

## DENSITY OF BACKWARD PATHS ON THE JULIA SET OF A SEMIGROUP

GERARDO R. CHACÓN, RENATO COLUCCI AND DANIELE D'ANGELI

ABSTRACT. A well-known result from the theory of dynamics of semigroups of rational functions is that the backward orbit of almost every complex number accumulates on the Julia set of the semigroup. In this article we significantly improve that result by giving a tree structure to the backward orbit and showing that almost every path of the tree is dense in the Julia set of the semigroup.

### 1. INTRODUCTION

We study the dynamical behavior of a semigroup of rational functions in the complex plane. This is a natural generalization of the study of the dynamics of a given rational function. The main idea in the classical case is to study the iteration of a fixed function, in the semigroup case we have a finite set of functions and in each iterative step we have several functions to compose with.

In [4] it is shown that if a complex number  $z$  belongs to the Julia set  $J(G)$  of a semigroup  $G$  of rational functions, then its backward orbit is dense in  $J(G)$ . This result allows us to make algorithms that approximate the graph of the Julia set.

In this article, we identify the backward orbit of a point  $z$  with a rooted  $k$ -ary tree  $T$  (where  $k$  corresponds to the sum of the degrees of the generators of the semigroup) and define the uniform measure on the boundary  $\partial T$  of the tree. Our main result states that the set of (infinite) paths in  $T$ , which correspond to a set in the complex plane which is dense in the Julia set of  $G$ , has full measure. That is, the full backward orbit is not needed for

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knowing the topology of the Julia set of  $G$  and almost every single (infinite) path of the backward tree is dense in  $J(G)$ . The study of the Julia set of a semigroup has also been considered under the context of time-series analysis in [2].

The article is organized as follows: In Section 2 we give the rigorous definitions and preliminary results. Section 3 is devoted to an example in which an approximate graph of the Julia set is constructed by the implementation of an algorithm that chooses just one path in the backward tree, this inspired the authors theoretical result. Finally, in Section 4 the main theorem is proven by using a combinatorial argument.

## 2. PRELIMINARIES

Given a finite set  $\{g_1, \dots, g_n\}$  of non-constant rational functions defined on the Riemann sphere  $\hat{\mathbb{C}}$ , we consider the semigroup  $G := \langle g_1, \dots, g_n \rangle$  generated by the family of non-constant rational functions endowed with the composition. In other words  $G$  is the set of all the possible compositions of the function  $g_1, \dots, g_n$ . The composition of two functions  $f$  and  $f'$  is denoted by  $ff'$ .

The study of dynamics of semigroups was initiated by Hinkkanen and Martin in a series of papers [4, 5, 6]. The main goal of these first works was to extend the classical theory of dynamics to the context of semigroups. The first step is then to generalize the notions of Fatou and Julia sets. Recall that a family of functions  $\mathcal{F}$  is normal in a region  $U \subset \hat{\mathbb{C}}$  if every sequence of elements of  $\mathcal{F}$  contains a subsequence that converges uniformly on every compact subset of  $U$ .

**Definition 1.** Let  $G := \langle g_1, \dots, g_n \rangle$  be a semigroup of rational functions. The *Fatou set*  $F(G)$  of  $G$ , is the set of points in which  $G$  is normal. The *Julia set*  $J(G)$  of  $G$  is the complement of  $F(G)$  in  $\hat{\mathbb{C}}$ ,

$$J(G) = \hat{\mathbb{C}} \setminus F(G).$$

Hence, the Fatou set is the set in which the dynamics of the semigroup is regular and the Julia set is the set in which the dynamics is irregular. Notice that in the case in which the semigroup is generated by a single map,  $G = \langle g \rangle$ , then  $F(G) = F(g)$  and  $J(G) = J(g)$  recovering the definitions of the classical case.

In [4] some fundamental properties of the Julia sets are shown. For example, it is shown that the Julia set is backward invariant, that is,  $g^{-1}(J(G)) \subset J(G)$  for every element  $g \in G$ . It is also shown that

$$J(G) = \overline{\bigcup_{g \in G} J(g)}, \tag{1}$$

and consequently, the topology of the Julia set is given by the topology of the Julia sets of the elements of the semigroup. A natural question is to study what information about the topology of the Julia set can be obtained from the knowledge of the topology of the Julia sets of the generators of the semigroup.

One way of obtaining an approximative picture of the Julia set of a single function is by calculating the *backward orbit*  $O_g^-(z)$  of an element of  $J(g)$ . It is well known (see [3]) that  $O_g^-(z)$  is dense in  $J(g)$  for  $z \in J(g)$  and hence a simple algorithm can be used to represent numerically a picture approximating  $J(g)$ .

For the case of a general semigroup  $G$  of rational functions, it is shown in [4] that if  $z \in J(G)$ , then

$$J(G) = \overline{O^-(z)}, \tag{2}$$

where

$$O^-(z) = \{w \in \hat{\mathbb{C}} : \text{there exists } g \in G \text{ such that } g(w) = z\}.$$

For any point  $z \in J(G)$ , we can give the set  $O^-(z)$  a tree structure by setting  $z$  at the root. For each point  $w \in O^-(z)$  we define  $A_j(w)$  be the set of all inverse images of  $w$  by the generator  $g_j$  with a chosen fixed order, then each element of  $A_j$  is a vertex of the tree having an edge connecting it to the vertex corresponding to  $w$ .

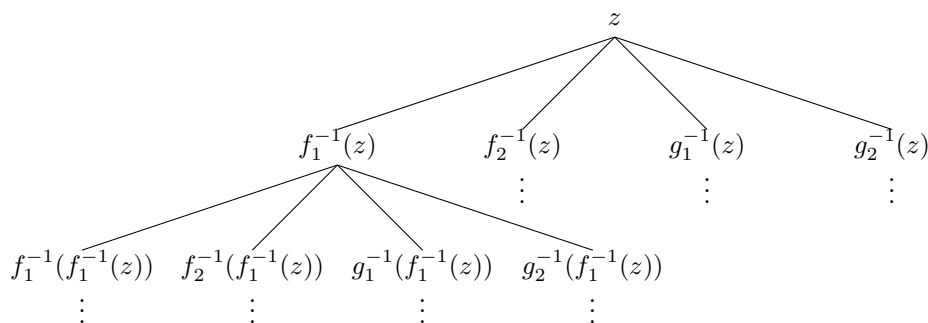


Figure 1: The backward tree for two quadratic generators

For example, consider a semigroup  $G$  generated by two quadratic polynomials  $f$  and  $g$ , then for  $z \in J(G)$  there are four different inverse images, then each of this four inverse images has respectively four inverse images and so on. We describe this construction in Figure 1.

Note that we can identify the set of all infinite paths of a  $k$ -ary tree with the interval  $[0, 1]$  by representing each real number in  $[0, 1]$  by its infinite

$k$ -ary expansion. This way, we can introduce a measure on the set of infinite paths of the tree by using the Lebesgue measure on  $[0, 1]$ .

### 3. AN EXAMPLE

We consider two simple quadratic functions

$$f(z) = z^2 - 1, \quad g(z) = z^2,$$

whose Julia sets are known. These are described in Figure 2. By the following numerical experiment (implemented in Matlab® and Maple®), it is possible to recover the whole Julia set of the semigroup in the following way.

We consider the backward orbits of any compositions of the inverses of  $f$  and  $g$ . Each function has two inverse images:

$$h_0(z) := \sqrt{1+z}, \quad h_1(z) := -\sqrt{1+z}, \quad h_2(z) := \sqrt{z}, \quad h_3(z) := -\sqrt{z},$$

where  $h_0, h_1$  are the inverse images of  $f$  and  $h_2, h_3$  of  $g$ .

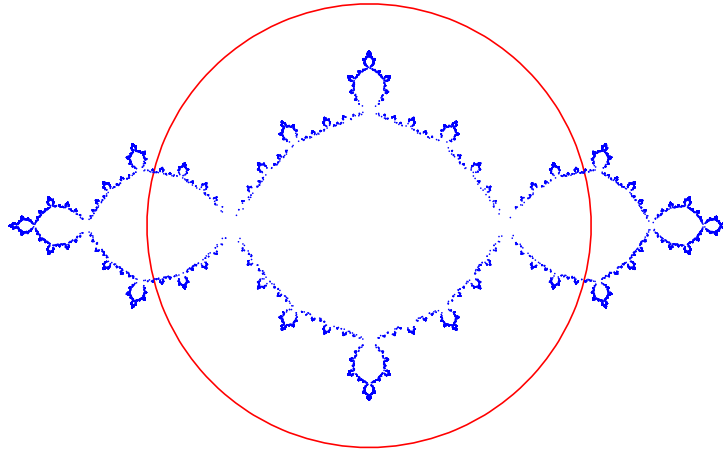


Figure 2: The Julia sets of  $f$  and  $g$

We consider sequences of length  $2^{18}$  of elements of  $H = \{h_0, h_1, h_2, h_3\}$  generated at each step by two random choices, the choice between the functions  $f$  and  $g$  and between their two pre-images.

After several experiments, the picture obtained (see Figure 3) is very similar to the one obtained by considering the whole backward orbit. Moreover it is clear that not every infinite path in the tree gives rise to an approximation of the Julia set of the semigroup. Thus, this allows us to conjecture

that almost every infinite path of the backward tree is dense in the Julia set.

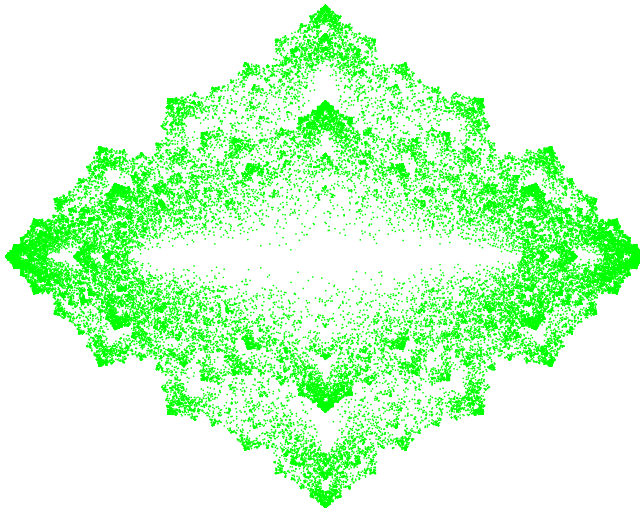


Figure 3: Approximation of the Julia set of the semigroup  $\langle f, g \rangle$

#### 4. MAIN THEOREM

We consider the definition of the backward tree as before and we reformulate the problem in the setting of graph theory.

Let  $G = \langle g_1, \dots, g_n \rangle$  be a semigroup of non-constant rational functions. Suppose the degree of each  $g_i$  is  $k_i$ . For any given  $z \in J(G)$  there is a natural graphical representation of the set

$$O^-(z) = \{w \in \hat{\mathbb{C}} : \text{there exists } g \in G \text{ such that } g(w) = z\},$$

by using a  $k$ -ary regular rooted tree  $T = T(G)$  as represented in Figure 1, where  $k = \sum_{i=1}^n k_i$  is the sum of the degrees of the generators of  $G$ . This is a graph without loops with one vertex of degree  $k$  (the root of  $T$ ) and any other vertex of degree  $k + 1$  (see Figure 1 for an example with  $k = 4$ ).

We define  $H = \{h_1, \dots, h_k\}$  as the ordered set of all inverse functions of the generators of  $G$ , more precisely if we define  $k_0 = 0$ , then the set of the inverse functions of  $g_j$  is given by

$$\left\{ h_{1+\sum_{i=0}^{j-1} k_i}, \dots, h_{\sum_{i=1}^j k_i} \right\}$$

For example, if  $G = \langle g_1, g_2 \rangle$ ,  $\deg(g_1) = 3$  and  $\deg(g_2) = 2$ , then the set of inverses of  $g_1$  is  $\{h_{1+\sum_{i=0}^{1-1} k_i}, \dots, h_{\sum_{i=1}^1 k_i}\} = \{h_1, h_2, h_3\}$  and the set of inverses of  $g_2$  is  $\{h_{1+\sum_{i=0}^{2-1} k_i}, \dots, h_{\sum_{i=1}^2 k_i}\} = \{h_4, h_5\}$ , consequently  $H = \{h_1, h_2, h_3, h_4, h_5\}$ .

The set of finite sequences of length  $l$  of elements in  $H$  is denoted by  $H^l = \{\xi_1 \cdots \xi_l : \xi_i \in H\}$ . It can be identified with the  $l$ -th level of the tree. The set  $H^* = \cup_{l \geq 0} H^l$  represents all the vertices in the tree. In order to simplify the notation, sometimes we will identify a vertex of  $T$  with its corresponding complex number in  $O^-(z)$  and we will denote them with the same symbol.

The boundary  $\partial T$  is the set of right-infinite sequences  $\xi = \xi_1 \xi_2 \cdots$ , such that  $\xi_i \in H$ .

In some cases it will be useful to associate to each vertex of  $T$  a word in a fixed alphabet  $X = \{0, 1, \dots, k-1\}$  in such a way that  $\{0, 1, \dots, k-1\}^l$  corresponds to the  $l$ -th level of  $T$  and  $\emptyset$  denotes the root of the tree. Using this identification, each element in  $\partial T$  has a unique representation as an infinite sequence of elements of  $\{0, 1, \dots, k-1\}$  and consequently it can be associated to a real number in the interval  $[0, 1]$ . Therefore, this gives rise to a bijection  $\Psi : \partial T \rightarrow [0, 1]$  and consequently we can define a measure  $m$  on  $\partial T$  as  $m(E) := \mu(\Psi(E))$  where  $\mu$  denotes the Lebesgue measure on  $[0, 1]$  and  $E$  is any subset of  $\partial T$  such that  $\Psi(E)$  is Lebesgue-measurable.

We denote by  $|v|$  the length of the word  $v$  in the alphabet  $X$  and by  $vH^*$  the set of words with  $v$  as prefix. Notice that  $vH^*$  is the subtree of  $H^*$  rooted at  $v$ . We will also need to define an order relation  $\prec$  in  $H^*$  as follows:

$$u \prec w \Leftrightarrow \exists v \in H^*, |v| > 0 : w = uv.$$

That is,  $u \prec w$  if  $u$  is an initial sub-word of  $w$ .

If we fix a point  $z \in J(G)$ , then to any path  $\xi = \xi_1 \xi_2 \cdots \in \partial T$  we can associate the backward orbit of  $z$  as the following set

$$B_\xi^-(z) := \{z\} \cup \bigcup_{j \geq 1} \{\xi_j \cdots \xi_2 \xi_1(z)\} = \{z\} \cup \bigcup_{j \geq 1} \{\xi_j(\cdots(\xi_2(\xi_1(z)))\cdots)\}$$

The following result is the first generalization of equation (2). It affirms that there exists at least one infinite path  $\xi \in \partial T$ , such that the set  $B_\xi^-(z)$  is dense  $J(G)$ . Although the result will be improved later, the argument might help to understand the idea of the proof for the general result.

**Theorem 1.** *Let  $G = \langle g_1, \dots, g_m \rangle$  be a finitely generated semigroup of non-constant rational functions, and let  $J(G)$  be the corresponding Julia set.*

Then for any  $z \in J(G)$  there exists  $\xi \in \partial T$  such that

$$\overline{B_\xi^-(z)} = J(G).$$

*Proof.* Let  $\{U_n\}_{n \geq 1}$  be a countable basis for  $\mathbb{C}$  covering  $J(G)$ . The idea of the proof consists of constructing the infinite path  $\xi$  that satisfies the conditions of the theorem. Let  $z \in J(G)$ , then since  $\overline{O^-(z)} = J(G)$  there exist  $f_1 \in G$  and  $r_1 \in U_1 \cap J(G)$  such that  $f_1(r_1) = z$ . Observe that since  $r_1 \in J(G)$  then we may use the same argument again to conclude that there exist  $f_2 \in G$  and  $r_2 \in U_2 \cap J(G)$  such that  $f_2(r_2) = r_1$ . We can iterate this method to get  $f_n \in G$  and  $r_n \in U_n \cap J(G)$  such that  $f_n(r_n) = r_{n-1}$  for every  $n$ . We remark that each one of the functions  $f_i$  is a finite composition of functions in the set  $\{g_1, \dots, g_m\}$ . Hence an inverse of the composition  $f_1 f_2 \cdots$  can be identified with an element  $\xi \in \partial T$ . Moreover the set  $\{z\} \cup \{r_n : n \geq 1\} \subseteq \overline{B_\xi^-(z)}$  is dense in  $J(G)$  by construction and therefore the assertion follows.  $\square$

Now we state and prove the main result of the article. It improves the previous theorem since it establishes that if  $z \in J(G)$ , then almost every path in the backward its tree is associated to a dense set  $J(G)$ .

We will need a result proved by Zhou [9] which is an extension of the so-called expansion property to the case of a semigroup, it sharpens a theorem previously proved by Boyd [1]. We define the set  $E(G)$  as the set of elements  $z \in \mathbb{C}$  such that  $O^-(z)$  contains at most two elements. It can be shown that there are at most two elements in  $E(G)$  (See [4]).

**Lemma 2.** *Let  $G = \langle g_1, \dots, g_m \rangle$  be a finitely generated semigroup of non-constant rational functions, and let  $J(G)$  be the corresponding Julia set. If  $U$  is open and intersects  $J(G)$ , and if  $K$  is a compact subset of the complement of  $E(G)$ , then there exists  $f \in G$  such that  $K \subset g(f(U))$  for all  $g \in G$ .*

**Theorem 3.** *Let  $G = \langle g_1, \dots, g_m \rangle$  be a finitely generated semigroup of non-constant rational functions, and let  $J(G)$  the corresponding Julia set. Then for every  $z \in J(G)$  the set of elements  $\xi \in \partial T$  for which  $\overline{B_\xi^-(z)} \neq J(G)$  has zero measure.*

*Proof.* Given  $z \in J(G)$ , we have that by the backward invariance of  $J(G)$ , it follows that  $O^-(z) \subset J(G)$ . Take  $K := \overline{O^-(z)}$  and let  $\{U_n\}_{n \geq 1}$  be a countable basis for  $\mathbb{C}$  covering  $J(G)$ . Then by Lemma 2 we have that for each non-negative integer  $j$  there exists a function  $f_j \in G$  (say of length  $N_j$ ) such that  $\overline{O^-(z)} \subset f_j(U_j)$ .

Now, by making the identification of  $O^-(z)$  with the tree  $T$ , we have that for every vertex  $v$  of  $T$  there exists an inverse of  $f_j$  such that  $f_j^{-1}(v) \in U_j$ . Note that the set of inverses of  $f_j$  can be identified with a subset of vertices of the level  $H^{N_j}$ .

The previous reasoning determines the existence of a function  $s_j : H^* \rightarrow H^{Nj}$  with the property that for every  $v \in H^*$  we have that the vertex  $vs_j(v)$  is associated to a point in  $J(G)$  that belongs to  $U_j$ .

Define the set

$$C_{s_j} := \{x = x_1x_2 \cdots \in \partial T : x_1 \cdots x_m s_j(x_1 \cdots x_m) \not\prec x \forall m\}.$$

Note that if an infinite path  $\xi \in \partial T$  is such that for every  $j$   $\xi \notin C_{s_j}$ , then for every  $j$  there exists a vertex  $v_j$  in  $\xi$  such that  $v_j s_j(v_j)$  is identified with an element of  $U_j$ . Consequently,  $\xi$  is identified with a dense set of  $J(G)$ . Therefore, we need to show that each set  $C_{s_j}$  has zero measure.

From now on we suppress the index  $j$  and rephrase the problem in a combinatorial setting.

Let us identify the backward orbit with a rooted tree  $T$  as described above. To compute the measure of  $C_s$  we can compute  $m(\partial T \setminus C_s) = 1 - m(C_s)$ , given by the sum of the measure of the subtrees rooted at  $ws(w), w \in H^*$ .

First, we reduce the problem to the case in which  $N = 1$ . If  $N > 1$  and  $|X| = k$ , we can define a new alphabet of  $k^N$  symbols and a new rooted tree  $\tilde{T}$  of degree  $k^N$ . Associate each of the vertices of a  $l$ -th level of  $\tilde{T}$  with the vertices in the  $lN$ -th level of  $T$  in a one-to-one manner. This so defined injective function will be denoted as  $\Upsilon$ .

Now define a function  $\tilde{s}$  on the set of the vertices of  $\tilde{T}$  as  $\tilde{s} := s \circ \Upsilon$ . Then it is clear that for every vertex  $v$  of  $\tilde{T}$  we have that  $|\tilde{s}(v)| = 1$  and that  $m(\partial \tilde{T} \setminus C_{\tilde{s}}) \leq m(\partial T \setminus C_s)$ .

Consequently, in what follows we will assume that for every  $v \in H^*$ ,  $|s(v)| = 1$ . Note that to calculate  $m(\partial T \setminus C_s)$ , we remove from  $T$  a tree of measure  $1/k$  rooted at  $s(\emptyset) = x_0$ . Then we remove  $(k-1)$  subtrees of measure  $1/k^2$  rooted at  $x \neq x_0$ . From each one of the subtree rooted at  $x \neq x_0$  we remove only  $(k-1)^2$  subtrees of measure  $1/k^3$  rooted at  $xx_1$  with  $x_1 \neq s(x)$ . Continuing this process we get

$$m(\partial T \setminus C_s) = \frac{1}{k} \sum_{i=0}^{\infty} \left( \frac{k-1}{k} \right)^i = 1.$$

This concludes the proof.  $\square$

#### FINAL REMARK

The present paper shows the utility of the use of combinatorial methods in complex dynamics. We observe that much more could be done by using this algebraic approach. In fact recently, Nekrashevych (see [7, 8]) found a relationship between complex dynamics and self similar groups, making available for the study of the dynamics of one complex function, many tools and techniques from group theory. A natural question to be addressed is



how this techniques generalize to the study of the dynamics of semigroups of rational functions.

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Gerardo R. Chacón and Renato Colucci  
 Departamento de Matemáticas  
 Pontificia Universidad Javeriana  
 Cr 7 # 40-62, Bogotá  
 Colombia  
 chacong@javeriana.edu.co  
 rcolucci@javeriana.edu.co

Daniele D'Angeli  
 Institut für Mathematische Strukturtheorie  
 (Math C)  
 Technische Universität Graz  
 Steyrergasse 30  
 8010 Graz  
 Austria  
 dangeli@math.tugraz.at