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CONTINUITY CONDITIONS FOR THE HILBERT TRANSFORM ON QUASI-HILBERT SPACES

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Abstract. We give necessary and sufficient conditions for the continuity of the Hilbert transform on complex quasi-Hilbert spaces, i.e. on complex, reflexive, strictly convex Banach spaces with Gâteaux-differentiable norm and with generalized inner product.

1. Introduction

In this paper X is a complex quasi-Hilbert space, i.e. X is a complex, reflexive, strictly convex Banach space with Gâteaux-differentiable norm and with quasi-inner product $(\cdot,\cdot)$ introduced in [3] as

$$(x,y) := \langle x,y\rangle - i\,\langle x,iy\rangle\,. \tag{1.1}$$

Here

$$\langle x,y\rangle := \lim_{t\searrow 0}\frac{\|x+ty\|^2-\|x\|^2}{2t} \quad \text{for} \quad x,y\in X,$$

$\|\cdot\|$ is a norm on X (see [5]).

In papers [3] and [6] we observed the Hilbert transform on complex quasi-Hilbert space X and we gave a connection between the Hilbert transform H and the operator A_+ (see Definition 1.3) on some set dense in X (Theorem 1.5). In this paper we show that the operator iA_+ being the infinitesimal generator of a bounded strongly continuous group of operators in $B(X)$ is both necessary and sufficient for the continuity of Hilbert transform on complex quasi-Hilbert space X (Theorem 2.1).

First we recall some notations, basic notions and claims that we will need in this paper. Let X be a complex Banach space, and let $B(X)$ denote the complex Banach algebra of all bounded linear operators on X.

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Definition 1.1. If $U(t)$ is an operator function on the real axis $\mathbb{R}$ to the Banach algebra $B(X)$ satisfying the following conditions:

i) $U(t_1+t_2) = U(t_1)U(t_2), \ \ (t_1, t_2 \in \mathbb{R})$,
ii) $U(0) = I, \ \ (I -$ identity operator),

then the family $U(t), t \in \mathbb{R}$ is called a *one-parameter group of operators in* $B(X)$. It is *strongly continuous* if it is continuous at the origin in the strong operator topology, i.e. if

$$\lim_{t\to 0} U(t)x = x \ \ (x \in X) \ \ (\text{in the } X - \text{norm, shorter, in } X).$$

If, in addition, there exists a constant M $(M \geq 1)$ such that

$$\|U(t)\| \leq M \ \text{ for all } \ t \in \mathbb{R},$$

then the strongly continuous group $U(t), t \in \mathbb{R}$ is said to be *bounded.*

The infinitesimal generator A of the group $U(t), t \in \mathbb{R}$ is defined by

$$Ax := \lim_{t\to 0} \frac{U(t)x - x}{x}$$

for all $x \in X$ for which the last limit in norm exists. A is a closed linear operator with dense domain $D(A)$ in X.

Definition 1.2. [1] Let $U(t), t \in \mathbb{R}$ be a strongly continuous group of operators in $B(X)$, and let $H_{\epsilon,N}$ $(0 < \epsilon < N < \infty)$ be a continuous linear operator on X defined by

$$H_{\epsilon,N}x := \frac{1}{\pi} \int_{\epsilon \leq |t| \leq N} \frac{U(t)x}{x} dt \ \ (x \in X).$$

If

$$\lim_{\substack{\epsilon\to 0 \\ N\to\infty}} H_{\epsilon,N}x$$

exists in X, it is denoted by Hx and called the *Hilbert transform of* x, i. e.

$$Hx = \lim_{\substack{\epsilon\to 0 \\ N\to\infty}} H_{\epsilon,N}x.$$

Definition 1.3. [3] The strong closure of the operator A_0 defined by

$$A_0 F_a x := aF_a x - F_a^2 x, \ x \in X, a \geq 0$$

is called *the positive square root of* $-A^2$ and it is denoted by A_+.

The family of operators $F_a, a \geq 0$ was introduced in [6] as

$$F_a x := \lim_{\alpha \searrow 0} F_{a,\alpha} x, \ \ x \in X, a \geq 0,$$

where

$$F_{a,\alpha}x := \frac{1}{\pi i}\int_0^a du \int_{\alpha+i0}^{\alpha+iu} \left[\lambda R(\lambda^2, -A^2) + \overline{\lambda} R(\overline{\lambda}^2, -A^2\right] d\lambda,$$

where $\lambda = \alpha + iy, \quad i = \sqrt{-1}$. Here the resolvent of $-A^2$ is denoted by $R(\lambda^2, -A^2)$, i.e.

$$R(\lambda^2, -A^2) = (\lambda^2 - (-A^2))^{-1} \in B(X),$$

$-A^2$ is the infinitesimal generator of the bounded strongly continuous cosine operator function $C(t)$ defined by

$$C(t) := \frac{U(t) + U(-t)}{2}, \quad t \in \mathbb{R},$$

where $U(t), t \in \mathbb{R}$ is a bounded strongly continuous group of operators in $B(X)$ with the infinitesimal generator iA.

Note that a family of bounded linear operators $F_a, a \geq 0$ exists for every bounded strongly continuous cosine operator function on X.
The following holds (proved in [2] and [3]):

1) The limit in the definition of operators $F_a, a \geq 0$ exists for all $x \in X$ and $a \geq 0$,
2) $F_a x = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin at}{t}\right)^2 C(2t)x\,dt = \frac{2a}{\pi}\int_0^\infty \left(\frac{\sin at}{t}\right)^2 C\left(\frac{2t}{a}\right) x\,dt$,
3) $\|F_a\| \leq a$ for all $a \geq 0$,
4) function $a \to F_a$ is strongly continuous on $[0, +\infty)$,
5) $\lim_{a\to+\infty} \frac{F_a x}{a} = x, \quad x \in X$,
6) $F_a F_b x = F_b F_a x = 2\int_0^a F_u x du + (b-a)F_a x, \quad x \in X, \ 0 \leq a \leq b$,
7)
$$F_a x \in D(A^{2k}), \ k = 1, 2, 3, \ldots, \ a \geq 0, \ x \in X, \tag{1.2}$$
8) $A^2 F_a x = F_a A^2 x$ for all $x \in D(A^2), \ a \geq 0$,
9)
$$\text{the set } \bigcup_{a\geq 0} \overline{F_a(X)} \text{ is dense in } X. \tag{1.3}$$
10) $\langle x, A_+ x\rangle \geq 0$ for all $x \in D(A_+)$,
11)
$${A_+}^2 x = A^2 x \text{ for all } x \in D(A^2). \tag{1.4}$$

More about operators $F_a, a \geq 0$ and A_+ and their properties can be seen in [2], [3], [4] and [7]. We will need the following theorems (proved in [3] and [7]):

Theorem 1.4. *Let X be a (complex) quasi-Hilbert space, and let $C(t), t \in \mathbb{R}$ be a bounded strongly continuous cosine operator function with the infinitesimal generator A. If the number 0 belongs to the point spectrum of the* *generator A, and if the set of all $x \in X$ for which $Ax = 0$ is denoted by L, then there exists the subspace M of the space X such that X is a direct and orthogonal sum of subspaces L and M (i.e. every $x \in X$ can be written in a unique way in the form*

$$x = l \overset{*}{+} y, \quad l \in L, m \in M,$$

and $(l, m) = 0$). The subspaces L and M have the following properties:

a) *$C(t)x = x$ for all $x \in L$ and $t \in \mathbb{R}$*
b) *M is invariant relative to all operators $C(t), t \in \mathbb{R}$,*
c) *0 does not belong to the point spectrum of the restriction of the operator A on the subspace M.*

Theorem 1.5. *Let X be a (complex) quasi-Hilbert space. If $U(t), t \in \mathbb{R}$ is a strongly continuous group of isometries in $B(X)$ with the infinitesimal generator iA, and if A_+ is the positive square root from $-A^2$, then the Hilbert transform H is defined on some set dense in X, and*

$$HAx = iA_+x \text{ for all } x \in D(A^2),$$
$$HA_+x = iAx \text{ for all } x \in D(A^2).$$

Theorem 1.6. *Let X be a (complex) quasi-Hilbert space. If A is the infinitesimal generator of a bounded strongly continuous cosine operator function, then 0 does not belong to the residual spectrum of the operator A.*

Theorem 1.7. *Let X be a (complex) quasi-Hilbert space, and let $U(t), t \in \mathbb{R}$ be a strongly continuous group of isometries in $B(X)$ with the infinitesimal generator iA. If 0 does not belong to the point spectrum of the operator A^2, then*

$$H^2x = -x \text{ for all } x \in D(A^2),$$

where H is the Hilbert transform.

Observe that the set $D(A^2)$ is dense in X.

Remark 1.8. Let X be a complex Banach space with quasi-inner product $(\cdot, \cdot)$ defined by (1.1), and let X^* be the dual space of space X. The following theorem holds (see [3] and [5]):

Given $\delta \in X^*$ there exists a unique $x_\delta \in X$ such that

$$\|x_\delta\| = \|\delta\| \text{ and } (x_\delta, y) = \delta(y) \text{ for all } x \in X,$$

and the mapping $\varphi : \delta \to x_\delta$ is continuous from the norm topology on X^* to the weak topology on X if and only if X is a quasi-Hilbert space.

Set

$$x \overset{*}{+} y := \varphi\left(\varphi^{-1}(x) + \varphi^{-1}(y)\right), \quad x, y \in X.$$

We have shown (see [3]):

a) the space X under the operation $\overset{*}{+}$ is isometrically isomorphic to X^*, which we denote by

$$(X, \overset{*}{+}) = (X^*, +)$$

b) if X is a quasi-Hilbert space, then X^* is also a quasi-Hilbert space.
c) if U is an isometry in $B(X)$, then

$$(x, Uy) = (U^{-1}x, y), x, y \in X,$$

and U is a linear operator in space $(X, +)$ and in space $(X, \overset{*}{+})$.

2. Continuity conditions for the Hilbert transform on quasi-Hilbert spaces

Theorem 2.1. *Let X be a complex quasi-Hilbert space with the quasi-inner product $(\cdot, \cdot)$ defined by (1.1). Let $U(t), t \in \mathbb{R}$ be a group of isometries in $B(X)$ with the infinitesimal generator iA, and let A_+ be the positive square root from $-A^2$. The Hilbert transform H is a continuous linear operator on X into itself if and only if the operator iA_+ is the infinitesimal generator of a bounded strongly continuous group of operators in $B(X)$.*

Proof. Suppose iA_+ is the infinitesimal generator of the bounded strongly continuous group $U_+(t), t \in \mathbb{R}$ in $B(X)$. Set

$$\|x\|_1 := \sup_{s \in \mathbb{R}} \|U_+(s)x\|, \quad x \in X.$$

Note that $\|\cdot\|_1$ is a new norm on X, and that the norm $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|$. Using Remark 1.8, it is easy to prove that the space X under the norm $\|\cdot\|_1$ denoted by $(X, \|\cdot\|_1)$ is a quasi-Hilbert space. Also it is easy to see that groups $U(t), t \in \mathbb{R}$ and $U_+(t), t \in \mathbb{R}$ are continuous groups of isometries in $B(X)$, where $X = (X, \|\cdot\|_1)$.

From here till the end of this part of the proof, let $X = (X, \|\cdot\|_1)$, and suppose that 0 does not belong to the point spectrum $\sigma_p(A^2)$ of A^2. Set

$$L^{'} := \{x \in X | U(t)x = U_+(t)x, \ \ t \in \mathbb{R}\},$$
$$L^{''} := \{x \in X | U(t)x = U_+(-t)x, \ \ t \in \mathbb{R}\}.$$

It is easy to see that both $L^{'}$ and $L^{''}$ are (closed) subspaces of X.

Furthermore, if $x_0 \in L^{'} \cap L^{''}$, then

$$U(t)x_0 = U(-t)x_0,$$

so

$$\frac{U(t)x_0 - x_0}{t} = \frac{U(-t)x_0 - x_0}{t}, \ \ t \neq 0.$$

Thus,

$$iA\left(\frac{1}{t}\int_0^t U(s)x_0 ds - \frac{1}{t}\int_0^t U(-s)x_0 ds\right) = 0, \quad t \neq 0.$$

Since

$$\frac{1}{t}\int_0^t U(s)x_0 ds - \frac{1}{t}\int_0^t U(-s)x_0 ds \to 2x_0 \text{ as } t \to 0,$$

and since iA is a closed operator,

$$x_0 \in D(iA) \text{ and } iAx_0 = 0,$$

so,

$$-A^2 x_0 = 0.$$

Since $0 \notin \sigma_p(A^2)$,

$$L^{'} \cap L^{''} = \{0\}.$$

Using the properties of isometries in $B(X)$ and the definition of subspaces $L^{'}$ i $L^{''}$, we get

$$(x, U(t)y) = (U(-t)x, y) = (U_+(t)x, y) = (x, U_+(-t)y) = (x, U(-t)y)$$

for all $t \in \mathbb{R}, x \in L^{''}, y \in L^{'}$. Hence,

$$(x, U(t)y) = (x, U(-t)y),\ t \in \mathbb{R},\ x \in L^{''},\ y \in L^{'}.$$

Since $L^{'}$ and $L^{''}$ are invariant relative to all operators $U(t)$, and $U(-t),\ t \in \mathbb{R}$, from the last equation, by replacing y with $U(t)y$, we obtain

$$(x, U(2t)y) = (x, y),\ t \in \mathbb{R},\ x \in L^{''},\ y \in L^{'}.$$

Hence,

$$\left(x, \frac{U(2t)y - y}{2t}\right) = 0,\ t \in \mathbb{R},\ x \in L^{''},\ y \in L^{'}.$$

Thus,

$$(x, iAy) = 0$$

for all $x \in L^{''},\ y \in L^{'},\ y \in D(A)$.

By Theorem 1.6, and since $0 \notin \sigma_p(A^2)$, the set $\left\{iAy | y \in L^{'},\ y \in D(A)\right\}$ is dense in $L^{'}$. Hence,

$$(x, y) = 0 \text{ for all } x \in L^{''},\ y \in L^{'}.$$

Now, in the usual way, we get that every $l \in L, L := \overline{L' \dot{+} L''}$, can be written in a unique way in the form

$$l = x + y,\ \ x \in L^{''},\ y \in L^{'}.$$

Let us prove that $L = X$.

First we shall prove that at least one of the spaces $L^{'}$ and $L^{''}$ is not equal to $\{0\}$.

For any $x \in K$, $K := \cup_{a\geq 0,\, b\geq 0} F_a F_b(X)$, $x \neq 0$ there exist A_+Ax and AA_+x, and

$$A_+Ax = AA_+x.$$

By (1.2) and by (1.4),

$$(A + A_+)(A - A_+)x = A^2x - A_+^2x = 0, \;\; x \in K.$$

If $Ax - A_+x = 0$, then

$$V_1(t)x := U(t)U_+(-t)x$$

is strongly differentiable with respect to t on $\mathbb{R}$ and

$$\frac{dV_1(t)x}{dt} = \frac{dU(t)}{dt}U_+(-t)x + U(t)\frac{dU_+(-t)x}{dt} = iU(t)U_+(-t)(Ax - A_+x) = 0.$$

Thus,

$$V_1(t)x = V_1(0)x \;\text{ for all }\; t \in \mathbb{R},$$

i.e. $x \in L^{'}$. So,

$$L' \neq \{0\}.$$

If $Ax - A_+x \neq 0$, then

$$y := Ax - A_+x \neq 0 \;\text{ and }\; Ay + A_+y = 0.$$

The function

$$V_2(t)y := U(t)U_+(t)y, \;\; t \in \mathbb{R}$$

is strongly differentiable with respect to t on $\mathbb{R}$ and

$$\frac{dV_2(t)y}{dt} = 0.$$

Thus, in a similar way, $y \in L^{''}$. So,

$$L'' \neq \{0\}.$$

Now, let us prove that $L = X$.

If $L \subset X, L \neq X$, and if the set of all $x \in X$ for which $(x, y) = 0$ (for all $y \in L$) is denoted by X_1, then $X_1 \neq \{0\}$, and $X_1 \cap L = \{0\}$. Moreover

$$X_1 \cap \left(L^{'} + L^{''}\right) = \{0\}. \tag{2.1}$$

By Remark 1.8, $(X_1, \overset{*}{+})$ is a quasi-Hilbert space. X_1 is invariant relative to operators $U(t)$ and $U_+(t), t \in \mathbb{R}$. $U(t), t \in \mathbb{R}$ and $U_+(t), t \in \mathbb{R}$ are strongly continuous groups of isometries in $B(X_1)$. Hence, the claim we have already proved for space X holds for X_1. That means that there exists $x_1 \in X_1$, $x_1 \neq 0$ such that $x_1 \in L'$ or $x_1 \in L''$. This is in contradiction

with (2.1), proving that $L = X$.
From the definition of subspaces $L^{'}$ and $L^{''}$ of X we easily obtain

$$Ay = A_+ y \text{ for } y \in D(A) \cap L^{'}, \text{ and}$$
$$Ax = -A_+ x \text{ for } x \in D(A) \cap L^{''}.$$

From this and from the Theorem 1.5, it follows that H is a continuous linear operator on some set dense in $L = X$. Consequently, H has a unique continuous linear extension on X into itself, which completes this part of the proof.

Observe that in the above proof, we have used the assumption that $0 \notin \sigma_p(A^2)$. In case $0 \in \sigma_p(A^2)$, the claim easily follows from the Theorem 1.4 together with the above proof.

Now, suppose that the Hilbert transform H is a continuous linear operator on X into itself. Then, by Theorem 1.7, $H^2 = -I$. From this it follows that the operator

$$H^{'} := iH$$

is also a continuous linear operator and

$$\left(H^{'}\right)^2 = I.$$

Lets prove that the group $U_+(t), t \in \mathbb{R}$ is bounded, where $U_+(t), t \in \mathbb{R}$ is a strongly continuous group of operators in $B(X)$ with the infinitesimal generator iA_+. Set

$$P := \frac{1}{2}(I + H^{'}), \;\; Q := \frac{1}{2}(I - H^{'}).$$

Operators P and Q are bounded, because the operator $H^{'}$ is bounded, and holds:

$$P^2 = P, \;\; Q^2 = Q, \;\; PQ = QP = 0, \;\; P + Q = I.$$

Thus, every $x \in X$ can be written in a unique way in the form

$$x = x^{'} + x^{''}, \;\; x^{'} = Px, \;\; x^{''} = Qx.$$

From this immediately follows

$$\left\|x^{'}\right\| \leq \|P\| \cdot \|x\| \text{ and } \left\|x^{''}\right\| \leq \|Q\| \cdot \|x\|. \tag{2.2}$$

Let $x = F_a z, z \in X, a \geq 0$ be arbitrary, but fixed. Then

$$x = x^{'} + x^{''}, \; x^{'} = Px, \; x^{''} = Qx,$$

and

$$x^{'} = \frac{1}{2}(x^{'} + H^{'} x^{'}), \; x^{'} \in D(A), \; x^{'} \in D(A_+),$$
$$AH^{'} x' = H^{'} Ax^{'}, \; A_+ Hx' = HA_+ x^{'}.$$

By Theorem 1.5. we have,

$$iAx^{'} = \frac{1}{2}(iAx^{'} + iH^{'}Ax^{'}) = \frac{1}{2}(iAx^{'} - HAx^{'}) = \frac{1}{2}(iAx^{'} - iA_{+}x^{'}),$$

and

$$iA_{+}x^{'} = \frac{1}{2}(iA_{+}x^{'} - HA_{+}x^{'}) = \frac{1}{2}(iA_{+}x^{'} - iAx^{'}).$$

Hence,

$$iAx^{'} = -iA_{+}x^{'}. \tag{2.3}$$

In a similar way it can be proved that

$$iAx^{''} = iA_{+}x^{''}. \tag{2.4}$$

The function

$$W(s)x^{''} := U(t-s)U_{+}(s)x^{''}$$

is strongly differentiable with respect to s in $0 < s < t$ for each fixed $t > 0$ and

$$\frac{dW(s)}{ds}x^{''} = -\frac{dU(t-s)}{d(t-s)} \cdot U_{+}(s)x^{''} + U(t-s) \cdot \frac{dU_{+}(s)}{ds}x^{''} =$$
$$= -U(t-s)U_{+}(s)iAx^{''} + U(t-s) \cdot U_{+}(s)iA_{+}x^{''}.$$

From this and from (2.4) we obtain

$$\frac{dW(s)}{ds}x^{''} = U(t-s)U_{+}(s) \cdot (iA_{+}x^{''} - iAx^{''}) = 0.$$

Thus,

$$W(s)x^{''} = W(t)x^{''} = W(0)x^{''},$$

which shows that

$$U(t)x^{''} = U_{+}(t)x^{''} \quad \text{for all} \;\; t \geq 0.$$

In a similar way it can be proved that

$$U(t)x^{''} = U_{+}(t)x^{''} \;\; \text{for all} \quad t < 0.$$

Hence,

$$U(t)x^{''} = U_{+}(t)x^{''} \quad \text{for all} \;\; t \in \mathbb{R}.$$

Similarly, by (2.3) we get

$$U(t)x^{'} = U_{+}(-t)x^{'} \;\; \text{for all} \;\; t \in \mathbb{R}.$$

Thus,

$$U_{+}(t)x = U_{+}(t)x^{'} + U_{+}(t)x^{''} =$$
$$= U(-t)x^{'} + U(t)x^{''}$$

for all $x \in \bigcup_{a \geq 0} \overline{F_a(X)},\; x = x^{'} + x^{''}, x^{'} = Px,\; x^{''} = Qx$ and for all $t \in \mathbb{R}$.

From this and from (2.2) it follows

$$\|U_+(t)x\| \leq \left\|U(-t)x^{'}\right\| + \left\|U(t)x^{''}\right\| = \left\|x^{'}\right\| + \left\|x^{''}\right\| \leq$$
$$\leq \|P\| \cdot \|x\| + \|Q\| \cdot \|x\| = (\|P\| + \|Q\|) \cdot \|x\|$$

for all $x \in \bigcup_{a\geq 0} \overline{F_a(X)}$ and for all $t \in \mathbb{R}$. Since the set $\bigcup_{a\geq 0} \overline{F_a(X)}$ is dense in X (by (1.3)),

$$\|U_+(t)x\| \leq (\|P\| + \|Q\|) \cdot \|x\|$$

for all $x \in X$ and for all $t \in \mathbb{R}$. By Definition 1.1, the group $U_+(t)$, $t \in \mathbb{R}$ is bounded. The theorem is proved. □

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