *The Riemann Liouville fractional derivative of order $\mu > 0$ of a continuous function is defined as:*

$$_{RL}D^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-s)^{n-\mu-1} f(s)\, ds,$$

where $n = [\mu] + 1$.

Definition 2.2. : *The Riemann Liouville fractional integral of order $\mu > 0$ for a continuous function f defined over $(0,b)$ is given by*

$$_{RL}I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_{0}^{t} (t-s)^{\mu-1} f(s)\, ds.$$

Definition 2.3. *The Hadamard fractional derivative of order μ for a function $f : (0,\infty) \to \mathbb{R}$ is defined as*

$$_{H}D^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \left(t\frac{d}{dt}\right)^{n} \int_{0}^{t} \left(\log\frac{t}{s}\right)^{n-\mu-1} \frac{f(s)}{s} ds,\ n-1 < \mu < n.$$

Definition 2.4. *The Hadamard fractional integral of order $\mu > 0$ of a function f defined on $(0,\infty)$ is given by:*

$$ {}_{H}I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \left(\log \frac{t}{s}\right)^{\mu-1} \frac{f(s)}{s} ds. $$

Lemma 2.1. *Let* $\alpha > 0, \beta > 0$. *Then, we have*

$$ {}_{H}I^{\alpha} t^{\beta} = \beta^{-\alpha} t^{\beta} \quad \text{and} \quad {}_{H}D^{\alpha} t^{\beta} = \beta^{\alpha} t^{\beta}. $$

Lemma 2.2. *Let* $\mu > 0$, *and* $x \in C(0,1) \cap L(0,1)$. *Then the equation* ${}_{RL}D^{\mu} x(t) = 0$, *has a unique solution* $x(t) = \sum_{i=1}^{n} a_i t^{\mu - i}$, *where* $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ *and* $n - 1 < \mu < n$.

Lemma 2.3. *Let* $\mu > 0$, *then for* $x \in C(0,1) \cap L(0,1)$, *we have that* ${}_{RL}I^{\mu}_{RL} D^{\mu} x(t) = x(t) + \sum_{i=1}^{n} a_i t^{\mu - i}$, *where* $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *and* $n - 1 < \mu < n$.

We also need the following lemma.

Lemma 2.4. *Given* $\varphi, \psi \in C([0,1], \mathbb{R})$, *the unique solution of the problem*

$$ \begin{cases} {}_{RL}D^{p} x(t) = \varphi(t), \ t \in [0,1], 1 < p \leq 2, \\ {}_{RL}D^{q} y(t) = \psi(t), \ t \in [0,1], 1 < q \leq 2, \\ x(0) = 0, x(1) = A_H I^{\rho} y(\xi), \rho > 0, \ 0 < \xi < 1, \\ y(0) = 0, y(1) = B_H I^{\sigma} x(\xi), \sigma > 0, \ 0 < \xi < 1, \end{cases} \tag{2.1} $$

is given by the expression

$$ \begin{aligned} x(t) := & \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \varphi(s)\, ds - \frac{t^{p-1}}{\Delta} \Bigg\{ \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} \varphi(s)\, ds \\ & - \frac{A}{\Gamma(\rho)\Gamma(q)} \int_0^{\xi} \int_0^t \left(\log \frac{\xi}{t}\right)^{\rho - 1} (t-s)^{q-1} \psi(s)\, ds \frac{dt}{t} \\ & + A(q-1)^{-\rho} \xi^{q-1} \Bigg(\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \psi(s)\, ds \\ & - \frac{B}{\Gamma(\sigma)\Gamma(p)} \int_0^{\xi} \int_0^t \left(\log \frac{\xi}{t}\right)^{\sigma - 1} (t-s)^{p-1} \varphi(s)\, ds \frac{dt}{t} \Bigg) \Bigg\} \end{aligned} \tag{2.2} $$

$$ \begin{aligned} y(t) := & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \psi(s)\, ds - \frac{t^{q-1}}{\Delta} \Bigg\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \psi(s)\, ds \\ & - \frac{B}{\Gamma(\sigma)\Gamma(p)} \int_0^{\xi} \int_0^t \left(\log \frac{\xi}{t}\right)^{\sigma - 1} (t-s)^{p-1} \varphi(s)\, ds \frac{dt}{t} \\ & + B(p-1)^{-\sigma} \xi^{p-1} \Bigg(\int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} \varphi(s)\, ds \\ & - \frac{A}{\Gamma(\sigma)\Gamma(q)} \int_0^{\xi} \int_0^t \left(\log \frac{\xi}{t}\right)^{\sigma - 1} (t-s)^{q-1} \psi(s)\, ds \frac{dt}{t} \Bigg) \Bigg\}, \end{aligned} \tag{2.3} $$

where $\Delta := 1 - AB(p-1)^{-\sigma}(q-1)^{-\rho} \xi^{p+q-2} \neq 0$.

Proof. By Lemma 2.6 and Lemma 2.7, we state that the two equations in (2.1) are equivalent to:

$$x(t) = {}_{RL}I^{p}\varphi(t) - a_1 t^{p-1} - a_2 t^{p-2}, \tag{2.4}$$

$$y(t) = {}_{RL}I^{q}\psi(t) - b_1 t^{q-1} - b_2 t^{q-2}, \tag{2.5}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
Applying the conditions $x(0) = 0$ and $y(0) = 0$, we get $a_2 = b_2 = 0$.
Now, using the property of the Hadamard fractional integral given by Lemma 2.5, we obtain:

$$\begin{cases} {}_{RL}I^{p}\varphi(1) = a_1 + A\,{}_{H}I^{\rho}\,{}_{RL}I^{q}\psi(\xi) - Ab_1\,{}_{H}I^{\rho}\xi^{q-1}, \\ {}_{RL}I^{q}\psi(1) = b_1 + B\,{}_{H}I^{\sigma}\,{}_{RL}I^{p}\varphi(\xi) - Ba_1\,{}_{H}I^{\sigma}\xi^{p-1}. \end{cases}$$

Hence, we have

$$\begin{aligned} a_1 = \tfrac{1}{\Delta}\big[&{}_{RL}I^{p}\varphi(1) - A\,{}_{H}I^{\rho}\,{}_{RL}I^{q}\psi(\xi) \\ &+ A(q-1)^{-\rho}\xi^{q-1}\left({}_{RL}I^{q}\psi(1) - B\,{}_{H}I^{\sigma}\,{}_{RL}I^{p}\varphi(\xi)\right)\big] \end{aligned}$$

and

$$\begin{aligned} b_1 = \tfrac{1}{\Delta}\big[&{}_{RL}I^{q}\psi(1) - B\,{}_{H}I^{\sigma}\,{}_{RL}I^{p}\varphi(\xi) \\ &+ B(p-1)^{-\sigma}\xi^{p-1}\left({}_{RL}I^{p}\varphi(1) - A\,{}_{H}I^{\rho}\,{}_{RL}I^{q}\psi(\xi)\right)\big]. \end{aligned}$$

Substituting a_1, a_2, b_1, b_2 in (2.4) and (2.5), we get (2.2) and (2.3). □

Let us now consider the quantities:

$$\begin{aligned} &{}_{RL}I^{\alpha}k\left(s, x(s), y(s), {}_{RL}D^{p-1}y(s)\right)(\eta) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta - s)^{\alpha-1}k\left(s, x(s), y(s), {}_{RL}D^{p-1}y(s)\right)ds, \end{aligned}$$

and

$$\begin{aligned} &{}_{H}I^{\beta}\,{}_{RL}I^{\alpha}k\left(s, x(s), y(s), {}_{RL}D^{p-1}y(s)\right)(\eta) \\ &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_0^{\eta}\int_0^{t}\left(\log\frac{\eta}{t}\right)^{\beta-1}(t-s)^{\alpha-1}k\left(s, x(s), y(s), {}_{RL}D^{p-1}y(s)\right)ds\frac{dt}{t}, \end{aligned}$$

where $\beta \in \{\rho, \sigma\}$, $\eta \in \{t, \xi, 1\}$, $\alpha \in \{p, q\}$ and $k \in \{f, g\}$.

3. Main Results

Let $E = \left\{x \mid x \in C([0,1], \mathbb{R}) \text{ and let } D^{q-1}x \in C([0,1], \mathbb{R})\right\}$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm $\|x\|_E = \sup_{t\in[0,1]}|x(t)| + \sup_{t\in[0,1]}\left|D^{q-1}x(t)\right|$, where $1 < q \leq 2$.
Also let

$$F = \left\{y \mid y \in C([0,1], \mathbb{R}) \text{ and } D^{p-1}y \in C([0,1], \mathbb{R})\right\}$$

be endowed with the norm $\|y\|_F = \sup_{t\in[0,1]}|y(t)| + \sup_{t\in[0,1]}\left|D^{p-1}y(t)\right|$, where $1 < p \leq 2$.

So, the product space $(E\times F,\|(x,y)\|)$ is also a Banach space with the norm
$$\|(x,y)\|_{E\times F}=\|x\|_E+\|y\|_F\,.$$
In view of Lemma 2.8, we can define the operator $\Lambda: E\times F\to E\times F$ by $\Lambda(x,y)(t)=\begin{pmatrix}\Lambda_1(x,y)(t)\\ \Lambda_2(x,y)(t)\end{pmatrix}$, where

$$\begin{aligned}\Lambda_1(x,y)(t):=&\,{}_{RL}I^p f\left(t,x(t),y(t),{}_{RL}D^{p-1}y(t)\right)\\&-\tfrac{t^{p-1}}{\Delta}\Big\{{}_{RL}I^p f\left(1,x(1),y(1),{}_{RL}D^{p-1}y(1)\right)\\&-A\,{}_HI^{\rho}\,{}_{RL}I^q g\left(\xi,x(\xi),y(\xi),{}_{RL}D^{q-1}x(\xi)\right)\\&+A(q-1)^{-\rho}\xi^{q-1}\Big[{}_{RL}I^q g\left(1,x(1),y(1),{}_{RL}D^{q-1}x(1)\right)\\&-B\,{}_HI^{\sigma}\,{}_{RL}I^p f\left(\xi,x(\xi),y(\xi),{}_{RL}D^{p-1}y(\xi)\right)\Big]\Big\}\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\Lambda_2(x,y)(t):=&\,{}_{RL}I^q g\left(t,x(t),y(t),{}_{RL}D^{q-1}x(t)\right)\\&-\tfrac{t^{q-1}}{\Delta}\Big\{{}_{RL}I^q g\left(1,x(1),y(1),{}_{RL}D^{q-1}x(1)\right)\\&-B\,{}_HI^{\sigma}\,{}_{RL}I^p f\left(\xi,x(\xi),y(\xi),{}_{RL}D^{p-1}y(\xi)\right)\\&+B(p-1)^{-\sigma}\xi^{p-1}\Big[{}_{RL}I^p f\left(1,x(1),y(1),{}_{RL}D^{p-1}y(1)\right)\\&-A\,{}_HI^{\rho}\,{}_{RL}I^q g\left(\xi,x(\xi),y(\xi),{}_{RL}D^{q-1}x(\xi)\right)\Big]\Big\}.\end{aligned}\tag{3.2}$$

Lemma 3.1. *Assume that $f,g\in C\left([0,1]\times\mathbb{R}^3,\mathbb{R}\right)$. Then $(x,y)\in E\times F$ is a solution of (1) if and only if (x,y) is the solution of the equation $\begin{pmatrix}\Lambda_1(x,y)(t)\\ \Lambda_2(x,y)(t)\end{pmatrix}=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$.*

Proof. The proof is immediate from Lemma 2.8. □

For convenience, we impose the hypothesis:

(H_1) There exist nonnegative functions $h(t),l(t)\in L^1(0,1)$, such that

$$|f(t,x_1,x_2,x_3)|\le h(t)+\sum_{i=1}^{3}\lambda_i|x_i|^{r_i},\ \lambda_i>0,\ 0<r_i<1,$$
$$|g(t,x_1,x_2,x_3)|\le l(t)+\sum_{i=1}^{3}\omega_i|x_i|^{\rho_i},\ \omega_i>0,\ 0<\rho_i<1.$$

We also consider the quantities:

$$\begin{aligned}M_1=&\left(1+\frac{1}{|\Delta|}+\frac{\Gamma(p)}{|\Delta|\Gamma(p-q+1)}\right)\,{}_{RL}I^p h(s)(1)+{}_{RL}I^{p-q+1}h(s)(1)\\&+\frac{|AB|(q-1)^{-\rho}\xi^{q-1}}{|\Delta|}\left(1+\frac{\Gamma(p)}{\Gamma(p-q+1)}\right)\,{}_HI^{\sigma}\,{}_{RL}I^p h(s)(\xi),\end{aligned}\tag{3.3}$$

$$M_2 = \frac{|A|\xi^{q-1}(q-1)^{-\rho}}{|\Delta|}\left(\frac{\Gamma(p)}{\Gamma(p-q+1)}+1\right) {}_{RL}I^{q}l(s)(1) + \left(\frac{1}{|\Delta|}+\frac{\Gamma(p)}{|\Delta|\Gamma(p-q+1)}\right)|A| \ {}_{H}I^{\rho}\ {}_{RL}I^{q}l(s)(\xi), \tag{3.4}$$

$$N_1 = \left(1+\frac{1}{|\Delta|}+\frac{\Gamma(q)}{|\Delta|\Gamma(q-p+1)}\right) {}_{RL}I^{q}l(s)(1) + {}_{RL}I^{q-p+1}l(s)(1) + \frac{|AB|(p-1)^{-\sigma}\xi^{p-1}}{|\Delta|}\left(1+\frac{\Gamma(q)}{\Gamma(q-p+1)}\right) {}_{H}I^{\rho}\ {}_{RL}I^{q}l(s)(\xi), \tag{3.5}$$

$$N_2 = \frac{|B|\xi^{p-1}(p-1)^{-\sigma}}{|\Delta|}\left(\frac{\Gamma(q)}{\Gamma(q-p+1)}+1\right) {}_{RL}I^{p}l(s)(1) + \left(\frac{1}{|\Delta|}+\frac{\Gamma(q)}{|\Delta|\Gamma(q-p+1)}\right)|B| \ {}_{H}I^{\sigma}\ {}_{RL}I^{p}l(s)(\xi), \tag{3.6}$$

$$K_1 = \frac{1}{\Gamma(p+1)}+\frac{1}{|\Delta|\Gamma(p+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{|\Delta|\Gamma(p+1)p^{\sigma}} + \frac{1}{\Gamma(p-q+2)}+\frac{1}{p|\Delta|\Gamma(p-q+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{p^{\sigma+1}|\Delta_1|\Gamma(p-q+1)}, \tag{3.7}$$

$$K_2 = \frac{|A|\xi^{q}}{|\Delta|\Gamma(q+1)q^{\rho}}+\frac{|A|(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)}+\frac{|A|\Gamma(p)\xi^{q}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)q^{\rho}} + \frac{|A|\Gamma(p)(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)}. \tag{3.8}$$

$$L_1 = \frac{1}{\Gamma(q+1)}+\frac{1}{|\Delta|\Gamma(q+1)}+\frac{|AB|\xi^{p+q-1}(p-1)^{-\sigma}}{|\Delta|\Gamma(q+1)q^{\rho}} + \frac{1}{\Gamma(q-p+2)}+\frac{1}{q|\Delta|\Gamma(q-p+1)}+\frac{|AB|\xi^{p+q-1}(p-1)^{-\sigma}}{q^{\rho+1}|\Delta|\Gamma(q-p+1)}. \tag{3.9}$$

$$L_2 = \frac{|B|\xi^{p}}{|\Delta|\Gamma(p+1)p^{\sigma}}+\frac{|B|(p-1)^{-\sigma}\xi^{p-1}}{|\Delta|\Gamma(p+1)}+\frac{|B|\Gamma(q)\xi^{p}}{|\Delta|\Gamma(p+1)\Gamma(q-p+1)p^{\sigma}} + \frac{|B|\Gamma(q)(p-1)^{-\sigma}\xi^{p-1}}{|\Delta|\Gamma(p+1)\Gamma(q-p+1)}. \tag{3.10}$$

Let us now define the set $\Omega_R := \left\{(x,y)\in(C([0,1],\mathbb{R}))^2; \|(x,y)\|_{E\times F}\le R\right\}$, where

$$max\left\{(16P\lambda_i)^{\frac{1}{1-r_i}},(16Q\omega_i)^{\frac{1}{1-\rho_i}}, P\in\{K_1,L_2\}, Q\in\{K_2,L_1\}, 16M_j, 16N_j\right\}\le R.$$

Our first main result is the following theorem.

Theorem 3.1. *If* (H_1) *holds, then the problem* (1.1) *has at least one solution on* $[0,1]$.

Proof. We prove that $\Lambda : \Omega_R \to \Omega_R$.
With some fastidious calculations, we obtain

$$\begin{aligned} ||\Lambda_1(x,y)|| &= \sup_{t\in[0,1]} |\Lambda_1(x,y)(t)| + \sup_{t\in[0,1]} \left|D^{q-1}\Lambda_1(x,y)(t)\right| \\ &\leq M_1 + M_2 + K_1\sum_{i=1}^{3}\lambda_i |x_i|^{r_i} + K_2\sum_{i=1}^{3}\omega_i |x_i|^{\rho_i} \leq \frac{R}{2}, \end{aligned}$$

and

$$\begin{aligned} ||\Lambda_2(x,y)|| &= \sup_{t\in[0,1]} |\Lambda_2(x,y)(t)| + \sup_{t\in[0,1]} \left|D^{p-1}\Lambda_2(x,y)(t)\right| \\ &\leq N_1 + N_2 + L_1\sum_{i=1}^{3}\omega_i R^{\rho_i} + L_2\sum_{i=1}^{3}\lambda_i R^{r_i} \leq \frac{R}{2}. \end{aligned}$$

Therefore, $\Lambda : \Omega_R \to \Omega_R$. Hence, we obtain,

$$||\Lambda(x,y)||_{E\times F} = ||\Lambda_1(x,y)|| + ||\Lambda_2(x,y)|| \leq R.$$

Since $\Lambda_1(x,y)(t)$, $\Lambda_2(x,y)(t)$, $D^{q-1}\Lambda_1(x,y)(t)$ and $D^{p-1}\Lambda_2(x,y)(t)$ are continuous functions on $[0,1]$, the operator Λ is also continuous.
Now, we show that Λ is a completely continuous operator.
Let

$$C_1 = \max_{t\in[0,1]} \left\{ \left| f\left(t, x(t), y(t), {}_{RL}D^{p-1}y(t)\right)\right|, (x,y)\in\Omega_R \right\}$$

and

$$C_2 = \max_{t\in[0,1]} \left\{ \left| g\left(t, x(t), y(t), {}_{RL}D^{q-1}x(t)\right)\right|, (x,y)\in\Omega_R \right\}.$$

For $t_1, t_2 \in [0,1], (t_1 < t_2)$, we have

$$\begin{aligned} &|\Lambda_1(x,y)(t_2) - \Lambda_1(x,y)(t_1)| \\ &\leq \frac{1}{\Gamma(p)}\int_0^{t_1}\left((t_2-s)^{p-1} - (t_1-s)^{p-1}\right)\left| f\left(s, x(s), y(s), {}_{RL}D^{p-1}y(s)\right)\right| ds \\ &\quad + \frac{1}{\Gamma(p)}\int_{t_1}^{t_2}(t_2-s)^{p-1}\left| f\left(s, x(s), y(s), {}_{RL}D^{p-1}y(s)\right)\right| ds \\ &\quad + \frac{t_2^{p-1} - t_1^{p-1}}{|\Delta|}\left\{ {}_{RL}I^p \left| f\left(1, x(1), y(1), {}_{RL}D^{p-1}y(1)\right)\right| \right. \\ &\quad + |A|\ {}_HI^{\rho}\ {}_{RL}I^q \left| g\left(\xi, x(\xi), y(\xi), {}_{RL}D^{q-1}x(\xi)\right)\right| \\ &\quad + |A|(q-1)^{-\rho}\xi^{q-1}\left[{}_{RL}I^q \left| g\left(1, x(1), y(1), {}_{RL}D^{q-1}x(1)\right)\right| \right. \\ &\quad \left.\left. + |B|\ {}_HI^{\sigma}\ {}_{RL}I^p \left| f\left(\xi, x(\xi), y(\xi), {}_{RL}D^{p-1}y(\xi)\right)\right| \right] \right\} \\ &\leq \frac{C_1}{\Gamma(p)}\left(\int_0^{t_1}\left((t_2-s)^{p-1} - (t_1-s)^{p-1}\right) ds + \int_{t_1}^{t_2}(t_2-s)^{p-1} ds\right) \\ &\quad + \frac{t_2^{p-1} - t_1^{p-1}}{|\Delta|}\left(\frac{C_1}{\Gamma(p+1)} + \frac{C_2|A|\xi^q}{\Gamma(q+1)q^{\rho}} + \frac{C_2|A|(q-1)^{-\rho}\xi^{q-1}}{\Gamma(q+1)} + \frac{C_1|B|\xi^p}{\Gamma(p+1)p^{\sigma}}\right), \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left(t_2^p - t_1^p\right) C_1}{\Gamma(p+1)} + \frac{t_2^{p-1} - t_1^{p-1}}{|\Delta|}\left[\frac{C_1}{\Gamma(p+1)} + \frac{C_2 |A| \xi^q}{\Gamma(q+1) q^\rho}\right. \\
&\quad \left. + |A| (q-1)^{-\rho} \xi^{q-1}\left(\frac{C_2}{\Gamma(q+1)} + \frac{C_1 |B| \xi^p}{\Gamma(p+1) p^\sigma}\right)\right] \\
&\quad + |A| \ {}_H I^\rho \ {}_{RL} I^q \left| g\left(s, x(s), y(s), {}_{RL} D^{q-1} x(s)\right)(\xi)\right| \\
&\quad + \frac{|A| (q-1)^{-\rho} \xi^{q-1}}{T^{q-1}}\left[{}_{RL} I^q \left| g\left(s, x(s), y(s), {}_{RL} D^{q-1} x(s)\right)(1)\right|\right. \\
&\quad \left. + \ |B| \ \ {}_H I^\sigma \ {}_{RL} I^p \left| f\left(s, x(s), y(s), {}_{RL} D^{p-1} y(s)\right)(\xi)\right|\right]\Big\} \\
&\leq \frac{C_1}{\Gamma(p-q+1)}\left(\int_0^{t_1}\left[(t_2-s)^{p-q} - (t_1-s)^{p-q}\right] ds + \int_{t_1}^{t_2}(t_2-s)^{p-q} ds\right) \\
&\quad + \frac{\Gamma(p)\left(t_2^{p-q} - t_1^{p-q}\right)}{|\Delta| \Gamma(p-q+1)}\left(\frac{C_1}{\Gamma(p+1)} + \frac{C_2 |A| \xi^q}{\Gamma(q+1) q^\rho}\right. \\
&\quad \left. + \frac{C_2 |A| (q-1)^{-\rho} \xi^{q-1}}{\Gamma(q+1)} + \frac{C_1 |B| \xi^p}{\Gamma(p+1) p^\sigma}\right) \\
&\leq \frac{\left(t_2^{p-q+1} - t_1^{p-q+1}\right) C_1}{\Gamma(p-q+2)} + \frac{\Gamma(p)\left(t_2^{p-q} - t_1^{p-q}\right)}{|\Delta| \Gamma(p-q+1)}\left[\frac{C_1}{\Gamma(p+1)} + \frac{C_2 |A| \xi^q}{\Gamma(q+1) q^\rho}\right. \\
&\quad \left. + |A| (q-1)^{-\rho} \xi^{q-1}\left(\frac{C_2}{\Gamma(q+1)} + \frac{C_1 |B| \xi^p}{\Gamma(p+1) p^\sigma}\right)\right].
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&|D^{q-1} \Lambda_1(x,y)(t_2) - D^{q-1} \Lambda_1(x,y)(t_1)| \\
&\quad \leq \frac{1}{\Gamma(p-q+1)} \int_0^{t_1}\left((t_2-s)^{p-q} - (t_1-s)^{p-q}\right)\left| f\left(s, x(s), y(s), {}_{RL} D^{p-1} y(s)\right)\right| ds \\
&\quad + \frac{1}{\Gamma(p-q+1)} \int_{t_1}^{t_2}(t_2-s)^{p+q}\left| f\left(s, x(s), y(s), {}_{RL} D^{p-1} y(s)\right)\right| ds \\
&\quad + \frac{\Gamma(p)\left(t_2^{p-q} - t_1^{p-q}\right)}{|\Delta| \Gamma(p-q+1)}\left\{{}_{RL} I^p \left| f\left(s, x(s), y(s), {}_{RL} D^{p-1} y(s)\right)(1)\right|\right.
\end{aligned}
$$

With the same arguments, we can prove that

$$
\begin{aligned}
&|\Lambda_2(x,y)(t_2) - \Lambda_2(x,y)(t_1)| \\
&\leq \frac{1}{\Gamma(q)} \int_0^{t_1}\left((t_2-s)^{q-1} - (t_1-s)^{q-1}\right)\left| g\left(s, x(s), y(s), {}_{RL} D^{q-1} x(s)\right)\right| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2}(t_2-s)^{q-1}\left| g\left(s, x(s), y(s), {}_{RL} D^{q-1} x(s)\right)\right| ds
\end{aligned}
$$

$$+\frac{t_2^{q-1}-t_1^{q-1}}{|\Delta|}\left(\frac{C_2}{\Gamma(q+1)}+\frac{C_1|B|\xi^p}{\Gamma(p+1)p^{\sigma}}+\frac{C_1|B|(p-1)^{-\sigma}\xi^{p-1}}{\Gamma(p+1)}+\frac{C_2|A|\xi^q}{\Gamma(q+1)q^{\rho}}\right)$$
$$\leq\frac{\left(t_2^q-t_1^q\right)C_2}{\Gamma(q+1)}+\frac{t_2^{q-1}-t_1^{q-1}}{|\Delta|}\left[\frac{C_2}{\Gamma(q+1)}+\frac{C_1|B|\xi^p}{\Gamma(p+1)p^{\sigma}}\right.$$
$$\left.+|B|(p-1)^{-\sigma}\xi^{p-1}\left(\frac{C_1}{\Gamma(p+1)}+\frac{C_2|A|\xi^q}{\Gamma(q+1)q^{\rho}}\right)\right]$$

and

$$\left|D^{p-1}\Lambda_2(x,y)(t_2)-D^{p-1}\Lambda_2(x,y)(t_1)\right|$$
$$\leq\frac{1}{\Gamma(q-p+1)}\int_0^{t_1}\left[(t_2-s)^{q-p}-(t_1-s)^{q-p}\right]\left|g\left(s,x(s),y(s),{}_{RL}D^{q-1}x(s)\right)\right|ds$$
$$+\frac{1}{\Gamma(q-p+1)}\int_{t_1}^{t_2}(t_2-s)^{q-p}\left|g\left(s,x(s),y(s),{}_{RL}D^{q-1}x(s)\right)\right|ds$$
$$+\frac{t_2^{q-p}-t_1^{q-p}}{|\Delta|}\left(\frac{C_2}{\Gamma(q+1)}+\frac{C_1|B|\xi^p}{\Gamma(p+1)p^{\sigma}}+\frac{C_1|B|(p-1)^{-\sigma}\xi^{p-1}}{\Gamma(p+1)}+\frac{C_2|A|\xi^q}{\Gamma(q+1)q^{\rho}}\right)$$
$$\leq\frac{\left(t_2^{q-p+1}-t_1^{q-p+1}\right)C_2}{\Gamma(q-p+2)}+\left[\frac{C_2}{\Gamma(q+1)}+\frac{C_1|B|\xi^p}{\Gamma(p+1)p^{\sigma}}\right.$$
$$\left.+|B|(p-1)^{-\sigma}\xi^{p-1}\left(\frac{C_1}{\Gamma(p+1)}+\frac{C_2|A|\xi^q}{\Gamma(q+1)q^{\rho}}\right)\right].$$

As $t_1\to t_2$, we obtain

$$|\Lambda_1(x,y)(t_2)-\Lambda_1(x,y)(t_1)|\to 0,\ \left|D^{q-1}\Lambda_1(x,y)(t_2)-D^{q-1}\Lambda_1(x,y)(t_1)\right|\to 0,$$
$$|\Lambda_2(x,y)(t_2)-\Lambda_2(x,y)(t_1)|\to 0,\ \left|D^{p-1}\Lambda_2(x,y)(t_2)-D^{p-1}\Lambda_2(x,y)(t_1)\right|\to 0.$$

Then, $|\Lambda(x,y)(t_2)-\Lambda(x,y)(t_1)|\to 0$.
Therefore, $\Lambda(\Omega_R)$ is equicontinuous, also it is uniformly bounded as $\Lambda(\Omega_R)\subset\Omega_R$. By Arzela-Ascoli theorem, we conclude that Λ is a completely continuous operator. Hence, by the Schauder fixed point theorem, we state that (1.1) has a solution on $[0,1]$. □

In the following second result, we prove the existence of solutions via the Leray-Schauder degree. We have

Theorem 3.2. *Let $f,g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ be two continuous functions and assume that:*

(H_2) *There exist constants $\gamma_i,\eta_i\geq 0, i=1,2,3$ and $D_1,D_2>0$, such that*

$$|f(t,x_1,x_2,x_3)|\leq\sum_{i=1}^{3}\gamma_i|x_i|+D_1 \text{ for all } (t,x_1,x_2,x_3)\in[0,1]\times\mathbb{R}^3,$$
$$|g(t,x_1,x_2,x_3)|\leq\sum_{i=1}^{3}\eta_i|x_i|+D_2 \text{ for all } (t,x_1,x_2,x_3)\in[0,1]\times\mathbb{R}^3.$$

In addition, assume that $(K_1+L_2)\sum_{i=1}^{3}\gamma_i+(K_2+L_1)\sum_{i=1}^{3}\eta_i<1$, *where* $K_j\,(j=1,2)$ *are given by* $(3.7),(3.8)$ *respectively and* $L_j\,(j=1,2)$ *are given by* $(3.9),(3.10)$ *respectively*.
Then, (1.1) *has a least one solution on* $[0,1]$.

Proof. We define the operator $\Lambda=(\Lambda_1,\Lambda_2):E\times F\to E\times F$ as in $(3.1),(3.2)$ and we consider the fixed point problem

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \Lambda_1(x,y) \\ \Lambda_2(x,y) \end{pmatrix}. \tag{3.11}$$

Now, we prove that there exists a fixed point $(x,y)\in E\times F$, satisfying (3.11).
It is sufficient to show that $\Lambda:\overline{\Omega}_R\to E\times F$ satisfies

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\neq\begin{pmatrix} \mu\Lambda_1(x,y) \\ \mu\Lambda_2(x,y) \end{pmatrix},\ \forall(x(t),y(t))\in\partial\Omega_R,\forall\mu\in[0,1], \tag{3.12}$$

where $\Omega_R=\left\{(x,y)\in E\times F;\ \|(x,y)\|_{E\times F}\leq R\right\}$. We define $S(\mu,(x(t),y(t)))=\mu\Lambda(x(t),y(t))$. As it is shown in theorem (3.2), the operator Λ is continuous, uniformly bounded and equicontinuous. Then by the Arzela-Ascoli Theorem, a continuous map T_μ defined by

$$\begin{aligned} T_\mu(x,y)&=(x,y)-S(\mu,(x(t),y(t)))\\ &=(x,y)-\mu\Lambda(x(t),y(t)), \end{aligned}$$

is completely continuous. If (3.12) is true, then the following Lery-Schauder degrees are well defined and by the homotopic invariance of topological degree, it follows that:

$$\begin{aligned} \deg(T_\mu,\Omega_R,0)&=\deg(I-\mu\Lambda,\Omega_R,0)=\deg(T_1,\Omega_R,0)\\ &=\deg(T_0,\Omega_R,0)=\deg(I,\Omega_R,0)=1\\ &\neq 0,\quad 0\in\Omega_R, \end{aligned}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $T_1(x,y)=(x,y)-\Lambda(x,y)=0$ for al least one $(x,y)\in\Omega_R$. In order to prove (3.12), we assume that $(x,y)=\mu\Lambda(x,y)$ for some $\mu\in[0,1]$ and for all $t\in[0,1]$. Then

$$\begin{aligned} &|\Lambda_1(x,y)(t)|\\ &\leq {}_{RL}I^p\left(\sum_{i=1}^{3}\gamma_i|x_i|+D_1\right)(1)+\tfrac{1}{|\Delta_1|}\left\{{}_{RL}I^p\left(\sum_{i=1}^{3}\gamma_i|x_i|+D_1\right)(1)\right.\\ &\quad+|A|\ {}_HI^\rho\ {}_{RL}I^q\left(\sum_{i=1}^{3}\eta_i|x_i|+D_2\right)(\xi) \end{aligned}$$

$$+|A|(q-1)^{-\rho}\xi^{q-1}\left[{}_{RL}I^{q}\left(\sum_{i=1}^{3}\eta_i|x_i|+D_2\right)(1)\right.$$
$$\left.\left.+|B|\ {}_{H}I^{\sigma}\ {}_{RL}I^{p}\left(\sum_{i=1}^{3}\gamma_i|x_i|+D_1\right)(\xi)\right)\right]\Bigg\}$$
$$\leq\sum_{i=1}^{3}\gamma_i|x_i|\left(\frac{1}{\Gamma(p+1)}+\frac{1}{|\Delta|\Gamma(p+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{|\Delta|\Gamma(p+1)p^{\sigma}}\right)$$
$$+D_1\left(\frac{1}{\Gamma(p+1)}+\frac{1}{|\Delta|\Gamma(p+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{|\Delta|\Gamma(p+1)p^{\sigma}}\right)$$
$$+\sum_{i=1}^{3}\eta_i|x_i|\left(\frac{|A|\xi^{q}}{|\Delta|\Gamma(q+1)q^{\rho}}+\frac{|A|(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)}\right)$$
$$+D_2\left(\frac{|A|\xi^{q}}{|\Delta|\Gamma(q+1)q^{\rho}}+\frac{|A|(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)}\right),$$

and

$$|D^{q-1}\Lambda_1(x,y)(t)|$$
$$\leq\sum_{i=1}^{3}\gamma_i|x_i|\left(\frac{1}{\Gamma(p-q+2)}+\frac{1}{p|\Delta|\Gamma(p-q+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{p^{\sigma+1}|\Delta|\Gamma(p-q+1)}\right)$$
$$+D_1\left(\frac{1}{\Gamma(p-q+2)}+\frac{1}{p|\Delta|\Gamma(p-q+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{p^{\sigma+1}|\Delta|\Gamma(p-q+1)}\right)$$
$$+\sum_{i=1}^{3}\eta_i|x_i|\left(\frac{|A|\Gamma(p)\xi^{q}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)q^{\rho}}+\frac{|A|\Gamma(p)(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)}\right)$$
$$+D_2\left(\frac{|A|\Gamma(p)\xi^{q}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)q^{\rho}}+\frac{|A|\Gamma(p)(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)}\right).$$

Thus,

$$||\Lambda_1(x,y)||$$
$$=\sup_{t\in[0,1]}|\Lambda_1(x,y)(t)|+\sup_{t\in[0,1]}\left|D^{q-1}\Lambda_1(x,y)(t)\right|$$
$$\leq\sum_{i=1}^{3}\gamma_i|x_i|\left[\frac{1}{\Gamma(p+1)}+\frac{1}{|\Delta|\Gamma(p+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{|\Delta|\Gamma(p+1)p^{\sigma}}\right.$$
$$\left.+\frac{1}{\Gamma(p-q+2)}+\frac{1}{p|\Delta|\Gamma(p-q+1)}+\frac{|AB|\xi^{p+q-1}(q-1)^{-\rho}}{p^{\sigma+1}|\Delta|\Gamma(p-q+1)}\right]$$
$$+\sum_{i=1}^{3}\eta_i|x_i|\left[\frac{|A|\xi^{q}}{|\Delta|\Gamma(q+1)q^{\rho}}+\frac{|A|(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)}\right.$$

$$
\begin{aligned}
&+\frac{|A|\Gamma(p)\xi^{q}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)q^{\rho}}+\frac{|A|\Gamma(p)(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)\Gamma(p-q+1)}\Bigg]\\
&+D_1\Bigg[\frac{1}{\Gamma(p+1)}+\frac{1}{|\Delta|\Gamma(p+1)}+\frac{|AB|(q-1)^{-\rho}\xi^{q-1}p^{-\sigma}\xi^{p}}{|\Delta_1|\Gamma(p+1)}\\
&+\frac{1}{\Gamma(p-q+2)}+\frac{1}{p|\Delta|\Gamma(p-q+1)}+\frac{|AB|p^{-\sigma}(q-1)^{-\rho}\xi^{p+q-1}}{p|\Delta|\Gamma(p-q+1)}\Bigg]\\
&+D_2\Bigg[\frac{q^{-\rho}\xi^{q}}{|\Delta|\Gamma(q+1)}+\frac{|A|(q-1)^{-\rho}\xi^{q-1}}{|\Delta|\Gamma(q+1)}\\
&+\frac{|A|\Gamma(p)}{|\Delta|\Gamma(p-q+1)}\left(\frac{\xi^{q}q^{-\rho}}{\Gamma(q+1)}+\frac{(q-1)^{-\rho}\xi^{q-1}}{\Gamma(q+1)}\right)\Bigg]\\
\le{}& K_1\left(\sum_{i=1}^{3}\gamma_i|x_i|+D_1\right)+K_2\left(\sum_{i=1}^{3}\eta_i|x_i|+D_2\right).
\end{aligned}
$$

Thus, $||(x,y)|| \le \dfrac{\sum_{j=1}^{2}\left(K_jD_j+L_jN_j\right)}{1-\left((K_1+L_2)\sum_{i=1}^{3}\gamma_i+(K_2+L_1)\sum_{i=1}^{3}\eta_i\right)}$.

If $R=\dfrac{\sum_{j=1}^{2}\left(K_jD_j+L_jN_j\right)}{1-\left((K_1+L_2)\sum_{i=1}^{3}\gamma_i+(K_2+L_1)\sum_{i=1}^{3}\eta_i\right)}+1$, then (3.12) is valid. □

4. Illustrative Examples

In this section, we give two examples illustrating our results.

Example 4.1

Consider the following problem:

$$
\begin{cases}
{}_{RL}D^{\frac{5}{3}}x(t)=\frac{x(t)^{\frac{1}{3}}}{38e^{t}}+\frac{y(t)^{\frac{1}{4}}}{49(t+1)}+\frac{\left({}_{RL}D^{\frac{2}{3}}y(t)\right)^{\frac{1}{5}}}{45e^{t+1}}+e^{t}, t\in[0,1],\\
{}_{RL}D^{\frac{3}{2}}y(t)=\frac{\sin(t)x(t)^{\frac{2}{3}}}{35\pi^{2}e^{t}}+\frac{y(t)^{\frac{2}{5}}}{27e\sqrt{t+1}}+\frac{\left({}_{RL}D^{\frac{1}{2}}x(t)\right)^{\frac{2}{7}}}{45e^{t^{2}-1}}+t+1, t\in[0,1],\\
x(0)=0,\ x(1)=\sqrt{e}\ {}_{H}I^{\frac{3}{5}}y\left(\frac{\sqrt{e}}{2}\right),\\
y(0)=0,\ y(1)=\frac{1}{e}\ {}_{H}I^{\frac{e}{4}}x\left(\frac{\sqrt{e}}{2}\right),
\end{cases}
\tag{4.1}
$$

where $p=\frac{5}{3}, q=\frac{3}{2}, \rho=\frac{3}{5}, \sigma=\frac{e}{4}, \xi=\frac{\sqrt{e}}{2}, A=\sqrt{e}, B=\frac{1}{e}, r_1=\frac{1}{3}, r_2=\frac{1}{4}, r_3=\frac{1}{5},$ $\rho_1=\frac{2}{3}, \rho_2=\frac{2}{5}, \rho_3=\frac{2}{7}$

and $f\left(t,x(t),y(t),D^{p-1}y(t)\right)=\frac{x(t)^{\frac{1}{3}}}{38e^{t}}+\frac{y(t)^{\frac{1}{4}}}{49\sqrt{t+1}}+\frac{\left({}_{RL}D^{\frac{2}{3}}y(t)\right)^{\frac{1}{5}}}{45e^{t+1}}+e^{t},$

$g\left(t,x(t),\, y(t),\, D^{q-1}x(t)\right) = \frac{\sin(t)x(t)^{\frac{2}{3}}}{35\pi^2 e^t} + \frac{y(t)^{\frac{2}{5}}}{27e\sqrt{t+1}} + \frac{\left({}_{RL}D^{\frac{1}{2}}x(t)\right)^{\frac{2}{7}}}{45e^{t^2+1}} + t + 1.$

Since $|f(t,x_1,x_2,x_3)| \leq h(t) + \sum_{i=1}^{3}\lambda_i |x_i|^{r_i}$, $|g(t,x_1,x_2,x_3)| \leq l(t) + \sum_{i=1}^{3}\omega_i |x_i|^{\rho_i}$,
where, $\lambda_1 = \frac{1}{38}, \lambda_2 = \frac{1}{49}, \lambda_3 = \frac{1}{45e}$, $\omega_1 = \frac{1}{35\pi^2}$, $\omega_2 = \frac{1}{27e}$, $\omega_3 = \frac{1}{45e}$
and $h(t) = e^t$, $l(t) = t+1$ are nonnegative functions
and $1 - AB(p-1)^{-\sigma}(q-1)^{-\rho}\xi^{p+q-2} = 3.3340 \times 10^{-2} \neq 0.$
Thus, Theorem 3.2 implies that (4.1) has at least one solution on $[0,1]$.

Example 4.2

Consider the following coupled problem:

$$\begin{cases} {}_{RL}D^{\sqrt{\frac{10}{7}}}x(t) = \frac{x(t)}{41e^{t+1}} + \frac{y(t)}{55e^2(t+1)} + \frac{{}_{RL}D^{\sqrt{\frac{10}{7}}-1}y(t)}{30(t^2+1)} + \sqrt{\pi}, t \in [0,1] \\ {}_{RL}D^{\frac{e}{2}}y(t) = \frac{\sin(t)x(t)}{70(t+2)} + \frac{y(t)}{57(t+1)} + \frac{{}_{RL}D^{\frac{e-2}{2}}x(t)}{66e^{t^2+1}} + \frac{1}{e}, t \in [0,1], \\ x(0) = 0, \;\; x(1) = {}_H I^{\frac{\sqrt{\pi}}{3}} y\left(\frac{2}{e}\right), \\ y(0) = 0, \;\; y(1) = {}_H I^{\frac{\sqrt{e}}{5}} x\left(\frac{2}{e}\right), \end{cases} \tag{4.2}$$

where $p = \sqrt{\frac{10}{7}}, q = \frac{e}{2}, \rho = \frac{\sqrt{\pi}}{3}, \sigma = \frac{\sqrt{e}}{5}, \xi = \frac{2}{e}, A = B = 1$

and $f\left(t,x(t),y(t),D^{p-1}y(t)\right) = \frac{x(t)}{41e^{t+1}} + \frac{y(t)}{55e^2(t+1)} + \frac{{}_{RL}D^{\sqrt{\frac{10}{7}}-1}y(t)}{30(t^2+1)} + \sqrt{\pi},$

$g\left(t,x(t),y(t),D^{p-1}y(t)\right) = \frac{\sin(t)x(t)}{70(t+2)} + \frac{y(t)}{57(t+1)} + \frac{{}_{RL}D^{\frac{e}{2}-1}x(t)}{66e^{t^2+1}} + \frac{1}{e},$

where $\sum_{i=1}^{3}\gamma_i = 4.4766 \times 10^{-2}$, $\sum_{i=1}^{3}\eta_i = 4.6982 \times 10^{-2}$, $K_1 = 4.0509$, $L_1 = 3.6062$, $K_2 = 1.9957, L_2 = 2.4533.$

Therefore, it is found that $(K_1 + L_2)\sum_{i=1}^{3}\gamma_i + (K_2 + L_1)\sum_{i=1}^{3}\eta_i = 0.55436 < 1.$

Hence, all conditions of theorem 3.3 are true, wich implies that (4.2) has at least one solution on $[0,1]$.

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