

ON PARAMETER CLASSES OF SOLUTIONS OF THE QUASILINEAR SECOND ORDER DIFFERENTIAL EQUATION

ALMA OMERSPAHIĆ

ABSTRACT. The quasilinear second order differential equation

$$\ddot{x} + (1 + p(x, t)) \dot{x} + p(x, t) x = f(x, t)$$

where $p, f \in C^1(D, \mathbb{R})$, $D = I_x \times I$, $I_x \subseteq \mathbb{R}$ open set, $I = \langle \bar{t}, \infty \rangle$ is under consideration. The paper presents some results on the existence and behavior of parameter classes of solutions of this equation. The qualitative analysis theory and topological retraction method are used. The general results are presented and subsequently certain examples are considered.

1. INTRODUCTION

Let us consider the quasilinear second order differential equation

$$\ddot{x} + (1 + p(x, t)) \dot{x} + p(x, t) x = f(x, t) \quad (1)$$

where $p, f \in C^1(D, \mathbb{R})$, $D = I_x \times I$, $I_x \subseteq \mathbb{R}$ open set, $I = \langle \bar{t}, \infty \rangle$. Let $r_1, r_2 \in C^1(I, \mathbb{R}^+)$. Moreover, p and f satisfy the sufficient conditions for existence and uniqueness of solutions of any Cauchy's problem for equation (1) in D .

By substitution

$$y = x + \dot{x}, \quad (2)$$

equation (1) is transformed into a quasilinear system of equations:

$$\left. \begin{aligned} \dot{x} &= -x + y, \\ \dot{y} &= -p(x, t) y + f(x, t), \\ \dot{t} &= 1. \end{aligned} \right\} \quad (3)$$

2010 *Mathematics Subject Classification.* 34C05.

Key words and phrases. Quasilinear differential equation, parameter classes of solutions, behavior of solutions.

We shall study the equation (1) by means of the equivalent system (3). Let $\Omega = I_x \times I_y \times I$, $I_y \subseteq \mathbb{R}$ be open set. We shall consider the behavior of the integral curve $(x(t), y(t), t)$ of system (3) with respect to the set

$$\omega = \{(x, y, t) \in \Omega : |x| < r_1(t), |y| < r_2(t)\}.$$

The boundary surfaces of ω are:

$$W_1^k = \{(x, y, t) \in Cl\omega \cap \Omega : B_1^k(x, y, t) := (-1)^k x - r_1(t) = 0\}, \quad k = 1, 2,$$

$$W_2^k = \{(x, y, t) \in Cl\omega \cap \Omega : B_2^k(x, y, t) := (-1)^k y - r_2(t) = 0\}, \quad k = 1, 2.$$

Let us denote by ∇B_1^k and ∇B_2^k a vectors of outer normals on surfaces W_i^k , $i = 1, 2$, $k = 1, 2$, respectively. We have:

$$\nabla B_1^k = \left((-1)^k, 0, -r_1'(t) \right),$$

$$\nabla B_2^k = \left(0, (-1)^k, -r_2'(t) \right).$$

The tangent vector field to an integral curve $(x(t), y(t), t)$ of (3) is

$$T = (-x + y, -p(x, t)y + f(x, t), 1).$$

By means of the scalar products $P_i^k(x, y, t) = (\nabla B_i^k, T)$ on surfaces W_i^k , $k = 1, 2$; $i = 1, 2$, we shall establish the behavior of integral curves of (3) with respect to the set ω .

Let us denote by $S^p(I)$, $p \in \{0, 1, 2\}$, a class of solutions $(x(t), y(t), t)$ of the system (3) defined on I , which depends on p parameters. We shall simply say that the class of solutions $S^p(I)$ belongs to the set ω if graphs of functions in $S^p(I)$ are contained in ω . In that case we shall write $S^p(I) \subset \omega$. For $p = 0$ we have the notation $S^0(I)$, which means that there exists at least one solution $(x(t), y(t), t)$ on I of the system (3), whose graph belongs to the set ω .

The results of this paper are based on the following Lemmas (see [3]-[7]).

Lemma 1. *If, for the system (3), the scalar product $P_i^k(x, y, t) < 0$ on $\partial\omega = W_1^1 \cup W_1^2 \cup W_2^1 \cup W_2^2$, then the system (3) has a class of solutions $S^2(I)$ belonging to the set ω for all $t \in I$, i.e. $S^2(I) \subset \omega$.*

Lemma 2. *If, for the system (3), the scalar product $P_i^k(x, y, t) < 0$ on $\partial\omega = W_1^1 \cup W_1^2 \cup W_2^1 \cup W_2^2$, then the system (3) has at least one solution on I whose graph belongs to the set ω for all $t \in I$, i.e. $S^0(I) \subset \omega$.*

Lemma 3. *If, for the system (3), the scalar product $P_i^k(x, y, t) < 0$ on $W_1^1 \cup W_1^2$, and $P_i^k(x, y, t) > 0$ on $W_2^1 \cup W_2^2$ (or reversely), then the system (3) has a class of solutions $S^1(I)$ belonging to the set ω for all $t \in I$, i.e. $S^1(I) \subset \omega$.*

2. MAIN RESULTS

Theorem 1. *Let $r_1, r_2 \in C^1(I, \mathbb{R}^+)$.*

a) *If the conditions*

$$|f(x, t)| < p(x, t) r_2(t) + r_2'(t), \quad (4)$$

$$r_2(t) < r_1(t) + r_1'(t), \quad t \in I \quad (5)$$

are satisfied on D , then the equation (1) has a two-parameters class of solutions $x(t)$ satisfying the conditions

$$|x(t)| < r_1(t), \quad |x(t) + \dot{x}(t)| < r_2(t) \quad \text{for all } t \in I. \quad (6)$$

b) *If the condition (4) and*

$$r_2(t) < -r_1(t) - r_1'(t), \quad t \in I \quad (7)$$

are satisfied on D , then the equation (1) has a one-parameters class of solutions $x(t)$ satisfying the conditions (6).

c) *If the conditions (7) and*

$$|f(x, t)| < -p(x, t) r_2(t) - r_2'(t), \quad (8)$$

are satisfied on D , then at least one solution of the equation (1) satisfies the conditions (6).

Proof. For the scalar products $P_1^k(x, y, t) = (\nabla B_1^k, T)$ on surfaces W_1^k , $k = 1, 2$, and $P_2^k(x, y, t) = (\nabla B_2^k, T)$ on surfaces W_2^k , $k = 1, 2$, we have:

$$P_1^k(x, y, t) = (-1)^k(-x + y) - r_1' = -r_1 + (-1)^k y - r_1',$$

$$P_2^k(x, y, t) = (-1)^k(f(x, t) - p(x, t)y) - r_2' = (-1)^k f(x, t) - p(x, t)r_2 - r_2'.$$

The following estimates for $P_i^k(x, y, t)$, $k = 1, 2$; $i = 1, 2$, are valid, respectively:

a)

$$P_1^k(x, y, t) \leq -r_1 + r_2 - r_1' < 0 \quad \text{on } W_1^k, \quad k = 1, 2,$$

$$P_2^k(x, y, t) \leq |f(x, y)| - p(x, y)r_2 - r_2' < 0 \quad \text{on } W_2^k, \quad k = 1, 2, \quad (9)$$

We have, $P_1^k(x, y, t) < 0$ and $P_2^k(x, y, t) < 0$ in all points of the boundary surfaces of ω . Accordingly, set $\partial\omega \cap \Omega$ is a set of points of strict entrance of integral curves of the system (3) with respect to the sets ω and Ω and system (3) has a class of solutions $S^2(I) \subset \omega$. Consequently, the equation (1) has a two-parameters class of solutions satisfying the conditions (6).

b) The estimates

$$P_1^k(x, y, t) \geq -r_1 - r_2 - r_1' > 0 \quad \text{on } W_1^k, \quad k = 1, 2 \quad (10a)$$

and (9) on W_2^k , $k = 1, 2$, are valid.

We have $P_1^k(x, y, t) > 0$ on W_1^k , $k = 1, 2$, and $P_2^k(x, y, t) < 0$ on W_2^k , $k = 1, 2$. Consequently, $U = (W_1^1 \cup W_1^2) \setminus (W_2^1 \cup W_2^2)$ is a set of strict exit and $V = (W_2^1 \cup W_2^2) \setminus (W_1^1 \cup W_1^2)$ is a set of strict entrance of integral curves of system (3) with respect to the sets ω and Ω . According to Lemma 3. system (3) has a class of solutions $S^1(I) \subset \omega$. Hence, the equation (1) has a one-parameters class of solutions $x(t)$ satisfying the conditions (6).

c) The estimates (10a) are valid on W_1^k , $k = 1, 2$, and

$$P_2^k(x, y, t) \geq -|f(x, y)| - p(x, y) r_2 - r_2' > 0, \quad \text{on } W_2^k, \quad k = 1, 2.$$

Accordingly, W_1^k and W_2^k , $k = 1, 2$, are sets of strict exit of integral curves of system (3) with respect to the sets ω and Ω . Hence, according to T. Ważewski's retraction method (see [8], Lemma 2.), the system (3) has at least one solution belonging to the set ω , i.e. $S^0(I) \subset \omega$. Consequently, the equation (1) has at least one solution satisfying conditions (6). \square

On the basis of Theorem 1. we have the next special results.

Let

$$D = \{(x, t) : |x| \leq c, t \in I = \langle t_0, \infty \rangle\},$$

where $c \in \mathbb{R}^+$.

Corollary 1. *Let $\theta \in \mathbb{R}^+$, $\theta < 1$ and*

$$p(x, t) > 1, \quad |f(x, t)| < ce^{-t} [p(x, t) - 1] \quad \text{on } D,$$

then the equation (1) has a two-parameters class of solutions $x(t)$ which are satisfying the conditions

$$|x(t)| < ce^{-\theta t}, \quad |x(t) + \dot{x}(t)| < ce^{-t} \quad \text{for } t > t_0 > \frac{\ln(1 - \theta)}{\theta - 1}.$$

Proof follows from Theorem 1. case a) for $r_1(t) = ce^{-\theta t}$ and $r_2(t) = ce^{-t}$.

Corollary 2. *If*

$$p(x, t) > \frac{1}{t}, \quad |f(x, t)| < \frac{c}{t} \left(p(x, t) - \frac{1}{t} \right)$$

on D , then the equation (1) has a two-parameters class of solutions $x(t)$ which are satisfying the conditions

$$|x(t)| < \frac{2c}{t}, \quad |x(t) + \dot{x}(t)| < \frac{c}{t}, \quad \text{for all } t \in I = \langle 2, \infty \rangle.$$

Proof follows from Theorem 1. case a) for $r_1(t) = \frac{2c}{t}$ and $r_2(t) = \frac{c}{t}$.

Corollary 3. *If*

$$p(x, t) > 3, \quad |f(x, t)| < ce^{-3t} [p(x, t) - 3] \quad (11)$$

on D , then the equation (1) has a one-parameters class of solutions $x(t)$ which are satisfying the conditions

$$|x(t)| < ce^{-3t}, \quad |x(t) + \dot{x}(t)| < ce^{-3t} \quad \text{for all } t \in \langle 0, \infty \rangle.$$

Proof follows from Theorem 1. case b) for $r_1(t) = r_2(t) = ce^{-3t}$.

Remark. The obtained results also give the possibility to discuss the stability (instability) of solutions of the system (3). For example, under the conditions of Theorem 1 (a), every solution of (3) with initial value in ω is r -stable (stable with the function of stability r), if $r(t)$ tends to zero as $t \rightarrow \infty$ and $r'(t) < 0$, $t \in I$. However, if we consider the case (b), then established solution in ω is r -unstable in case where $r'(t) > 0$, $t \in I$.

REFERENCES

- [1] Zh. Athanassov, *Existence, uniqueness, and stability theorems for bounded solutions of nonlinear differential systems*, Ann. Mat. Pura Appl., 140 (4) (1985), 301-330.
- [2] S. I. Nenov, *Asymptotic behavior of the solutions of a class of second order differential systems*, Electron. J. Differential Equations 2001, No. 12, 9 pp.
- [3] A. Omerspahić, *Retraction method in the qualitative analysis of the solutions of the quasilinear second order differential equation*, Applied Mathematics and Computing (edited by M. Rogina et al.), Department of Mathematics, University of Zagreb, Zagreb, 2001, 165–173.
- [4] B. Vrdoljak and A. Omerspahić, *Qualitative analysis of some solutions of quasilinear system of siferencijal equations*, Applied Mathematics and Scietific Computing (edited by Drmać et al.), Kluwer Academic/Plenum Publishers, 2002, 323–332.
- [5] B.Vrdoljak and A. Omerspahić, *Existence and approximation of solutions of a system of differential equations of Volterra type*, Math. Commun., 9 (2) 2004, 125–139.
- [6] A. Omerspahić and B.Vrdoljak, *On parameter classes of solutions for system of quasilinear differential equations*, Proceeding of the Conference on Applied Mathematics and Scientific Computing, Springer, Dordrecht, 2005, 263–272.
- [7] A. Omerspahić, *Existence and behavior solutions of a system of quasilinear differential equations*, Creat. Math. Inform., 17 (3) 2008, 487–492.
- [8] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des integrales des equations differentielles ordinaires*, Ann. Soc. Polon. Math., 20 (1947), 279–313.

(Received: April 15, 2013)

(Revised: June 19, 2013)

Faculty of Mechanical Engineering
 University of Sarajevo
 Vilsonovo šetalište 9
 71000 Sarajevo
 Bosnia and Herzegovina
 alma.omerspahic@mef.unsa.ba