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SOME PRESERVATION PROPERTIES OF MKZ-STANCU TYPE OPERATORS

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Abstract. In this work, we construct Stancu type modification of the generalization of Meyer-König and Zeller operators (MKZ) defined in [12]. We show that the Lipschitz constant of a Lipschitz continuous function and the properties of the function of modulus of continuity can be retained by these operators.

1. Introduction

In [13], by means of the probabilistic methods Stancu constructed a generalization of Bernstein polynomials as follows:

$$S_n^{\alpha}(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right) s_{n,k}(x,\alpha), \quad x\in[0,1], n\in\mathbb{N},$$

where $\alpha\geq 0$ is a parameter and

$$s_{n,k}(x,\alpha):=\binom{n}{k}\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)}.$$

Some works related to Stancu type generalization of some linear positive operators can be found in [2], [6], [7], [11], [14] and [19].

The Meyer-König and Zeller operators [10] are defined by

$$M_n(f;x)=\begin{cases}(1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k & , x\in[0,1)\\ f(x) & , x=1\end{cases}.$$

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In order to get the monotonicity properties, Cheney and Sharma [4] introduced the following slight modification of these operators

$$T_n(f;x)=\begin{cases}(1-x)^{n+1}\sum_{k=0}^{\infty}f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k & ,\ x\in[0,1)\\ f(x) & ,\ x=1\end{cases}.$$

In 2007, Rempulska and Skorupka [12] defined a generalization of MKZ operators on an unbounded interval as follows:

$$M_n^*(f;x)=\begin{cases}\left(1-\frac{x}{b_n}\right)^{n+1}\sum_{k=0}^{\infty}f\left(\frac{k}{n+k}b_n\right)\binom{n+k}{k}\left(\frac{x}{b_n}\right)^k & ,\ x\in[0,b_n)\\ f(x) & ,\ x\geq b_n\end{cases},\tag{1.1}$$

where $n\in\mathbb{N}$ and (b_n) is a sequence of real numbers having the properties

$$1\leq b_n<b_{n+1},\ \lim_{n\to\infty}b_n=\infty,\ \lim_{n\to\infty}\frac{b_n}{n}=0.\tag{1.2}$$

The authors investigated some approximation properties of these operators for differentiable functions in polynomial weighted spaces. Very recently, Erençin, İnce and Olgun [5] proposed a modification of the operators given by (1.1) based on q-integers and obtained some convergence properties of these operators in weighted spaces of continuous functions on $[0,\infty)$ with the help of a weighted Korovkin type theorem. Furthermore, they also gave an application to functional differential equations and Stancu type remainder.

In this work, we introduce the following Stancu type generalization of the operators M_n^* defined by (1.1)

$$S_{n,\alpha}(f;x)=\begin{cases}\sum_{k=0}^{\infty}f\left(\frac{k}{n+k}b_n\right)P_{n,k}(\alpha,x) & ,\ x\in[0,b_n)\\ f(x) & ,\ x\geq b_n\end{cases},\tag{1.3}$$

where $n\in\mathbb{N}$, f is a continuous function on $[0,\infty)$, (b_n) is a sequence of real numbers satisfying the conditions given in (1.2), $\alpha\geq 0$ is a parameter and

$$P_{n,k}(\alpha,x):=\binom{n+k}{k}\frac{\prod_{s=0}^{k-1}\left(\frac{x}{b_n}+\alpha s\right)\prod_{s=0}^{n}\left(1-\frac{x}{b_n}+\alpha s\right)}{\prod_{s=0}^{n+k}(1+\alpha s)}.\tag{1.4}$$

An empty product in (1.4) will be taken as 1. It is obvious that for $\alpha=0$ the operators $S_{n,\alpha}$ reduce to the operators M_n^*. We remark that since $P_{n,0}(\alpha,0)=1$ and $P_{n,k}(\alpha,0)=0$ for $k\geq 1$ we have $S_{n,\alpha}(f;0)=f(0)$. By taking into consideration this fact, a simple computation shows that the operators $S_{n,\alpha}$ for $\alpha > 0$ can also be represented as

$$S_{n,\alpha}(f;x) = \begin{cases} f(0) & , x = 0 \\ \int_0^1 \psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t) M_n^*(f, tb_n) dt & , x \in (0, b_n), \\ f(x) & , x \geq b_n \end{cases} \tag{1.5}$$

where the operators M_n^* are given by (1.1) and

$$\psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t) = \frac{t^{\frac{x}{\alpha b_n}-1}(1-t)^{\frac{b_n-x}{\alpha b_n}-1}}{B\left(\frac{x}{\alpha b_n}, \frac{b_n-x}{\alpha b_n}\right)} \tag{1.6}$$

in which B is the beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, (x, y > 0).$$

Now let us consider the following operators

$$B_{\alpha,b_n}(f;x) = \int_0^1 \psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t) f(tb_n) dt, \quad x \in (0, b_n), \tag{1.7}$$

where $\psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t)$ is defined by (1.6). Straightforward computation yields

$$B_{\alpha,b_n}(1;x) = 1, B_{\alpha,b_n}(s;x) = x. \tag{1.8}$$

For future discussions, we recall the following definitions.
Let f be a real valued continuous function defined on $[0, \infty)$. Then f is said to be Lipschitz continuous of order μ on $[0, \infty)$ if

$$|f(x) - f(y)| \leq A |x - y|^{\mu}$$

for $x, y \in [0, \infty)$ with $A > 0$ and $0 < \mu \leq 1$. The set of Lipschitz continuous functions is denoted by $Lip_A(\mu)$.

A real valued continuous and non-negative function ω defined on $[0, \infty)$ is called the function of modulus of continuity if each of the following conditions is satisfied:

a) $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$ i.e., ω is semi-additive,
b) $\omega(t_1) \geq \omega(t_2)$ for $t_1 \geq t_2$, i.e., ω is non-decreasing,
c) $\lim_{t\to 0^+} \omega(t) = \omega(0) = 0$.

We now reproduce the following properties of the operators $M_n^*(f;x) := M_n^*(f)$ given by (1.1).

Lemma A ([18]). *If $f \in Lip_A(\mu)$, then for all $n \in \mathbb{N}$, $M_n^*(f) \in Lip_A(\mu)$.*

Lemma B ([18]). *If ω is a function of modulus of continuity, then for all $n \in \mathbb{N}$, $M_n^*(\omega)$ is also a function of modulus of continuity.*

2. Main results

It is well known that first elementary proof regarding preservation of Lipschitz constant of a Lipschitz continuous function was given by Brown, Elliott and Paget in [1] for Bernstein polynomials. Similar results for other univariate or multivariate operators can be found in [3], [8],[15] [16] and [17]. On the other hand, in [9] Li established that Bernstein polynomials also preserve the properties of the general function of the modulus of continuity which is related to smoothness. Cao, Ding and Xu [3] extended for the first time this result to the multivariate Baskakov operators.

Our objective in this work is to show that the operators $S_{n,\alpha}(f;x) := S_{n,\alpha}(f)$ defined by (1.5) preserve the properties of the function of modulus of continuity and Lipschitz condition. For this purpose, let $\triangle$ be a set defined by

$$\triangle := \{(u,v) : u \geq 0, v \geq 0, u+v \leq 1, u,v \in \mathbb{R}\}.$$

For $\alpha > 0$ and $x, y \in (0, b_n)$ such that $x < y$ we define the following auxiliary function

$$h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v) := \begin{cases} \frac{\Gamma\left(\frac{1}{\alpha}\right) u^{\frac{x}{\alpha b_n}-1} v^{\frac{y-x}{\alpha b_n}-1} (1-u-v)^{\frac{b_n-y}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} & , (u,v) \in \Delta \\ 0 & , \text{otherwise} \end{cases}. \tag{2.1}$$

Theorem 1. *For any $t \in (0,1)$ we have*

(a) $\psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t) = \int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(t,v)\,dv,$

(b) $\psi_{\frac{1}{\alpha},\frac{y}{b_n}}(t) = \int_0^{t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t-u)\,du,$

(c) $\psi_{\frac{1}{\alpha},\frac{y-x}{b_n}}(t) = \int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t)\,du,$

where $\psi_{\frac{1}{\alpha},\frac{x}{b_n}}$ is given by (1.6).

Proof. (a) From (2.1), one has

$$\begin{aligned}
&\int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(t,v)dv \\
&= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{x}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} \int_0^{1-t} v^{\frac{y-x}{\alpha b_n}-1}(1-t-v)^{\frac{b_n-y}{\alpha b_n}-1}dv \\
&= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{x}{\alpha b_n}-1}(1-t)^{\frac{b_n-x}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} \int_0^{1-t} \left(\frac{v}{1-t}\right)^{\frac{y-x}{\alpha b_n}-1} \left(1-\frac{v}{1-t}\right)^{\frac{b_n-y}{\alpha b_n}-1} \frac{dv}{1-t}.
\end{aligned}$$

Now making the substitution $\xi = \frac{v}{1-t}$ and using the definition of beta function, we find

$$\int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(t,v)dv$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{x}{\alpha b_n}-1}(1-t)^{\frac{b_n-x}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}\int_0^1 \xi^{\frac{y-x}{\alpha b_n}-1}(1-\xi)^{\frac{b_n-y}{\alpha b_n}-1}d\xi$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{x}{\alpha b_n}-1}(1-t)^{\frac{b_n-x}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}B\left(\frac{y-x}{\alpha b_n},\frac{b_n-y}{\alpha b_n}\right)$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{x}{\alpha b_n}-1}(1-t)^{\frac{b_n-x}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-x}{\alpha b_n}\right)}$$
$$= \frac{t^{\frac{x}{\alpha b_n}-1}(1-t)^{\frac{b_n-x}{\alpha b_n}-1}}{B\left(\frac{x}{\alpha b_n},\frac{b_n-x}{\alpha b_n}\right)}$$
$$= \psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t).$$

(b) Similarly,

$$\int_0^{t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t-u)du$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right)(1-t)^{\frac{b_n-y}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}\int_0^t u^{\frac{x}{\alpha b_n}-1}(t-u)^{\frac{y-x}{\alpha b_n}-1}du$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y}{\alpha b_n}-1}(1-t)^{\frac{b_n-y}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}\int_0^t \left(\frac{u}{t}\right)^{\frac{x}{\alpha b_n}-1}\left(1-\frac{u}{t}\right)^{\frac{y-x}{\alpha b_n}-1}\frac{du}{t}.$$

If we make the substitution $\xi = \frac{u}{t}$, then we get

$$\int_0^{t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t-u)du$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y}{\alpha b_n}-1}(1-t)^{\frac{b_n-y}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}\int_0^1 \xi^{\frac{x}{\alpha b_n}-1}(1-\xi)^{\frac{y-x}{\alpha b_n}-1}d\xi$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y}{\alpha b_n}-1}(1-t)^{\frac{b_n-y}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}B\left(\frac{x}{\alpha b_n},\frac{y-x}{\alpha b_n}\right)$$

$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y}{\alpha b_n}-1}(1-t)^{\frac{b_n-y}{\alpha b_n}-1}}{\Gamma\left(\frac{y}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)}$$
$$= \psi_{\frac{1}{\alpha},\frac{y}{b_n}}(t),$$

which is the required result.
(c) By means of the equality (2.1) we can write

$$\int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t)du$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y-x}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} \int_0^{1-t} u^{\frac{x}{\alpha b_n}-1}(1-t-u)^{\frac{b_n-y}{\alpha b_n}-1}du$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y-x}{\alpha b_n}-1}(1-t)^{\frac{b_n-(y-x)}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} \int_0^{1-t}\left(\frac{u}{1-t}\right)^{\frac{x}{\alpha b_n}-1}\left(1-\frac{u}{1-t}\right)^{\frac{b_n-y}{\alpha b_n}-1}\frac{du}{1-t}.$$

Thus letting $\xi = \frac{u}{1-t}$ we find

$$\int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t)du$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y-x}{\alpha b_n}-1}(1-t)^{\frac{b_n-(y-x)}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} \int_0^1 \xi^{\frac{x}{\alpha b_n}-1}(1-\xi)^{\frac{b_n-y}{\alpha b_n}-1}d\xi$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y-x}{\alpha b_n}-1}(1-t)^{\frac{b_n-(y-x)}{\alpha b_n}-1}}{\Gamma\left(\frac{x}{\alpha b_n}\right)\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-y}{\alpha b_n}\right)} B\left(\frac{x}{\alpha b_n}, \frac{b_n-y}{\alpha b_n}\right)$$
$$= \frac{\Gamma\left(\frac{1}{\alpha}\right) t^{\frac{y-x}{\alpha b_n}-1}(1-t)^{\frac{b_n-(y-x)}{\alpha b_n}-1}}{\Gamma\left(\frac{y-x}{\alpha b_n}\right)\Gamma\left(\frac{b_n-(y-x)}{\alpha b_n}\right)}$$
$$= \psi_{\frac{1}{\alpha},\frac{y-x}{b_n}}(t),$$

which completes the proof. □

Theorem 2. *If $f \in Lip_A(\mu)$, then $S_{n,\alpha}(f) \in Lip_A(\mu)$ for all $n \in \mathbb{N}$ and $\alpha > 0$.*

Proof. Let $x, y \in (0, b_n)$ such that $x < y$. Using items (b) and (a) of Theorem 1, and taking into consideration the representation of the operators $S_{n,\alpha}$

given by (1.5) one gets

$$
\begin{aligned}
S_{n,\alpha}(f;y) - S_{n,\alpha}(f;x) &= \int_0^1 \left[\psi_{\frac{1}{\alpha},\frac{y}{b_n}}(t) - \psi_{\frac{1}{\alpha},\frac{x}{b_n}}(t)\right] M_n^*(f;tb_n)dt \\
&= \int_0^1 \int_0^t h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t-u)M_n^*(f;tb_n)dudt \\
&\quad - \int_0^1 \int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(t,v)M_n^*(f;tb_n)dvdt \\
&= \int_0^1 \int_u^1 h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,t-u)M_n^*(f;tb_n)dtdu \\
&\quad - \int_0^1 \int_0^{1-t} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(t,v)M_n^*(f;tb_n)dvdt.
\end{aligned}
$$

Now letting $t-u=v$ and $t=u$ in the first and second integral, respectively, on the right side of the above equality we obtain

$$
\begin{aligned}
&S_{n,\alpha}(f;y) - S_{n,\alpha}(f;x) \\
&= \int_0^1 \int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)M_n^*(f;(u+v)b_n)dvdu \\
&\quad - \int_0^1 \int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)M_n^*(f;ub_n)dvdu \\
&= \int_0^1 \int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)\left[M_n^*(f;(u+v)b_n) - M_n^*(f;ub_n)\right]dvdu
\end{aligned}
\tag{2.2}
$$

so that

$$
\begin{aligned}
&|S_{n,\alpha}(f;y) - S_{n,\alpha}(f;x)| \\
&\le \int_0^1 \int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)\left|M_n^*(f;(u+v)b_n) - M_n^*(f;ub_n)\right|dvdu.
\end{aligned}
$$

From Lemma A we have $M_n^*(f) \in Lip_A(\mu)$. So using this fact and the item (c) of Theorem 1 it follows that

$$
\begin{aligned}
|S_{n,\alpha}(f;y) - S_{n,\alpha}(f;x)| &\le A \int_0^1 \int_0^{1-u} (vb_n)^\mu h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)dvdu \\
&= A \int_0^1 \int_0^{1-v} (vb_n)^\mu h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)dudv \\
&= A \int_0^1 (vb_n)^\mu \left[\int_0^{1-v} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)du\right] dv \\
&= A \int_0^1 (vb_n)^\mu \psi_{\frac{1}{\alpha},\frac{y-x}{b_n}}(v)dv.
\end{aligned}
$$

Finally by Hölder's inequality and the equations (1.7) and (1.8) we obtain that

$$\begin{aligned}|S_{n,\alpha}(f;y)-S_{n,\alpha}(f;x)| &\leq A\left(\int_0^1 \psi_{\frac{1}{\alpha},\frac{y-x}{b_n}}(v)(vb_n)dv\right)^{\mu}\\ &= A\left(B_{\alpha,b_n}\left(s;y-x\right)\right)^{\mu}\\ &= A(y-x)^{\mu}.\end{aligned}$$

This is the desired result. Similarly, by using (1.5) it can be shown that for the cases $x=0, y\in(0,b_n)$; $x=0, y\geq b_n$ and $x\in(0,b_n), y\geq b_n$ our claim is true. Thus the proof is completed. □

Theorem 3. *Let ω be a modulus of continuity function, then the operators $S_{n,\alpha}(\omega)$ have the same property for all $n\in\mathbb{N}$ and $\alpha>0$.*

Proof. By (2.2), we have

$$\begin{aligned}&S_{n,\alpha}(\omega;y)-S_{n,\alpha}(\omega;x)\\ &=\int_0^1\int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)\left[M_n^*(\omega;(u+v)b_n)-M_n^*(\omega;ub_n)\right]dvdu.\end{aligned}$$

From Lemma B the above equality leads to

$$S_{n,\alpha}(\omega;y)-S_{n,\alpha}(\omega;x)\geq 0$$

for $y>x$. Again from Lemma B we know that $M_n^*(\omega)$ is semi-additive. So, we may write in view of Theorem 1, (c) and (1.5) that

$$\begin{aligned}&S_{n,\alpha}(\omega;y)-S_{n,\alpha}(\omega;x)\\ &=\int_0^1\int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)\left[M_n^*(\omega;(u+v)b_n)-M_n^*(\omega;ub_n)\right]dvdu\\ &\leq\int_0^1\int_0^{1-u} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)M_n^*(\omega;vb_n)dvdu\\ &=\int_0^1\int_0^{1-v} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)M_n^*(\omega;vb_n)dudv\\ &=\int_0^1 M_n^*(\omega;vb_n)\left[\int_0^{1-v} h_{\frac{1}{\alpha},\frac{x}{b_n},\frac{y}{b_n}}(u,v)du\right]dv\\ &=\int_0^1 \psi_{\frac{1}{\alpha},\frac{y-x}{b_n}}(v)M_n^*(\omega;vb_n)dv\\ &=S_n(\omega;y-x)\end{aligned}$$

which shows the semi-additivity of $S_{n,\alpha}(\omega;x)$. Moreover, by the definition of $S_{n,\alpha}$ we have $\lim_{x\longrightarrow 0^+} S_{n,\alpha}(\omega;x)=\omega(0)=0$. This completes the proof. □

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