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FOURIER SERIES OF FUNCTIONS WITH INFINITE DISCONTINUITIES

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Abstract. Using the total H_1-integrability concept we shall show that functions, which take on infinite values in the interval $(-\pi, \pi)$ at only finitely many places, can be expanded into a *Fourier* series over this interval.

1. Introduction

As is well-known, significant progress in *Fourier* analysis has gone hand in hand with progress in theories of integration, [4, 11]. Perhaps this can be best exemplified by using the so-called total value of the generalized *Riemann* integrals introduced by *Saric* in his works [6, 7, 8, 9]. This brand new theory of integration, which takes the notion of residues of real valued functions into account, gives us the opportunity to integrate real valued functions that are not integrable in any of the known integration methods until now. Accordingly, in the main part of this paper, we shall see that real-valued functions, with infinite discontinuities within the interval $(-\pi, \pi)$, can be expanded into a *Fourier* series over $[-\pi, \pi]$.

2. Preliminaries

The *Lebesgue* measure on the set of all real numbers $\mathbb{R}$ is denoted by μ, however, for $E \subset \mathbb{R}$ we write $|E|$ instead of $\mu(E)$. By $\mathbb{N}$ we denote the set of natural numbers. Given the compact interval $[-\pi, \pi]$ let the collection $\mathcal{I}([-\pi, \pi])$ be a family of all compact subintervals I of $[-\pi, \pi]$. Any real valued function defined on $\mathcal{I}([-\pi, \pi])$ is an interval function. For $f : [-\pi, \pi] \mapsto \mathbb{R}$ the associated interval function of f is an interval function $f : \mathcal{I}([-\pi, \pi]) \mapsto \mathbb{R}$, again denoted by f, [10]. A partition $P[-\pi, \pi]$ of $[-\pi, \pi]$ is a finite set (collection) of interval-point pairs $([a_i, b_i], x_i)$, $i = 1, \ldots, \nu$,

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such that the subintervals $[a_i, b_i]$ are non-overlapping ($(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$, where (a_i, b_i) is the interior of $[a_i, b_i]$), $\cup_{i \leq \nu} [a_i, b_i] = [-\pi, \pi]$ and $x_i \in (a_i, b_i)$ if x_i is an interior point of $[-\pi, \pi]$. The points $\{x_i\}_{i \leq \nu}$ are the tags of $P[-\pi, \pi]$, [1, 3]. If E is a subset of $[-\pi, \pi]$, then the restriction of $P[-\pi, \pi]$ to E is a finite subset of $([a_i, b_i], x_i) \in P[-\pi, \pi]$ such that each pair of sets $[a_i, b_i]$ and E intersects in at least one point. In symbols, $P[-\pi, \pi]|_E = \{([a_i, b_i], x_i) \in P[-\pi, \pi] \mid [a_i, b_i] \cap E \neq \emptyset\}$. It is evident that a given partition of $[-\pi, \pi]$ can be tagged in infinitely many ways by choosing different points as tags. Given $\delta : [-\pi, \pi] \mapsto \mathbb{R}_+$, named a gauge, a partition $P[-\pi, \pi]$ is called *δ-fine* if $[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$. Let $\mathcal{P}[-\pi, \pi]$ be the family of all partitions $P[-\pi, \pi]$ of $[-\pi, \pi]$. Then, by $\mathcal{P}_\delta[-\pi, \pi]$ we denote the family of all *δ-fine* partitions $P[-\pi, \pi]$ of $[-\pi, \pi]$ for some given $\delta : [-\pi, \pi] \mapsto \mathbb{R}_+$.

For the infinite set of partitions $\{P_n[-\pi, \pi] \mid P_n[-\pi, \pi] = \{([a_{i_n}, b_{i_n}], x_{i_n})\}, n \in \mathbb{N}\}$, denoted by $\langle P_n[-\pi, \pi]\rangle$, we write $\langle P[-\pi, \pi]\rangle \in (\mathcal{P}[-\pi, \pi], \prec)$, if $P_n[-\pi, \pi] \prec P_{n+1}[-\pi, \pi]$ for each $n \in \mathbb{N}$. The statement $P_n[-\pi, \pi] \prec P_{n+1}[-\pi, \pi]$ means that for each interval-point pair $\left(\left[a_{i_{n+1}}, b_{i_{n+1}}\right], x_{i_{n+1}}\right) \in P_{n+1}[-\pi, \pi]$ there exists a corresponding interval-point pair $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[-\pi, \pi]$ such that $\left[a_{i_{n+1}}, b_{i_{n+1}}\right] \subset [a_{i_n}, b_{i_n}]$, and

$$\{x_{i_n} \mid ([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[-\pi, \pi]\} \subset \left\{x_{i_{n+1}} \mid \left(\left[a_{i_{n+1}}, b_{i_{n+1}}\right], x_{i_{n+1}}\right) \in P_{n+1}[-\pi, \pi]\right\}.$$

Clearly, for any $x \in [-\pi, \pi]$ there exists a directed set $\langle P_n[-\pi, \pi]\rangle \in (\mathcal{P}[-\pi, \pi], \prec)$ so that x is a tag for it.

In what follows we will use the following notations $\Delta F(I) = F(v) - F(u)$, where u and v are the endpoints of I, $\sum_{i_n} \Delta F([a_{i_n}, b_{i_n}]) = \Delta F(P_n[-\pi, \pi])$ and $\sum_{i_n} f(x_{i_n}) |[a_{i_n}, b_{i_n}]| = (f\Delta x)(P_n[-\pi, \pi])$, whenever $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[-\pi, \pi]$. In addition, if $t \in (0, \pi)$ and $\gamma(I, t)$ is an interval function associated to $\gamma(x, t)$, then

$$\sum_{i_n} [\gamma([a_{i_n}, b_{i_n}], t)\, \Delta F([a_{i_n}, b_{i_n}]) - \gamma(x_{i_n}, t) |[a_{i_n}, b_{i_n}]|] = (\boldsymbol{\gamma}\Delta F - \gamma\Delta x)(P_n[-\pi, \pi], t).$$

The following two definitions come from [2] and [5].

Definition 1. Let $f : [-\pi, \pi] \mapsto \mathbb{R}$. The point function f is H_1-integrable to a real point $\mathcal{A}$ on $[-\pi, \pi]$, if there exists a gauge δ on $[-\pi, \pi]$ such that for every $\varepsilon > 0$ there exists a *δ-fine* partition $P_{n_\varepsilon}[-\pi, \pi]$ such that

$$|(f\Delta x)(P_n[-\pi, \pi]) - \mathcal{A}| < \varepsilon,$$

whenever $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle \in (\mathcal{P}_\delta \;\; [-\pi,\pi], \prec)$ and $P_{n_\varepsilon}[-\pi,\pi] \prec P_n[-\pi,\pi]$. In symbols, $\mathcal{A} := \mathrm{H}_1 - \int_{-\pi}^{\pi} f dx$.

Definition 2. Let $\gamma : \mathcal{I}[-\pi,\pi] \mapsto \mathbb{R}$ be an arbitrary interval function. Then, a point function $g : [-\pi,\pi] \mapsto \mathbb{R}$ is the *Moore-Smith* limit of γ on $E \subseteq [-\pi,\pi]$, if there exists a gauge δ on $[-\pi,\pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$|\gamma([a_{i_n}, b_{i_n}]) - g(x_{i_n})| < \varepsilon,$$

whenever $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[-\pi,\pi]|_E$, $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle \in (\mathcal{P}_\delta[-\pi,\pi], \prec)$ and $P_{n_\varepsilon}[-\pi,\pi] \prec P_n[-\pi,\pi]$.

For a primitive $F : [-\pi,\pi] \mapsto \mathbb{R}$, the derivative f on $[-\pi,\pi]$ could be defined as the *Moore-Smith* limit of the interval function $\phi : \mathcal{I}[-\pi,\pi] \mapsto \mathbb{R}$ defined by

$$\phi(I) = \frac{\Delta F(I)}{\Delta x(I)} = \frac{\Delta F}{\Delta x}(I), \tag{2.1}$$

where $\Delta x(I) = |I|$. In this case, according to *Definition 2*, given $\varepsilon > 0$ there exists a gauge δ on $[-\pi,\pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that $|\Delta F([a_{i_n}, b_{i_n}]) - f(x_{i_n})\Delta x([a_{i_n}, b_{i_n}])| \leq \varepsilon \Delta x([a_{i_n}, b_{i_n}])$, whenever $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[-\pi,\pi]|_E$, $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle \in (\mathcal{P}_\delta[-\pi,\pi], \prec)$ and $P_{n_\varepsilon}[-\pi,\pi] \prec P_n[-\pi,\pi]$. Accordingly, if $E \subset [-\pi,\pi]$, more precisely if $F : [-\pi,\pi] \mapsto \mathbb{R}$ is a function that is not differentiable on $[-\pi,\pi]$, then for a given $\varepsilon > 0$ in the set

$$\Omega_\varepsilon^{\mathcal{KH}} = \{(x, I) : x \in [-\pi,\pi] \text{ is inside } I \text{ and } |\Delta F(I)| \geq \varepsilon |I|\}$$

we isolate two subsets:

$$\Omega_{<\varepsilon}^{\mathcal{KH}} = \{(x, I) : x \in [-\pi,\pi] \text{ is inside } I \text{ and } \varepsilon |I| \leq |\Delta F(I)| < \varepsilon\} \text{ and}$$

$$\Omega_{\geq\varepsilon}^{\mathcal{KH}} = \{(x, I) : x \in [-\pi,\pi] \text{ is inside } I \text{ and } |\Delta F(I)| \geq \varepsilon\}.$$

Definition 3. Let $F : [-\pi,\pi] \mapsto \mathbb{R}$. The set $(vss)[-\pi,\pi] = \{x \in [-\pi,\pi]$: for every $\varepsilon > 0$ there exists a δ-fine $(x, I) \in \Omega_{<\varepsilon}^{\mathcal{KH}}\}$ is said to be the set of apparent singular points of F on $[-\pi,\pi]$.

Definition 4. Let $F : [-\pi,\pi] \mapsto \mathbb{R}$. The set $(vs)[-\pi,\pi] = \{x \in [-\pi,\pi]$: for every $\varepsilon > 0$ there exists a δ-fine $(x, I) \in \Omega_{\geq\varepsilon}^{\mathcal{KH}}\}$ is said to be the set of singular points of F on $[-\pi,\pi]$.

When working with functions, which have a finite number of discontinuities on $[-\pi,\pi]$, it does not really matter, from the point of view of totalization of the H_1 integral, how these functions will be defined on the set E of discontinuities. Hence, we adopt the convention that such functions are equal to 0 at all points at which they can take values $\pm\infty$ or not be defined at all. Accordingly, we may define point functions $F_{ex} : [-\pi,\pi] \mapsto \mathbb{R}$ and

$f_{ex} : [-\pi, \pi] \mapsto \mathbb{R}$ by extending F and its derivative f from $[-\pi, \pi] \setminus E$ to E by $F_{ex}(x) = 0$ and $f_{ex}(x) = 0$ for $x \in E$, so that

$$F_{ex}(x) = \begin{cases} F(x), & \text{if } x \in [-\pi, \pi] \setminus E \\ 0, & \text{if } x \in E \end{cases} \quad \text{and}$$

$$f_{ex}(x) = \begin{cases} f(x), & \text{if } x \in [-\pi, \pi] \setminus E \\ 0, & \text{if } x \in E \end{cases}. \tag{2.2}$$

The following two definitions come from [9].

Definition 5. Let $\gamma : \mathcal{I}[-\pi, \pi] \mapsto \mathbb{R}$ be an arbitrary interval function and for $F : [-\pi, \pi] \mapsto \mathbb{R}$ let $\phi : \mathcal{I}[-\pi, \pi] \mapsto \mathbb{R}$ be an interval function defined by (2.1), that converge, according to Definition 2, to $g(x)$ and $f(x)$, respectively, almost everywhere on $[-\pi, \pi]$. A point function $g(x)$ is totally H_1-integrable, with respect to the differential form $dF(x) = f(x)\,dx$, to a real point $\mathcal{F}$ on $[-\pi, \pi]$ if there exists a gauge δ on $[-\pi, \pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi, \pi]$ such that

$$|(\boldsymbol{\gamma}\Delta F)(P_n[-\pi, \pi]) - \mathcal{F}| < \varepsilon, \tag{2.3}$$

whenever $P_n[-\pi, \pi] \in \langle P_n[-\pi, \pi] \rangle$, $\langle P_n[-\pi, \pi] \rangle \in (\mathcal{P}_\delta[-\pi, \pi], \prec)$ and $P_{n_\varepsilon}[-\pi, \pi] \prec P_n[-\pi, \pi]$. In symbols, $\mathcal{F} := \mathrm{H}_1 - vt \int_{-\pi}^{\pi} g dF$.

Remark 1. In case, any of the point functions g and f above is the *Moore-Smith* limit of the corresponding interval function on $[-\pi, \pi]$, then in the previous definition (2.3) can be replaced by

$$|(g\Delta F)(P_n[-\pi, \pi]) - \mathcal{F}| < \varepsilon \text{ or } |(\boldsymbol{\gamma} f \Delta x)(P_n[-\pi, \pi]) - \mathcal{F}| < \varepsilon,$$

respectively.

Definition 6. Let $F : [-\pi, \pi] \mapsto \mathbb{R}$ and $E \subset (-\pi, \pi)$ be a set of *Lebesgue* measure zero such that $E = (vs)[-\pi, \pi]$. The linear differential form $dF(x) = f(x)\,dx$, as the *Moore-Smith* limit of $\Delta F(I) = \phi(I)\Delta x(I)$ on $[-\pi, \pi]$, where $I \in \mathcal{I}([-\pi, \pi])$, is said to be basically summable (BS_δ) to a real number $\Re$ on E if there exists a gauge δ on $[-\pi, \pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi, \pi]$ such that

$$|(\Delta F - f_{ex}\Delta x)(P_n[-\pi, \pi]|_E) - \Re| < \varepsilon,$$

whenever $P_n[-\pi, \pi] \in \langle P_n[-\pi, \pi] \rangle$, $\langle P_n[-\pi, \pi] \rangle \in (\mathcal{P}_\delta[-\pi, \pi], \prec)$ and $P_{n_\varepsilon}[-\pi, \pi] \prec P_n[-\pi, \pi]$. If in addition E can be written as a countable union of sets on each of which the linear differential form $f(x)\,dx$ is BS_δ, then $f(x)\,dx$ is said to be BSG_δ on E. In symbols, $\Re := \sum_{x \in E} f(x)\,dx$.

3. Main results

It is an old result (see [9]) that if $\gamma : [-\pi, \pi] \mapsto \mathbb{R}$ is a point function defined by $\gamma(x,t) = \sum_{k=1}^{+\infty} \Gamma_k(x,t) + (x-t)/2$, where $\Gamma_k(x,t) = \sin[k(x-t)]/k$, for every fixed $t \in (0,\pi)$, then the dispersion of function values on $[-\pi,\pi]$ is as follows

$$\gamma(x,t) = \begin{cases} -\frac{\pi}{2}, & \text{if } x \in [-\pi, t) \\ 0, & \text{if } x = t \\ \frac{\pi}{2}, & \text{if } x \in (t, \pi] \end{cases}. \tag{3.1}$$

Let $\gamma : \mathcal{I}([-\pi,\pi]) \mapsto \mathbb{R}$ be the associated interval function of γ. If $I \in \mathcal{I}([-\pi,\pi])$ and $\gamma(I,t) = \gamma(v,t) - \gamma(u,t)$, where u and v are the endpoints of I, then

$$\gamma(I,t) = \sum_{k=1}^{+\infty} \Gamma_k(I,t) + \frac{I}{2} = \begin{cases} \pi, & \text{if } t \in int.I \\ \frac{\pi}{2}, & \text{if } t \text{ is the endpoint of } I \\ 0, & \text{if } t \notin I \end{cases}, \tag{3.2}$$

where $\Gamma_k(I,t) = \Gamma_k(v,t) - \Gamma_k(u,t)$. In addition, let $E \subset (-\pi,\pi)$ be a set of *Lebesgue* measure zero at whose points an arbitrary point function F, defined and differentiable to f on $[-\pi,\pi] \backslash E$, can take values $\pm\infty$ or not be defined at all and $t \notin E$. If we introduce the analysis of the interval function $\gamma(I,t)\,\Delta F_{ex}(I)$, as the product of the two interval functions $\gamma(I,t)$ defind by (3.2) and $\Delta F_{ex}(I)$, whenever $I \in \mathcal{I}([-\pi,\pi])$, then, according to Definition 5,

$$\mathrm{H}_1 - vt \int_{-\pi}^{\pi} \left[\sum_{k=1}^{+\infty} G_k(x,t) + \frac{1}{2} \right] f(x)\,dx = \pi f(t), \tag{3.3}$$

where $G_k(x,t) = \cos[k(x-t)]$ is the *Moore-Smith* limit of $\Gamma_k(I,t)/\Delta x(I)$, since $f(t)$ is the *Moore-Smith* limit of interval function $\phi(I) = (\Delta F_{ex}/\Delta x)$ (I) at the point t and therefore there exists a gauge δ on $[-\pi,\pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$|(\boldsymbol{\gamma\phi}_{ex})(P_n[-\pi,\pi],t) - \pi f(t)| = \pi\,|\phi_{ex}([a_{i_n}, b_{i_n}]_t) - f(t)| < \pi\varepsilon, \tag{3.4}$$

whenever $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi] \rangle$, $\langle P_n[-\pi,\pi] \rangle \in (\mathcal{P}_\delta[-\pi,\pi], \prec)$ and P_{n_ε} $[-\pi,\pi] \prec P_n[-\pi,\pi]$, where $[a_{i_n}, b_{i_n}]_t$ are the subintervals $[a_{i_n}, b_{i_n}]$ to which the point t belongs.

On the other hand, considering the fact that $\sum_{k=1}^{+\infty} \Gamma_k(I,t)$ converges on $[-\pi,\pi]$ it follows that $\sum_{k=1}^{+\infty} (\mathbf{\Gamma}_k/\Delta x)(P[-\pi,\pi],t) = (\sum_{k=1}^{+\infty} \mathbf{\Gamma}_k/\Delta x)(P$ $[-\pi,\pi],t)$, for every $P[-\pi,\pi] \in \mathcal{P}[-\pi,\pi]$. Hence, (3.3) becomes the *Fourier* series of f at the point t, as follows

$$\frac{1}{\pi}\sum_{k=1}^{+\infty}\mathrm{H}_1-vt\int_{-\pi}^{\pi}G_k\left(x,t\right)f\left(x\right)dx+\frac{1}{2\pi}\mathrm{H}_1-vt\int_{-\pi}^{\pi}f\left(x\right)dx=f\left(t\right). \quad (3.5)$$

The following theorem gives us the opportunity to compute the *Fourier* coefficients for a function that can take not only finite but infinite values within $[-\pi,\pi]$, using the H_1-integral, [9].

Theorem 1. *For* $[-\pi,\pi]\in\mathbb{R}$ *let* $E\subset(-\pi,\pi)$ *be a set of Lebesgue measure zero such that a primitive* F *is defined and differentiable on* $[-\pi,\pi]\setminus E$ *and its derivative* f *can take values* $\pm\infty$ *or not be defined at all and let* $t\in(0,\pi)/E$*. If* $G_k(x,0)\,dF(x)=G_k(x,0)\,f(x)\,dx$*, as the Moore-Smith limit of* $G_k(x,0)\,\Delta F_{ex}(I)$ *on* $[-\pi,\pi]$*, where* $I\in\mathcal{I}([-\pi,\pi])$*, is basically summable* (BS_δ) *on* E *to the sum* $\Re_k$ *and* $G_k(x,0)\,f_{ex}(x)$ *is* H_1*-integrable to a real number* $\mathcal{A}_k$ *on* $[-\pi,\pi]$*, for each* $k\in\mathbb{N}$*, then*

$$H_1-vt\int_{-\pi}^{\pi}G_k\left(x,t\right)f\left(x\right)dx=H_1-\int_{-\pi}^{\pi}G_k\left(x,t\right)f\left(x\right)dx+\Re_k. \quad (3.6)$$

Proof. Let F_{ex} and f_{ex} be defined by (2.3). Since the point function $G_k(x,0)$ $f_{ex}(x)$ is H_1-integrable to a real number $\mathcal{A}_k$ on $[-\pi,\pi]$ and $G_k(x,0)\,dF(x)$ is (BS_δ) on E to $\Re_k$, for each $k\in\mathbb{N}$, it follows from Definitions 2 and 6 that there exists a gauge δ_1 on $[-\pi,\pi]$ such that for every $\varepsilon>0$ there exists a δ_1-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$\left|\left(G_kf_{ex}\Delta x\right)\left(P_n\left[-\pi,\pi\right],0\right)-\mathcal{A}_k\right|<\varepsilon,$$

whenever $P_n[-\pi,\pi]\in\langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle\in(\mathcal{P}_{\delta_1}[-\pi,\pi],\prec)$ and P_{n_ε} $[-\pi,\pi]\prec P_n[-\pi,\pi]$, as well as a gauge δ_2 on $[-\pi,\pi]$ such that for every $\varepsilon>0$ there exists a δ_2-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$\left|\left(G_k\Delta F_{ex}-G_kf_{ex}\Delta x\right)\left(P_n\left[-\pi,\pi\right]|_E,0\right)-\Re_k\right|<\varepsilon,$$

whenever $P_n[-\pi,\pi]\in\langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle\in(\mathcal{P}_{\delta_2}[-\pi,\pi],\prec)$ and P_{n_ε} $[-\pi,\pi]\prec P_n[-\pi,\pi]$. In addition $f_{ex}(x)\equiv 0$ on E and $G_k(x,0)\,dF(x)$ is the *Moore-Smith* limit of $G_k(x,0)\,\Delta F(I)$ on $[-\pi,\pi]\setminus E$, that means that there exits a gauge δ_3 on $[-\pi,\pi]$ such that for every $\varepsilon>0$ there exists a δ_2-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$\left|\left(G_k\Delta F-G_kf\Delta x\right)\left(P_n\left[-\pi,\pi\right]\setminus P_n\left[-\pi,\pi\right]|_E,0\right)\right|\leq 2\pi\varepsilon,$$

whenever $P_n[-\pi,\pi]\in\langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle\in(\mathcal{P}_{\delta_3}[-\pi,\pi],\prec)$ and P_{n_ε} $[-\pi,\pi]\prec P_n[-\pi,\pi]$. A gauge δ may be chosen so that $\delta(x)=\min(\delta_1(x),$ $\delta_2(x),\delta_3(x))$ on $[-\pi,\pi]$. Hence, there exists a gauge δ on $[-\pi,\pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$\begin{aligned}|(G_k\Delta F_{ex})(P_n[-\pi,\pi],0) - \mathcal{A}_k - \Re_k| &\le |(G_k f_{ex}\Delta x)(P_n[-\pi,\pi],0) - \mathcal{A}_k| \\ &+ |(G_k\Delta F_{ex} - G_k f_{ex}\Delta x)(P_n[-\pi,\pi],0) - \Re_k|,\end{aligned}$$

that is,

$$\begin{aligned}&|(G_k\Delta F_{ex} - G_k f_{ex}\Delta x)(P_n[-\pi,\pi],0) - \Re_k| \\ &\qquad \le |(G_k\Delta F - G_k f\Delta x)(P_n[-\pi,\pi]\backslash P_n[-\pi,\pi]|_E\,,0)| \\ &\qquad + |(G_k\Delta F_{ex} - G_k f_{ex}\Delta x)(P_n[-\pi,\pi]|_E\,,0) - \Re_k| \le (2\pi+1)\varepsilon,\end{aligned}$$

whenever $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle \in (\mathcal{P}_\delta[-\pi,\pi],\prec)$ and $P_{n_\varepsilon}[-\pi,\pi] \prec P_n[-\pi,\pi]$, so that there exists a gauge δ on $[-\pi,\pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that

$$|(G_k\Delta F_{ex})(P_n[-\pi,\pi],0) - \mathcal{A}_k - \Re_k| \le 2(1+\pi)\varepsilon,$$

whenever $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle \in (\mathcal{P}_\delta[-\pi,\pi],\prec)$ and $P_{n_\varepsilon}[-\pi,\pi] \prec P_n[-\pi,\pi]$. Therefore,

$$\mathrm{H}_1 - vt\int_{-\pi}^{\pi} G_k(x,0)\,f(x)\,dx = \mathrm{H}_1 - \int_{-\pi}^{\pi} G_k(x,0)\,f_{ex}(x)\,dx + \Re_k.$$

□

If $H(x)$ is the *Heaviside* (unit) step function, then $dH(x) = \delta(x)\,dx$, where $\delta(x)$ is the *Dirac* delta function that is zero everywhere except at zero, is the *Moore-Smith* limit of the interval function $\Delta H(I)$, associated to $H(x)$, on $[-\pi,\pi]\backslash E_0$, where $E_0 = \{0\}$. Since there exists a gauge δ on $[-\pi,\pi]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[-\pi,\pi]$ such that $|\Delta H(P_n[-\pi,\pi]) - 1| < \varepsilon$ and $|(G_k\Delta H)(P_n[-\pi,\pi],0) - 1| < \varepsilon$, whenever $P_n[-\pi,\pi] \in \langle P_n[-\pi,\pi]\rangle$, $\langle P_n[-\pi,\pi]\rangle \in (\mathcal{P}_{\delta_3}[-\pi,\pi],\prec)$, $P_{n_\varepsilon}[-\pi,\pi] \prec P_n[-\pi,\pi]$ and $k \in \mathbb{N}$, it follows from Definition 5 that

$$\mathrm{H}_1 - vt\int_{-\pi}^{\pi} G_k(x,0)\,dH(x) = 1, \tag{3.7}$$

for each $k \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

By Theorem 1 and Definition 6, we see that $\mathrm{H}_1 - \int_{-\pi}^{\pi} G_k(x,0)\,\delta_{ex}(x)\,dx \equiv 0$, for each $k \in \mathbb{N}_0$, since $G_k(x,0)\,dH(x)$ are basically summable (BS_δ) to 1 on E_0, for each $k \in \mathbb{N}_0$. Finally, it follows from (3.5) that

$$\sum_{k=1}^{+\infty} G_k(0,t) + \frac{1}{2} = \pi\delta(t), \tag{3.8}$$

at every point t belonging to the set $[-\pi,\pi]\backslash E_0$. This confirms that $\sum_{k=1}^{+\infty} G_k(x,t)$ is the *Moore-Smith* limit of $\sum_{k=1}^{+\infty}\Gamma_k(I,t)/\Delta x(I)$ on $[-\pi,\pi]\backslash E_t$, where

$E_t = \{t\}$ and $t \in (-\pi, \pi)$. In addition, for any real-valued periodic function $f(x)$ of period 2π, which is defined at a point $t \in (-\pi, \pi)$, it follows from (3.5) that

$$\pi f(t) = \mathrm{H}_1 - vt \int_{-\pi}^{\pi} \pi\delta(x-t) f(x)\, dx$$

$$= \mathrm{H}_1 - vt \int_{-\pi}^{\pi} [\sum_{k=1}^{+\infty} G_k(x,t) + \frac{1}{2}] f(x)\, dx = \sum_{k=1}^{+\infty} \mathrm{H}_1 - vt \int_{-\pi}^{\pi} G_k(x,t) f(x)\, dx$$

$$+ \frac{1}{2}\mathrm{H}_1 - vt \int_{-\pi}^{\pi} f(x)\, dx. \quad (3.9)$$

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