SARAJEVO JOURNAL OF MATHEMATICS
Vol.10 (22) (2014), 55–60 DOI: 10.5644/SJM.10.1.07

ON UNIFIED CLASS OF γ-SPIRALLIKE FUNCTIONS OF COMPLEX ORDER

T. M. SEOUDY

ABSTRACT. In this paper, we obtain a necessary and sufficient condition for functions to be in an unified class of γ-spirallike functions of complex order. Some of our results generalize previously known results.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{\,n}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that f and g are analytic in $\mathbb{U}$. We say that the function f is subordinate to g in $\mathbb{U}$, or g superordinate to$f$ in $\mathbb{U}$, and we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function ω in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. If g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true (see [5] and [6]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on ϕ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $\mathcal{S}^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \tag{1.2}$$

and $\mathcal{C}(\phi)$ class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z). \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30D30, 33D20.
Key words and phrases. Analytic function, univalent, starlike, convex, subordination.

These classes were introduced and studied by Ma and Minda [4]. Ravichandran et al. [10] defined classes $\mathcal{S}_b^*(\phi)$ and $\mathcal{C}_b(\phi)$ of complex order defined as follows :

$$\mathcal{S}^*(\phi;b)=\left\{f\in\mathcal{A}:1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\phi(z)\quad(b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\})\right\} \tag{1.4}$$

and

$$\mathcal{C}(\phi;b)=\left\{f\in\mathcal{A}:1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\prec\phi(z)\quad(b\in\mathbb{C}^*)\right\}. \tag{1.5}$$

From (1.4) and (1.5), we have

$$f\in\mathcal{C}(\phi;b)\Longleftrightarrow zf'\in\mathcal{S}^*(\phi;b).$$

Now, we introduce a more general class of γ-spirallike functions of complex order $\mathcal{T}(\phi;\lambda,b)$ as follow:

Definition 1. Let $\phi(z)$ be an analytic function with positive real part on ϕ with $\phi(0)=1,\phi'(0)>0$ which maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $\mathcal{T}^\gamma(\phi;\lambda,b)$ consists of all analytic functions $f\in\mathcal{A}$ satisfying:

$$1+\frac{e^{i\gamma}}{b\cos\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\prec\phi(z) \tag{1.6}$$

$$\left(|\gamma|<\frac{\pi}{2},b\in\mathbb{C}^*;\lambda\geq 0\right).$$

We note that

(i) $\mathcal{T}^\gamma\left(\frac{1+z}{1-z};0,b\right)=\mathcal{S}^\gamma(b)$ and $\mathcal{T}^\gamma\left(\frac{1+z}{1-z};1,b\right)=\mathcal{C}^\gamma(b)\left(|\gamma|<\frac{\pi}{2},b\in\mathbb{C}^*\right)$ (Al-Oboudi and Haidan [1] and Aouf et al. [2]);

(ii) $\mathcal{T}^0(\phi;0,b)=\mathcal{S}^*(\phi;b)$ and $\mathcal{T}^0(\phi;1,b)=\mathcal{C}(\phi;b)$ (Ravichandran et al. [10]);

(iii) $\mathcal{T}^0(\phi;0,1)=\mathcal{S}^*(\phi)$ and $\mathcal{T}^0(\phi;1,1)=\mathcal{C}(\phi)$ (Ma and Minda [4]);

(iv) $\mathcal{T}^0\left(\frac{1+(1-2\alpha)z}{1-z};0,b\right)=\mathcal{S}_\alpha^*(b)$ and $\mathcal{T}^0\left(\frac{1+(1-2\alpha)z}{1-z};1,b\right)=\mathcal{C}_\alpha(b)\,(0\leq\alpha<1;b\in\mathbb{C}^*)$ (Frasin [3]);

(v) $\mathcal{T}^0\left(\frac{1+z}{1-z};0,b\right)=\mathcal{T}^0\left(\frac{1+(2b-1)z}{1-z};0,1\right)=\mathcal{S}^*(b)\,(b\in\mathbb{C}^*)$ (Nasr and Aouf [8] and Wiatrowski [15]);

(vi) $\mathcal{T}^0\left(\frac{1+z}{1-z};1,b\right)=\mathcal{T}^0\left(\frac{1+(2b-1)z}{1-z};1,1\right)=\mathcal{C}(b)\,(b\in\mathbb{C}^*)$ (Nasr and Aouf [7] and Wiatrowski [15]);

(vii) $\mathcal{T}^0\left(\frac{1+z}{1-z};0,1-\alpha\right)=\mathcal{T}^0\left(\frac{1+(1-2\alpha)z}{1-z};0,1\right)=\mathcal{S}^*(\alpha)$ and $\mathcal{T}^0\left(\frac{1+z}{1-z};1,1-\alpha\right)=\mathcal{T}^0\left(\frac{1+(1-2\alpha)z}{1-z};1,1\right)=\mathcal{C}(\alpha)\,(0\leq\alpha<1)$ (Robertson [11]).

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $\mathcal{T}^{\gamma}(\phi;\lambda,b)$ which we have defined. The motivation of this paper is to generalize the results obtained by Ravichandran et al. [10], Aouf et al. [2], Srivastava and Lashin [14] and also Obradovic et al. [9].

2. Main results

Unless otherwise mentioned, we assume throughout the sequel that $|\gamma| < \frac{\pi}{2}, b \in \mathbb{C}^*, \lambda \geq 0$ and all powers are understood as principle values. To prove our main result, we need the following lemmas.

Lemma 1 ([12]). *Let ϕ be a convex function defined on $\mathbb{U}$, $\phi(0) = 1$. Define $F(z)$ by*

$$F(z) = z\ \exp\left(\int_0^z \frac{\phi(t)-1}{t}dt\right). \tag{2.1}$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in $\mathbb{U}$. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z) \tag{2.2}$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\frac{p(tz)}{p(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.3}$$

Lemma 2 ([6]). *Let $q(z)$ be univalent in $\mathbb{U}$ and let $\varphi(z)$ be analytic in a domain containing $q(\mathbb{U})$. If $\frac{zq'(z)}{q(z)}$ is starlike, then*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1. Let $\phi(z)$ and $F(z)$ be as in Lemma 1. The function $f \in \mathcal{T}^{\gamma}(\phi;\lambda,b)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\left[\left(\frac{sf(tz)}{tf(sz)}\right)^{1-\lambda}\left(\frac{f'(tz)}{f'(sz)}\right)^{\lambda}\right]^{\frac{e^{i\gamma}}{b\cos\gamma}} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.4}$$

Proof. Define the function $p(z)$ by

$$p(z) = \left[\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\right]^{\frac{e^{i\gamma}}{b\cos\gamma}} \quad (z \in \mathbb{U}). \tag{2.5}$$

Taking logarithmic derivative of (2.5), we get

$$1+\frac{zp'(z)}{p(z)}=1+\frac{e^{i\gamma}}{b\cos\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right].$$

Since $f\in\mathcal{T}(\phi;\lambda,b)$, then we have

$$1+\frac{zp'(z)}{p(z)}\prec\phi(z)$$

and the result now follows from Lemma 1. □

Putting $\lambda=\gamma=0$ in Theorem 1, we obtain the following result of Shanmugam et al. [13].

Corollary 1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1. The function $f\in\mathcal{S}^*(\phi;b)$ if and only if for all $|s|\le 1$ and $|t|\le 1$, we have*

$$\left(\frac{sf(tz)}{tf(sz)}\right)^{\frac{1}{b}}\prec\frac{sF(tz)}{tF(sz)}. \tag{2.6}$$

For $\lambda=1$ and $\gamma=0$ in Theorem 1, we obtain the following result of Shanmugam et al. [13].

Corollary 2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1. The function $f\in\mathcal{C}(\phi;b)$ if and only if for all $|s|\le 1$ and $|t|\le 1$, we have*

$$\left(\frac{f'(tz)}{f'(sz)}\right)^{\frac{1}{b}}\prec\frac{sF(tz)}{tF(sz)}. \tag{2.7}$$

Theorem 2. Let $\phi(z)$ be starlike with respect to 1 and $F(z)$ given by (2.1) be starlike. If $f\in\mathcal{T}^{\gamma}(\phi;\lambda,b)$, then we have

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\prec\left(\frac{F(z)}{z}\right)^{\frac{b\cos\gamma}{e^{i\gamma}}}. \tag{2.8}$$

Proof. Let $p(z)$ be given by (2.5) and $q(z)$ be given by

$$q(z)=\frac{F(z)}{z}\qquad(z\in\mathbb{U}). \tag{2.9}$$

After a simple computation we obtain

$$1+\frac{zp'(z)}{p(z)}=1+\frac{e^{i\gamma}}{b\cos\gamma}\left[\left(1-\lambda\right)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{zf'(z)}\right)-1\right]$$

and

$$\frac{zq'(z)}{q(z)}=\frac{zF'(z)}{F(z)}-1=\phi(z)-1.$$

Since $f \in \mathcal{T}^{\gamma}(\phi; \lambda, b)$, we have

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}.$$

The result now follows by an application of Lemma 2. □

For $\phi(z) = \dfrac{1+z}{1-z}$ and $\lambda = 0$ in Theorem 2, we get the following result of Aouf et al. [2].

Corollary 3. *If $f \in \mathcal{S}^{\gamma}(b)$, then we have*

$$\frac{f(z)}{z} \prec (1-z)^{-2be^{-i\gamma}\cos\gamma}.$$

For $\phi(z) = \dfrac{1+z}{1-z}$ and $\lambda = 1$ in Theorem 2, we get the following result of Aouf et al. [2].

Corollary 4. *If $f \in \mathcal{C}^{\gamma}(b)$, then we have*

$$f'(z) \prec (1-z)^{-2be^{-i\gamma}\cos\gamma}.$$

Putting $\lambda = \gamma = 0$ in Theorem 2, we obtain the following results of Shanmugam et al. [13].

Corollary 5. *Let $\phi(z)$ be starlike with respect to 1 and $F(z)$ given by* (2.1) *be starlike. If $f \in \mathcal{S}^{*}(\phi; b)$, then we have*

$$\frac{f(z)}{z} \prec \left(\frac{F(z)}{z}\right)^{\frac{b\cos\gamma}{e^{i\gamma}}}.$$

Taking $\phi(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we get the following corollary:

Corollary 6. *If $f \in \mathcal{T}^{\gamma}\left(\dfrac{1+Az}{1+Bz}; \lambda, b\right)$ $(-1 \leq B < A \leq 1)$, then we have*

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec (1+Bz)^{\frac{(A-B)b\cos\gamma}{Be^{i\gamma}}} \quad (B \neq 0).$$

For $\phi(z) = \dfrac{1+z}{1-z}$ and $\lambda = \gamma = 0$ in Theorem 2, we get the following result of Obradovic et al. [9], and Srivastava and Lashin [14].

Corollary 7. *If $f \in \mathcal{S}^{*}(b)$, then we have*

$$\frac{f(z)}{z} \prec (1-z)^{-2b}.$$

For $\phi(z) = \dfrac{1+z}{1-z}, \gamma = 0$ and $\lambda = 1$ in Theorem 2, we get the following result of Obradovic et al. [9], and Srivastava and Lashin [14].

Corollary 8. *If $f \in \mathcal{C}(b)$, then we have*

$$f'(z) \prec (1-z)^{-2b} .$$

REFERENCES

[1] F. M. Al-Oboudi and M. M. Haidan, *Spirallike functions of complex order*, J. Nat. Geom., 19 (2000), 53–72.

[2] M. K. Aouf, F. M. Al-Oboudi and M. M. Haidan, *On some results for λ-spirallike and λ-Robertson functions of complex order*, Publ. Instit. Math. Belgrade, 77 (91) (2005), 93–98.

[3] B. A. Frasin, *Family of analytic functions of complex order*, Acta Math. Acad. Paedagog. Nyházi. (N. S.), 22 (2) (2006), 179–191.

[4] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Internat. Press, Cambridge, MA.

[5] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), 157–171.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications, Serieson Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[7] M. A. Nasr and M. K. Aouf, *On convex functions of complex order*, Mansoura Bull. Sci., 8 (1982), 565–582.

[8] M. A. Nasr and M. K. Aouf, *Starlike function of complex order*, J. Nat. Sci. Math., 25 (1985), 1–12

[9] M. Obradovic, M. K. Aouf and S. Owa, *On some results for starlike functions of complex order*, Publ. Inst. Math., Nouv. Ser. 46 (60) (1989), 79–85.

[10] V. Ravichandran, Y. Polatoglu, M. Bolcal and A. Sen, *Certain subclasses of starlike and convex functions of complex order*, Hacettepe J. Math. Stat., 34 (2005), 9–15.

[11] M. S. Robertson, *On the theory of univalent functions*, Ann. Math., 37 (1936), 374–408.

[12] St. Ruscheweyh, *Convolutions in Geometric Function Theory*, Presses Univ. Montreal, Montreal, Que., 1982.

[13] T. N. Shanmugam, S. Sivasubramanian and S. Kavitha, *On certain subclasses of functions of complex order*, South. Asian Bull. Math., 33 (2009), 535–541.

[14] H. M. Srivastava and A.Y. Lashin, *Some applications of the Briot-Bouquetdifferential subordination*, J. Inequal. Pure Appl. Math., 6 (2) (2005), Art. 41, 1–7.

[15] P. Wiatrowski, *On the coefficients of some family of holomorphic functions*, Zeszyty Nauk. Uniw. Łódz Nauk. Mat.-Przyrod., 39 (1970), 75–85.

(Received: May 12, 2013)
(Revised: December 13, 2013)

Department of Mathematics
Faculty of Science, Fayoum University
Fayoum 63514, Egypt
tmseoudy@gmail.com