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COINCIDENCE POINT OF FOUR COMPLETELY RANDOM OPERATORS SATISFYING GENERALIZED WEAK CONTRACTIVE CONDITIONS

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Abstract. Contractive conditions were investigated by various authors (see, e.g [4], [9], [13], [24]). In [22], we introduced the notion of completely random operators and proved some properties of such operators. The purpose of this paper is to present some results on the existence of random coincidence points of four completely random operators satisfying generalized weak contractive conditions. Some applications to random fixed point theorems and random equations are given.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, X, Y be separable metric spaces and $f : \Omega \times X \to Y$ be a random operator in the sense that for each fixed x in X, the mapping $f(., x) : \omega \mapsto f(\omega, x)$ is measurable. The random operator f is said to be continuous if for each ω in Ω, the mapping $f(\omega, .) : x \mapsto f(\omega, x)$ is continuous. An X-valued random variable ξ is said to be a random fixed point of the random operator $f : \Omega \times X \to X$ if $f(\omega, \xi(\omega)) = \xi(\omega)$ a.s. and an X-valued random variable ξ is said to be a random coincidence point of the random operators $f, g : \Omega \times X \to X$ if $f(\omega, \xi(\omega)) = g(\omega, \xi(\omega))$ a.s.

The theory of random fixed points and random coincidence points is an important topic of the stochastic analysis and has been investigated by various authors (see, e.g. [5], [6], [7], [8], [10], [14], [15], [16], [17], [18]).

In this paper, we are concerned with mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$. Since a random operator f can be viewed as an action which transforms each deterministic input x in X into a random output $f(x)$ in $L_0^Y(\Omega)$ while $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ can be viewed as an action which transforms each random input u in $L_0^X(\Omega)$ into a random output Φu in $L_0^Y(\Omega)$, we call Φ a completely random operator. In the Section 2, we recall some properties of

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completely random operators. Section 3 deals with the notion of random coincidence points of completely random operators and gives some conditions ensuring the existence of a random coincidence point of four completely random operators satisfying generalized weak contractive conditions. It should be noted that the existence of a random coincidence point of completely random operators does not follow from the existence of corresponding deterministic coincidence point theorem as in the case of the random operator. In the Section 4, some applications to random fixed point theorems and random equations are presented.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and X be a separable Banach space. A mapping $\xi : \Omega \to X$ is called an X-valued random variable if ξ is $(\mathcal{F}, \mathcal{B}(X))$-measurable, where $\mathcal{B}(X)$ denotes the Borel σ-algebra of X. The set of all (equivalent classes) X-valued random variables is denoted by $L_0^X(\Omega)$ and it is equipped with the topology of convergence in probability. For each $p > 0$, the set of X-valued random variables ξ such that $E\|\xi\|^p < \infty$ is denoted by $L_p^X(\Omega)$.

At first, recall that (see, e.g. [20]).

Definition 1. Let X, Y be two separable Banach spaces.

(1) A mapping $f : \Omega \times X \to Y$ is said to be a random operator if for each fixed x in X, the mapping $\omega \mapsto f(\omega, x)$ is measurable.

(2) The random operator $f : \Omega \times X \to Y$ is said to be continuous if for each ω in Ω the mapping $x \mapsto f(\omega, x)$ is continuous.

(3) Let $f, g : \Omega \times X \to Y$ be two random operators. The random operator g is said to be a modification of f if for each x in X, we have $f(\omega, x) = g(\omega, x)$ a.s.

Noting that the exceptional set can depend on x.

The following is the notion of the completely random operator.

Definition 2. Let X, Y be two separable Banach spaces.

(1) A mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is called a completely random operator.

(2) The completely random operator Φ is said to be continuous if for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim u_n = u$ a.s., we have $\lim \Phi u_n = \Phi u$ a.s.

(3) The completely random operator Φ is said to be continuous in probability if for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim u_n = u$ in probability, we have $\lim \Phi u_n = \Phi u$ in probability.

(4) The completely random operator Φ is said to be an extension of a random operator $f : \Omega \times X \to Y$ if for each x in X

$$\Phi x(\omega) = f(\omega, x) \text{ a.s.}$$

where for each x in X, x denotes the random variable u in $L_0^X(\Omega)$ given by $u(\omega) = x$ a.s.

Introducing the definition of completely random operator, we proved the following theorems.

Theorem 1. [22, Theorem 2.3] *Let $f : \Omega \times X \to Y$ be a random operator admitting a continuous modification. Then, there exists a continuous completely random operator*

$$\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$$

such that Φ is an extension of f.

Proposition 1. [22, Proposition 2.4] *Let $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ be a completely random operator. Then, the continuity of Φ implies the continuity in probability of Φ.*

3. Random coincidence points of four completely random operators

Let $f, g : \Omega \times X \to X$ be random operators. Recall that (see, e.g. [3], [5], [7], [18]), an X-valued random variable ξ is said to be a random fixed point of the random operator f if

$$f(\omega, \xi(\omega)) = \xi(\omega) \quad \text{a.s.}$$

An X-valued random variable u^* is said to be a random coincidence point of two random operators f, g if

$$f(\omega, u^*(\omega)) = g(\omega, u^*(\omega)) \quad \text{a.s.}$$

Assume that f, g are continuous. Then, by Theorem 1 the mappings $\Phi, \Psi : L_0^X(\Omega) \to L_0^X(\Omega)$ defined respectively by

$$\Phi u(\omega) = f(\omega, u(\omega))$$
$$\Psi u(\omega) = g(\omega, u(\omega))$$

are completely random operators extending f and g, respectively. For each random fixed point ξ of f, we get

$$\Phi\xi(\omega) = \xi(\omega) \quad \text{a.s.}$$

and for each random coincidence point u^* of two random operators f, g, we have

$$\Phi u^*(\omega) = \Psi u^*(\omega) \quad \text{a.s.}$$

This led us to the following definition.

Definition 3. (1) Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator. An X-valued random variable ξ in $L_0^X(\Omega)$ is called a random fixed point of Φ if

$$\Phi\xi = \xi.$$

(2) Let $\Phi_1, \Phi_2, \dots, \Phi_n : L_0^X(\Omega) \to L_0^X(\Omega)$ be completely random operators. An X-valued random variable u^* in $L_0^X(\Omega)$ is called a random coincidence point of $\Phi_1, \Phi_2, \dots, \Phi_n$ if

$$\Phi_1 u^* = \Phi_2 u^* = \cdots = \Phi_n u^*. \tag{1}$$

(3) Let $\Phi, T : L_0^X(\Omega) \to L_0^X(\Omega)$ be completely random operators. A pair of completely random operators (Φ, T) is called weakly compatible pair if they commute at coincidence points.

In this section, we present some conditions ensuring the existence of a random coincidence point of four completely random operators.

Theorem 2. *Let $f : [0,\infty) \to [0,\infty)$ be a function such that $f(0) = 0, f(t) < t$, $\alpha : [0,\infty) \to [0,\infty)$ be continuous, nondecreasing function satisfying $\alpha(t+u) \le \alpha(t) + \alpha(u)$ for all $t, u \ge 0$ and $\alpha(t) = 0$ if and only if $t = 0$. Put*

$$h(t) = \inf_{s \ge t} \frac{f(s)}{\alpha(s)} \quad \forall t > 0. \tag{2}$$

Assume that $h(t) > 0 \quad \forall t > 0$, Φ, Ψ, S and T be four probabilistic completely random operators satisfying following conditions

(a) *$S(L_0^X(\Omega)), T(L_0^X(\Omega))$ are closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega)) \subset T(L_0^X(\Omega))$, $\Psi(L_0^X(\Omega)) \subset S(L_0^X(\Omega))$;*
(c) *for any random variables u, v in $L_0^X(\Omega)$ and $t > 0$, we have*

$$P\left(\alpha\left(\|\Phi u - \Psi v\|\right) > t\right) \le P\left(\alpha\left(\|Su - Tv\|\right) - f\left(\alpha\left(\|Su - Tv\|\right)\right) > t\right). \tag{3}$$

Then, the pairs (Φ, S) and (Ψ, T) have a random coincidence point if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that

$$M = E[\alpha\left(\|\Phi u_0 - \Psi u_1\|\right)]^p < \infty \tag{4}$$

where u_1 is a random variable satisfying $Tu_1 = \Phi u_0$.

And Φ, Ψ, S and T have a random coincidence point if the following condition is satisfied

(d) *the pairs (Φ, S) and (Ψ, T) are weakly compatible.*

Proof. Suppose that $E[\alpha\left(\|\Phi u_0 - \Psi u_1\|\right)]^p < \infty$ where u_1 is the random variable satisfying $Tu_1 = \Phi u_0$ for some random variable u_0 in $L_0^X(\Omega)$ and $p > 0$. By the assumption (b), there exists a random variable u_2 in $L_0^X(\Omega)$

such that $Su_2 = \Psi u_1$. By induction, there exists a sequence (u_n) in $L_0^X(\Omega)$ such that

$$Tu_1 = \Phi u_0, Su_2 = \Psi u_1, \ldots, Tu_{2n+1} = \Phi u_{2n}, Su_{2n+2} = \Psi u_{2n+1} \quad n = 0, 1, \ldots \tag{5}$$

We will show that (ξ_n) given by

$$\xi_{2n} = \Phi u_{2n} = Tu_{2n+1}; \quad \xi_{2n+1} = \Psi u_{2n+1} = Su_{2n+2} \quad n = 0, 1, \ldots \tag{6}$$

in (5) is a Cauchy sequence in $L_0^X(\Omega)$. Define the function $g(t), t > 0$ by

$$g(t) = 1 - \frac{f(t)}{\alpha(t)}.$$

So, we have

$$f(t) = (1 - g(t))\, \alpha(t).$$

Since $f(t) > 0 \quad \forall t > 0$, we get $g(t) < 1 \quad \forall t > 0$. For any random variables u, v in $L_0^X(\Omega)$, we have

$$P\left(\alpha\left(\|\Phi u - \Psi v\|\right) > t\right) \leq P\left(\alpha\left(\|Su - Tv\|\right) - f\left(\alpha\left(\|Su - Tv\|\right)\right) > t\right).$$

Equivalently,

$$P\left(\alpha\left(\|\Phi u - \Psi v\|\right) > t\right) \leq P(g\left(\alpha\left(\|Su - Tv\|\right)\right) \alpha\left(\|Su - Tv\|\right) > t). \tag{7}$$

Fix $t > 0$. For each $s \geq t > 0$, we have

$$g(s) = 1 - \frac{f(s)}{\alpha(s)} \leq 1 - h(t) = q(t).$$

Since $g(t) < 1$, we get

$$\{g\left(\alpha\left(\|Su - Tv\|\right)\right) \alpha\left(\|Su - Tv\|\right) > t\} \subset \{\alpha\left(\|Su - Tv\|\right) > t\}.$$

Hence,

$$\begin{aligned}
P(\alpha\left(\|\Phi u - \Psi v\|\right) > t) &\leq P(g(\alpha\left(\|Su - Tv\|\right))\alpha\left(\|Su - Tv\|\right) > t) \\
&= P(g(\alpha\left(\|Su - Tv\|\right))\alpha\left(\|Su - Tv\|\right) > t, \alpha\left(\|Su - Tv\|\right) > t) \\
&\leq P(q(t)\alpha\left(\|Su - Tv\|\right) > t, \alpha\left(\|Su - Tv\|\right) > t) \\
&\leq P(q(t)\alpha\left(\|Su - Tv\|\right) > t) \\
&= P(\alpha\left(\|Su - Tv\|\right) > t/q(t)) = P(\alpha\left(\|Su - Tv\|\right) > t/q)
\end{aligned}$$

where $q = q(t)$. Noting that $q < 1$ since $h(t) > 0$.

From this for each n, we obtain

$$\begin{aligned}
P(\alpha\left(\|\xi_{2n+2} - \xi_{2n+1}\|\right) > t) &= P(\alpha\left(\|\Phi u_{2n+1} - \Psi u_{2n+1}\|\right) > t) \\
&\leq P(\alpha\left(\|Su_{2n+2} - Tu_{2n+1}\|\right) > t/q) \\
&= P(\alpha\left(\|\xi_{2n+1} - \xi_{2n}\|\right) > t/q),
\end{aligned}$$

and

$$\begin{aligned}P(\alpha(\|\xi_{2n+1}-\xi_{2n}\|)>t) &= P(\alpha(\|\Phi u_{2n}-\Psi u_{2n+1}\|)>t)\\ &\le P(\alpha(\|Su_{2n}-Tu_{2n+1}\|)>t/q)\\ &= P(\alpha(\|\xi_{2n}-\xi_{2n-1}\|)>t/q).\end{aligned}$$

By induction and the Chebyshev inequality, we get

$$\begin{aligned}P(\alpha(\|\xi_{n+1}-\xi_n\|)>t) &\le P(\alpha(\|\xi_n-\xi_{n-1}\|)>t/q)\\ &\le \cdots\\ &\le P(\alpha(\|\xi_1-\xi_0\|)>t/q^n)\\ &= P(\alpha(\|\Psi u_1-\Phi u_0\|)>t/q^n)\\ &\le E[\alpha(\|\Phi u_0-\Psi u_1\|)]^p\frac{(q^n)^p}{t^p}=M\frac{q^{np}}{t^p}.\end{aligned}$$

Let $r=\frac{x}{q}$ where $q<x<1$. Then, $r>1$ and $(r-1)(\frac{1}{r}+\frac{1}{r^2}+\cdots+\frac{1}{r^m})+\frac{1}{r^m}=1 \quad \forall m\ge 1$. Thus, for any $t>0$, $n\ge 2$ and m in N, we have

$$\begin{aligned}P(\alpha(\|\xi_{n+m}-\xi_n\|)>t) &\le P(\alpha(\|\xi_{n+m}-\xi_n\|)>(1-\frac{1}{r^m})t)\\ &\le P(\alpha(\|\xi_{n+m}-\xi_{n+m-1}\|)>t(r-1)/_{r^m})\\ &+\cdots+P(\alpha(\|\xi_{n+1}-\xi_n\|)>t(r-1)/_r)\\ &\le \frac{M}{[(r-1)t]^p}\left[(r^m)^p(q^{n+m-1})^p+\cdots+r^p(q^n)^p\right]\\ &= \frac{M}{[(r-1)t]^p}(q^n)^p r^p\left[(qr)^{p(m-1)}+\cdots+(qr)^p+1\right]\\ &= \frac{M}{[(r-1)t]^p}(q^n)^p r^p\frac{1-(qr)^{mp}}{1-(qr)^p}\\ &< \frac{Mr^p}{[(r-1)t]^p[1-(qr)^p]}q^{np} \quad n\ge 2\end{aligned}$$

which tends to 0 as $n\to\infty$. This implies that (ξ_n) is a Cauchy sequence in $L_0^X(\Omega)$. Hence, there exists ξ in $L_0^X(\Omega)$ such that p-$\lim \xi_n=\xi$. From the assumption (a), there exists u^* in $L_0^X(\Omega)$ such that $Su^*=\xi$. So, we have

$$\begin{aligned}P(\alpha(\|\Phi u^*-\xi_{2n+1}\|)>t) &= P(\alpha(\|\Phi u^*-\Psi u_{2n+1}\|)>t)\\ &\le P(\alpha(\|Su^*-Tu_{2n+1}\|)>t)\\ &\le P(\alpha(\|Su^*-Tu_{2n+1}\|)-f(\alpha(\|Su^*-Tu_{2n+1}\|))>t)\\ &\le P(\alpha(\|Su^*-Tu_{2n+1}\|)>t/q)\\ &= P(\alpha(\|\xi-\xi_{2n}\|)>t/q).\end{aligned}$$

Let $n \to \infty$, we get $P\left(\alpha\left(\|\Phi u^* - \xi\|\right) > t\right) = 0$ for all $t > 0$ implying $\Phi u^* = \xi$ a.s. Hence, $\xi = \Phi u^* = S u^*$ a.s and u^* is the random coincidence of Φ, S.

From the assumption (a), there exists v^* in $L_0^X(\Omega)$ such that $Tv^* = \xi$. So, we have

$$\begin{aligned}
P\left(\alpha\left(\|\xi_{2n} - \Psi v^*\|\right) > t\right) &= P\left(\alpha\left(\|\Phi u_{2n} - \Psi v^*\|\right) > t\right) \\
&\leq P\left(\alpha\left(\|S u_{2n} - T v^*\|\right) > t\right) \\
&\leq P\left(\alpha\left(\|S u_{2n} - T v^*\|\right) - f\left(\alpha\left(\|S u_{2n} - T v^*\|\right)\right) > t\right) \\
&\leq P\left(\alpha\left(\|S u_{2n} - T v^*\|\right) > t/q\right) \\
&= P\left(\alpha\left(\|\xi_{2n-1} - \xi\|\right) > t/q\right).
\end{aligned}$$

Let $n \to \infty$, we get $P\left(\alpha\left(\|\xi - \Psi v^*\|\right) > t\right) = 0$ for all $t > 0$ implying $\Psi v^* = \xi$ a.s. Hence, $\xi = \Psi v^* = T v^*$ a.s and u^* is the random coincidence of Ψ, T.

If the condition (d) is satisfied, because Φ and S are weakly compatible, we have $\Phi\xi = \Phi S u^* = S\Phi u^* = S\xi$. Agains, Ψ and T are weakly compatible, then $\Psi\xi = \Psi T u^* = T\Psi u^* = T\xi$. So we have

$$\begin{aligned}
P\left(\alpha\left(\|\Phi\xi - \Psi\xi\|\right) > t\right) &\leq P\left(\alpha\left(\|S\xi - T\xi\|\right) - f\left(\alpha\left(\|S\xi - T\xi\|\right)\right) > t\right) \\
&= P\left(\alpha\left(\|\Phi\xi - \Psi\xi\|\right) - f\left(\alpha\left(\|\Phi\xi - \Psi\xi\|\right)\right) > t\right) \\
&= P\left(\alpha\left(\|\Phi\xi - \Psi\xi\|\right) > t/q\right)
\end{aligned}$$

for all $t > 0$. Then we have $P\left(t < \alpha\left(\|\Phi\xi - \Psi\xi\|\right) \leq t/q\right)$ for all $t > 0$. Thus, $\alpha\left(\|\Phi\xi - \Psi\xi\|\right) = 0$ a.s., which implies ξ is random coincidence point of Φ, Ψ, S and T. □

Theorem 3. *Let $\Phi, \Psi, S, T : L_0^X(\Omega) \to L_0^X(\Omega)$ be completely random operators, $f : [0, \infty) \to [0, \infty)$ be a continuous, increasing function such that $f(0) = 0, \lim_{t\to\infty} f(t) = \infty$, q be a positive number in $(0, 1)$ and $\alpha : [0, \infty) \to [0, \infty)$ be continuous, nondecreasing function satisfying $\alpha(t+u) \geq \alpha(t)+\alpha(u)$ for all $t, u \geq 0$ and $\alpha(t) = 0$ if and only if $t = 0$. If*

(a) *$S(L_0^X(\Omega)), T(L_0^X(\Omega))$ is closed in $L_0^X(\Omega)$;*
(b) *$\Phi(L_0^X(\Omega)) \subset T(L_0^X(\Omega)), \Psi(L_0^X(\Omega)) \subset S(L_0^X(\Omega))$;*
(c) *for any random variables u, v in $L_0^X(\Omega)$ and $t > 0$, we have*

$$P\left(\alpha\left(\|\Phi u - \Psi v\|\right) > f\left(t\right)\right) \leq P\left(\alpha\left(\|S u - T v\|\right) > f\left(t/q\right)\right). \tag{8}$$

Then,

(1) *Assume that there exists a number c in $(q, 1)$ such that*

$$\sum_{n=1}^{\infty} f(c^n) < \infty. \tag{9}$$

Then, the pairs (Φ, S) *and* (Ψ, T) *have a random coincidence point if there exist a random variable* u_0 *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that*

$$M = \sup_{t>0} t^p P\left(\alpha\left(\|\Phi u_0 - \Psi u_1\|\right) > f(t)\right) < \infty \tag{10}$$

where u_1 *is a random variable satisfying* $Tu_1 = \Phi u_0$.

(2) *Assume that for each* $t, s > 0$

$$f(t+s) \geq f(t) + f(s). \tag{11}$$

Then, the condition (10) *is also sufficient for the pairs* (Φ, S) *and* (Ψ, T) *have a random coincidence point.*

(3) *If the pairs* (Φ, S) *and* (Ψ, T) *are weakly compatible and* (9) *or* (11) *are satisfied, then* Φ, Ψ, S *and* T *have a random coincidence point.*

Proof. Let $g = f^{-1}$ be the inverse function of f. Then, $g : [0, \infty) \to [0, \infty)$ is increasing with $g(0) = 0, \lim_{t\to\infty} g(t) = \infty$. The condition (8) is equivalent to the following

$$P\left(g\left(\alpha\left(\|\Phi u - \Psi v\|\right)\right) > t\right) \leq P\left(g\left(\alpha\left(\|Su - Tv\|\right)\right) > t/q\right). \tag{12}$$

Let u_0 be a random variable in $L_0^X(\Omega)$ such that (10) holds and u_1 is the random variable such that $Tu_1 = \Phi u_0$. By the assumption (b), there exists a random variable u_2 in $L_0^X(\Omega)$ such that $Su_2 = \Psi u_1$. By induction, there exists a sequence (u_n) in $L_0^X(\Omega)$ by

$$Tu_1 = \Phi u_0, Su_2 = \Psi u_1, \ldots, Tu_{2n+1} = \Phi u_{2n}, Su_{2n+2} = \Psi u_{2n+1} \quad n = 1, 2, \ldots \tag{13}$$

Put

$$\xi_{2n} = \Phi u_{2n} = Tu_{2n+1}; \quad \xi_{2n+1} = \Psi u_{2n+1} = Su_{2n+2}. \tag{14}$$

From (12), for each n, we obtain

$$\begin{aligned} P\left(g\left(\alpha\left(\|\xi_{2n+2} - \xi_{2n+1}\|\right)\right) > t\right) &= P\left(g\left(\alpha\left(\|\Phi u_{2n+2} - \Psi u_{2n-1}\|\right)\right) > t\right) \\ &\leq P\left(g\left(\alpha\left(\|Su_{2n+2} - Tu_{2n+1}\|\right)\right) > t/q\right) \\ &= P\left(g\left(\alpha\left(\|\xi_{2n+1} - \xi_{2n}\|\right)\right) > t/q\right), \end{aligned}$$

and

$$\begin{aligned} P\left(g\left(\alpha\left(\|\xi_{2n+1} - \xi_{2n}\|\right)\right) > t\right) &= P\left(g\left(\alpha\left(\|\Phi u_{2n} - \Psi u_{2n+1}\|\right)\right) > t\right) \\ &\leq P\left(g\left(\alpha\left(\|Su_{2n} - Tu_{2n+1}\|\right)\right) > t/q\right) \\ &= P\left(g\left(\alpha\left(\|\xi_{2n-1} - \xi_{2n}\|\right)\right) > t/q\right) \\ &= P\left(g\left(\alpha\left(\|\xi_{2n} - \xi_{2n-1}\|\right)\right) > t/q\right). \end{aligned}$$

By induction, we obtain for each n

$$\begin{aligned} &P\left(g\left(\alpha\left(\|\xi_{n+1} - \xi_n\|\right)\right) > t\right) \\ &\quad \leq P\left(g\left(\alpha\left(\|\xi_1 - \xi_0\|\right)\right) > t/q^n\right) = P\left(g\left(\alpha\left(\|\Phi u_0 - \Psi u_1\|\right)\right) > t/q^n\right). \end{aligned} \tag{15}$$

(1) From (10), we have

$$P\left(g\left(\alpha\left(\|\Phi u_0-\Psi u_1\|\right)\right)>s\right)=P\left(\alpha\left(\|\Phi u_0-\Psi u_1\|\right)>f\left(s\right)\right)\leq\frac{M}{s^p}. \quad (16)$$

From (15) and (16), we get

$$P\left(g\left(\alpha\left(\|\xi_{n+1}-\xi_n\|\right)\right)>t\right)\leq\frac{Mq^{np}}{t^p}. \quad (17)$$

Taking $t=c^n$, from (17), we get

$$P\left(g\left(\alpha\left(\|\xi_{n+1}-\xi_n\|\right)\right)>c^n\right)\leq M\frac{q^{np}}{c^{np}} \quad (18)$$

i.e.

$$P\left(\alpha\left(\|\xi_{n+1}-\xi_n\|\right)>f\left(c^n\right)\right)\leq M\frac{q^{np}}{c^{np}}\,. \quad (19)$$

Since

$$\sum_{n=1}^{\infty}P\left(\alpha\left(\|\xi_{n+1}-\xi_n\|\right)>f(c^n)\right)\leq M\sum_{n=1}^{\infty}\frac{q^{np}}{c^{np}}<\infty,$$

by the Borel-Cantelli Lemma, there is a set D with probability one such that for each ω in D there is $N(\omega)$

$$\alpha\left(\|\xi_{n+1}-\xi_n\|\right)\leq f(c^n)\quad\forall n>N(\omega).$$

By (9), we conclude that $\sum_{n=1}^{\infty}\alpha\left(\|\xi_{n+1}-\xi_n\|\right)<\infty$ for all ω in D, which implies that there exists $\lim\xi_n(\omega)$ for all ω in D. Consequently, the sequence (ξ_n) converges a.s. to ξ in $L_0^X(\Omega)$. From the assumption (a), there exists u^* in $L_0^X(\Omega)$ such that $Su^*=\xi$. So, we have

$$\begin{aligned}P\left(\alpha\left(\|\Phi u^*-\xi_{2n+1}\|\right)>f(t)\right)&=P\left(\alpha\left(\|\Phi u^*-\Psi u_{2n+1}\|\right)>f(t)\right)\\&\leq P\left(\alpha\left(\|Tu^*-Tu_{2n+1}\|\right)>f(t/q)\right)\\&=P\left(\alpha\left(\|\xi-\xi_{2n}\|\right)>f(t/q)\right).\end{aligned}$$

Let $n\to\infty$, we get $P\left(\alpha\left(\|\Phi u^*-\xi\|\right)>f(t)\right)=0$ for all $t>0$ implying $\Phi u^*=\xi=Su^*$ a.s. and u^* is the random coincidence of Φ,S.

Again, from the assumption (a), there exists v^* in $L_0^X(\Omega)$ such that $Tv^*=\xi$. So, we have

$$\begin{aligned}P\left(\alpha\left(\|\xi_{2n}-\Psi v^*\|\right)>f(t)\right)&=P\left(\alpha\left(\|\Phi u_{2n}-\Psi v^*\|\right)>f(t)\right)\\&\leq P\left(\alpha\left(\|Su_{2n}-\Psi v^*\|\right)>f(t/q)\right)\\&=P\left(\alpha\left(\|\xi_{2n-1}-\xi\|\right)>f(t/q)\right).\end{aligned}$$

Let $n\to\infty$, we get $P\left(\alpha\left(\|\xi_{2n}-\Psi v^*\|\right)>f(t)\right)=0$ for all $t>0$ implying $\Psi v^*=\xi=Tv^*$ a.s. and v^* is the random coincidence of Ψ,T.

(2) It is easy to see that for each $t,s>0$

$$g(s+t)\leq g(t)+g(s).$$

Hence, for $a = \sum_{i=1}^{m} s_i$, we have

$$\begin{aligned} P\left(g\left(\alpha\left(\|\xi_{n+1}-\xi_n\|\right)\right)>a\right) &\leq P\left(g\left(\sum_{i=1}^{m}\alpha\left(\|\xi_{n+i}-\xi_{n+i-1}\|\right)\right)>a\right) \\ &\leq P\left(\sum_{i=1}^{m} g\left(\alpha\left(\|\xi_{n+i}-\xi_{n+i-1}\|\right)\right)>a\right) \\ &\leq \sum_{i=1}^{m} P\left(g\left(\alpha\left(\|\xi_{n+i}-\xi_{n+i-1}\|\right)\right)>s_i\right). \end{aligned}$$

From (10), we have

$$P\left(g\left(\alpha\left(\|\xi_{n+i}-\xi_{n+i-1}\|\right)\right)>s_i\right) \leq \frac{Mq^{(n+i-1)p}}{s_i^p}. \tag{20}$$

Put $r = \frac{x}{q}$ where $q < x < 1$ and $s_i = s(r-1)/r^i$. An argument similar to that in the proof of Theorem 2 yields

$$\lim_{n\to\infty} P\left(g\left(\alpha\left(\|\xi_{n+m}-\xi_n\|\right)\right)>s\right)=0 \quad \forall s>0,$$

so

$$\lim_{n\to\infty} P\left(\alpha\left(\|\xi_{n+m}-\xi_n\|\right)>f\left(s\right)\right)=0 \quad \forall s>0.$$

Thus, we obtain

$$\lim_{n\to\infty} P\left(\alpha\left(\|\xi_{n+m}-\xi_n\|\right)>t\right)=0 \quad \forall t>0.$$

Consequently, the sequence (ξ_n) converges in probability to ξ in $L_0^X(\Omega)$. By the same argument as above, we receive that the pairs (Φ, S) and (Ψ, T) have a random coincidence point.

(3) Because Φ and S are weakly compatible, we have $\Phi\xi = \Phi S u^* = S\Phi u^* = S\xi$. Agains, Ψ and T are weakly compatible, then $\Psi\xi = \Psi T u^* = T\Psi u^* = T\xi$. So we have

$$\begin{aligned} P\left(\alpha\left(\|\Phi\xi-\Psi\xi\|\right)>f(t)\right) &\leq P\left(\alpha\left(\|S\xi-T\xi\|\right)-f\left(\alpha\left(\|S\xi-T\xi\|\right)\right)>f(t/q)\right) \\ &= P\left(\alpha\left(\|\Phi\xi-\Psi\xi\|\right)>f(t/q)\right) \end{aligned}$$

for all $t > 0$. Then we have $P\left(f(t) < \alpha\left(\|\Phi\xi-\Psi\xi\|\right) \leq f(t/q)\right)$ for all $t > 0$. Thus, $\alpha\left(\|\Phi\xi-\Psi\xi\|\right) = 0$ a.s., which implies ξ is random coincidence point of Φ, Ψ, S and T.

□

4. Applications to random fixed point theorems and random equations

In this section, we present some applications to random fixed point theorems and random equations.

Theorem 4. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator, $f : [0,\infty) \to [0,\infty)$ be a continuous, increasing function such that $f(0) = 0, \lim_{t\to\infty} f(t) = \infty$ and q be a positive number in $(0,1)$ and $\alpha : [0,\infty) \to [0,\infty)$ be continuous, nondecreasing function satisfying $\alpha(t+u) \leq \alpha(t)+\alpha(u)$ for all $t,u \geq 0$ and $\alpha(t) = 0$ if and only if $t = 0$. Assume that for each pair u, v in $L_0^X(\Omega)$*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > f(t)\right) \leq P\left(\alpha\left(\|u - v\|\right) > f(t/q)\right). \tag{21}$$

Then

(1) *Assume that there exists a number c in $(q,1)$ such that*

$$\sum_{n=1}^{\infty} f(c^n) < \infty. \tag{22}$$

Then, Φ have a unique random fixed point if there exist a random variable u_0 in $L_0^X(\Omega)$ and $p > 0$ such that

$$M = \sup_{t>0} t^p P\left(\alpha\left(\left\|\Phi u_0 - \Phi^2 u_0\right\|\right) > f(t)\right) < \infty. \tag{23}$$

(2) *Assume that for each $t, s > 0$*

$$f(t+s) \geq f(t) + f(s). \tag{24}$$

Then, the condition (23) is also sufficient for Φ to have a unique random fixed point.

Proof. Consider the completely random operators Ψ, S and T given by $\Psi u = \Phi u, Su = Tu = u$. By Theorem 3, Φ and Ψ have a random coincidence point ξ which is exactly the random fixed point of Φ.

Let ξ, η be two random fixed points of Φ. Then, for each $t > 0$, we have

$$\begin{aligned} P\left(\alpha\left(\|\xi - \eta\|\right) > f(t)\right) &= P\left(\alpha\left(\|\Phi\xi - \Phi\eta\|\right) > f(t)\right) \\ &\leq P\left(\alpha\left(\|\xi - \eta\|\right) > f(t/q)\right). \end{aligned}$$

By induction, it follows that

$$P\left(\alpha\left(\|\xi - \eta\|\right) > f(t)\right) \leq P\left(\alpha\left(\|\xi - \eta\|\right) > f(t/q^n)\right) \quad \forall n.$$

Since $\lim_{n\to\infty} f(t/q^n) = +\infty$, we conclude that $P\left(\alpha\left(\|\xi - \eta\|\right) > f(t)\right)$ for each $t > 0$. Hence, $g(\alpha\left(\|\xi - \eta\|\right)) = 0$ a.s., with g is the inverse function of f. So, we have $\xi = \eta$ a.s. as claimed. □

Theorem 5. *Let* $\Phi, S : L_0^X(\Omega) \to L_0^X(\Omega)$ *be completely random operators,* $f : [0,\infty) \to [0,\infty)$ *be a mapping such that* $f(t) = 0$ *if and only if* $t = 0$, $f(t) < t \quad \forall t > 0$ *and* $\alpha : [0,\infty) \to [0,\infty)$ *be a continuous, nondecreasing function satisfying* $\alpha(t+u) \le \alpha(t) + \alpha(u)$ *for all* $t, u \ge 0$ *and* $\alpha(t) = 0$ *if and only if* $t = 0$. *For each* $t > 0$, *define*

$$h(t) = \inf_{s \ge t} \frac{f(s)}{\alpha(s)}. \tag{25}$$

Assume that $h(t) > 0 \quad \forall t > 0$ *and*

(a) $S(L_0^X(\Omega))$ *is closed in* $L_0^X(\Omega)$;
(b) $\Phi(L_0^X(\Omega)) \subset S(L_0^X(\Omega))$;
(c) *For each pair* u, v *in* $L_0^X(\Omega)$ *and* $t > 0$, *we have*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > t\right) \le P\left(\alpha\left(\|Su - Sv\|\right) - f\left(\alpha\left(\|Su - Sv\|\right)\right) > t\right); \tag{26}$$

(d) *the pair* (Φ, S) *is weakly compatible.*

Then, Φ *and* S *have a unique common random fixed point if there exist a random variable* u_0 *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that*

$$M = E[\alpha\left(\|\Phi u_0 - \Phi u_1\|\right)]^p < \infty \tag{27}$$

where u_1 *is the random variable satisfying* $Su_1 = \Phi u_0$.

Proof. Suppose that (27) holds. By Theorem 2, there exists u^* such that $\Phi u^* = Su^* = \xi$. For $t > 0$, we have

$$\begin{aligned} P\left(\alpha\left(\|\Phi\xi - \xi\|\right) > t\right) &= P\left(\alpha\left(\Phi\xi - \Phi u^*\right) > t\right) \le P\left(\alpha\left(\|S\xi - Su^*\|\right) > t/q\right) \\ &= P\left(\alpha\left(\|S\Phi u^* - \xi\|\right) > t/q\right) = P\left(\alpha\left(\Phi S u^* - \xi\right) > t/q\right) \\ &= P\left(\alpha\left(\|\Phi\xi - \xi\|\right) > t/q\right). \end{aligned}$$

By induction, it follows that $P\left(\alpha\left(\|\Phi\xi - \xi\|\right) > t\right) \le P\left(\alpha\left(\|\Phi\xi - \xi\|\right) > t/q^n\right)$ for any $n \in N$. Let $n \to \infty$, we have $P\left(\alpha\left(\|\Phi\xi - \xi\|\right) > t\right) = 0$ for any $t > 0$. Thus, $\Phi\xi = \xi$ i.e. ξ is a random fixed point of Φ. We have $S\xi = S\Phi u^* = \Phi S u^* = \Phi\xi = \xi$. So ξ is also a random fixed point of S.

Let ξ_1 and ξ_2 be two common random fixed points of Φ and S. For each $t > 0$, we have

$$\begin{aligned} P\left(\alpha\left(\|\xi_1 - \xi_2\|\right) > t\right) &= P\left(\alpha\left(\|\Phi\xi_1 - \Phi\xi_2\|\right) > t\right) \le P\left(\alpha\left(\|S\xi_1 - S\xi_2\|\right) > t/q\right) \\ &= P\left(\alpha\left(\|\xi_1 - \xi_2\|\right) > t/q\right) > t/q) \\ &\le \cdots \le P\left(\alpha\left(\|\xi_1 - \xi_2\|\right) > t/q^n\right). \end{aligned}$$

Let $n \to \infty$, we have $P\left(\alpha\left(\|\xi_1 - \xi_2\|\right) > t\right) = 0$ for all $t > 0$. Hence, $\xi_1 = \xi_2$. □

Corollary 1. *Let* $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ *be a completely random operator. Assume that there exists a number* q *in* $(0,1)$ *such that*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > t\right) \leq P\left(\alpha\left(\|u - v\|\right) > t/q\right)$$

for all random variables u, v *in* $L_0^X(\Omega)$ *and* $t > 0$*, and* $\alpha : [0,\infty) \to [0,\infty)$ *be a continuous, nondecreasing function satisfying* $\alpha(t+u) \leq \alpha(t) + \alpha(u)$ *for all* $t, u \geq 0$ *and* $\alpha(t) = 0$ *if and only if* $t = 0$*. Then,* Φ *has a unique random fixed point if there exist a random variable* u_0 *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that*

$$E\left[\alpha\left(\left\|\Phi u_0 - \Phi^2 u_0\right\|\right)\right]^p < \infty.$$

Proof. Consider $S : L_0^X(\Omega) \to L_0^X(\Omega)$ given by $Su = u$, the function $f(t) = (1-q)t$ and $h(t) = 1 - q > 0$. Then Φ, S and $f(t)$ satisfy the conditions stated in the Theorem 5 and Φ, S commute. Thus, Φ and S have a common random fixed point ξ i.e. Φ has a random fixed point ξ. □

Theorem 6. *Let* $\Phi, S : L_0^X(\Omega) \to L_0^X(\Omega)$ *be probabilistic completely random operators where*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > f(t)\right) \leq P\left(\alpha\left(\|Su - Sv\|\right) > f(t/q)\right) \tag{28}$$

for all u, v *in* $L_0^X(\Omega)$*,* $t > 0$ *and* $f : [0,\infty) \to [0,\infty)$ *be a continuous, increasing function such that* $f(0) = 0, \lim_{t\to\infty} f(t) = \infty$ *satisfying either* (22) *or* (24) *and* q *is a positive number,* $\alpha : [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing function satisfying* $\alpha(t+u) \leq \alpha(t) + \alpha(u)$ *for all* $t, u \geq 0$ *and* $\alpha(t) = 0$ *if and only if* $t = 0$*. Consider random equation of the form*

$$\Phi u - \lambda S u = \eta \tag{29}$$

where λ *is a real number and* η *is a random variable in* $L_0^X(\Omega)$*.*

Assume that

$$S(L_0^X(\Omega)) \text{ is closed in } L_0^X(\Omega) \tag{30}$$

$$\Phi(L_0^X(\Omega)) \subset \lambda S(L_0^X(\Omega)) + \eta \tag{31}$$

$$|\lambda| \geq \sup_{t>0} \frac{f\left(\frac{q}{q'} t\right)}{f\left(t\right)} \tag{32}$$

where q' *in* $(0,1)$*. Then the equation* (29) *has a unique random solution if there exist a random variable* u_0 *in* $L_0^X(\Omega)$ *and a number* $p > 0$ *such that*

$$M = \sup_{t>0} t^p P\left(\alpha\left(|\lambda| \|\Phi u_0 - \Phi u_1\|\right) > f\left(t\right)\right) < \infty \tag{33}$$

where u_1 *is a random variable satisfying* $Su_1 = \frac{\Phi u_0 - \eta}{\lambda}$*.*

Proof. Suppose that the condition (33) holds. Define a completely random operator Θ by

$$\Theta u = \frac{\Phi u - \eta}{\lambda}.$$

From (31) and (33) it follows that

$$\Theta(L_0^X(\Omega)) \subset S(L_0^X(\Omega))$$
$$M = \sup_{t>0} t^p P\left(\alpha\left(\|\Theta u_0 - \Theta u_1\|\right) > f(t)\right) < \infty.$$

where u_1 is a random variable satisfying $Su_1 = \frac{\Phi u_0 - \eta}{\lambda} = \Theta u_0$. Let $g = f^{-1}$ be the inverse function of f. Then, $g : [0, \infty) \to [0, \infty)$ is continuous, increasing with $g(0) = 0, \lim_{t\to\infty} g(t) = \infty$. For each $t > 0$, there exists t' so that $f(t') = |\lambda| f(t)$ i.e. $t' = g(|\lambda| f(t))$. So, we have

$$\begin{aligned} P\left(\alpha\left(\|\Theta u - \Theta v\|\right) > f\left(t\right)\right) &= P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > |\lambda| f\left(t\right)\right) \\ &= P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > f\left(t'\right)\right) \\ &\le P\left(\alpha\left(\|Su - Sv\|\right) > f\left(t'/q\right)\right) \\ &= P\left(\alpha\left(\|Su - Sv\|\right) > f\left(\frac{t}{q'}\frac{q't'}{qt}\right)\right). \end{aligned}$$

From (32), we receive $|\lambda| f\left(t\right) \ge f\left(\frac{q}{q'}t\right)$. Then, we deduce $g\left(|\lambda| f\left(t\right)\right) \ge \frac{q}{q'}t$. So, $t' \ge \frac{q}{q'}t$ and $\frac{q't'}{qt} \ge 1$. Hence,

$$P\left(\alpha\left(\|Su - Sv\|\right) > f\left(\frac{t}{q'}\frac{q't'}{qt}\right)\right) \le P\left(\alpha\left(\|Su - Sv\|\right) > f\left(t/q'\right)\right)$$

which implies

$$P\left(\alpha\left(\|\Theta u - \Theta v\|\right) > f\left(t\right)\right) \le P\left(\alpha\left(\|Su - Sv\|\right) > f\left(t/q'\right)\right).$$

Consequently, Θ and S satisfy the conditions stated in the Theorem 3. Hence, Θ and S has a random coincidence point ξ i.e. the equation (29) has a random solution ξ. ☐

Corollary 2. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator satisfying the following condition*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > f(t)\right) \le P\left(\alpha\left(\|u - v\|\right) > f(t/q)\right) \tag{34}$$

for all u, v in $L_0^X(\Omega)$, $t > 0$, where $f : [0, \infty) \to [0, \infty)$ is a continuous, increasing function such that $f(0) = 0, \lim_{t\to\infty} f(t) = \infty$ satisfying either (22) or (24) and q be a positive number, $\alpha : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function satisfying $\alpha(t + u) \le \alpha(t) + \alpha(u)$ for all $t, u \ge 0$ and $\alpha(t) = 0$ if and only if $t = 0$. Consider random equation of the form

$$\Phi u - \lambda u = \eta \tag{35}$$

where λ is a real number and η is a random variable in $L_0^X(\Omega)$.

Assume that

$$|\lambda| \geq \sup_{t>0} \frac{f\left(\frac{q}{q'}t\right)}{f(t)} \tag{36}$$

where q' in $(0,1)$. Then the equation (35) has a unique random solution if there exist a random variable u_0 in $L_0^X(\Omega)$ and a number $p>0$ such that

$$M = \sup_{t>0} t^p P\left(\alpha\left(|\lambda| \|\Phi u_0 - \Phi u_1\|\right) > f(t)\right) < \infty \tag{37}$$

where $u_1 = \frac{\Phi u_0 - \eta}{\lambda}$.

Proof. Apply Theorem 6 for the completely random operator S given by $Su = u$. □

Corollary 3. *Let $\Phi, S : L_0^X(\Omega) \to L_0^X(\Omega)$ be two completely random operators satisfying the following condition*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > t\right) \leq P\left(\alpha\left(\|Su - Sv\|\right) > t/q\right) \tag{38}$$

where $\alpha : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing function satisfying $\alpha(t+u) \leq \alpha(t) + \alpha(u)$ for all $t, u \geq 0$ and $\alpha(t) = 0$ if and only if $t = 0$. Consider the random equation

$$\Phi u - \lambda S u = \eta \tag{39}$$

where λ is a real number and η is a random variable in $L_p^X(\Omega)$, $p > 0$.

Assume that

$$S(L_0^X(\Omega)) \text{ is closed in } L_0^X(\Omega)$$
$$\Phi(L_0^X(\Omega)) \subset \lambda S(L_0^X(\Omega)) + \eta$$
$$|\lambda| > q.$$

Then, the random equation (39) has a solution if there exists a random variable u_0 in $L_0^X(\Omega)$ such that

$$E[\alpha\left(|\lambda| \|\Phi u_0 - \Phi u_1\|\right)]^p < \infty \tag{40}$$

where u_1 is a random variable satisfying $Su_1 = \frac{\Phi u_0 - \eta}{\lambda}$.

Proof. Suppose that there exists a random variable u_0 in $L_0^X(\Omega)$ such that (40) holds. So, Φ and S satisfy (34) where $f(t) = t$. Take $q < s < |\lambda|$, then $q' = q/s < 1$ and

$$|\lambda| > s = \frac{q}{q'} = \frac{f\left(\frac{q}{q'}t\right)}{f(t)}.$$

Moreover, for each $t > 0$, by the Chebyshev inequality, we have

$$t^p P\left(\alpha\left(|\lambda| \|\Phi u_0 - \Phi u_1\|\right) > t\right) \leq E[\alpha\left(|\lambda| \|\Phi u_0 - \Phi u_1\|\right)]^p < \infty.$$

Hence, the condition (33) is satisfied. By Theorem 6, we conclude that the equation (39) has a random solution. □

Taking the completely random operator S given by $Su = u$, we obtain:

Corollary 4. *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator satisfying the following condition*

$$P\left(\alpha\left(\|\Phi u - \Phi v\|\right) > t\right) \leq P\left(\alpha\left(\|u - v\|\right) > t/q\right) \tag{41}$$

where $\alpha : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function satisfying $\alpha(t + u) \leq \alpha(t) + \alpha(u)$ for all $t, u \geq 0$ and $\alpha(t) = 0$ if and only if $t = 0$. Consider the random equation

$$\Phi u - \lambda u = \eta \tag{42}$$

where λ is a real number satisfying $|\lambda| > q$ and η is a random variable in $L_p^X(\Omega)$, $p > 0$. Then, the random equation (42) has a unique random solution if there exists a random variable u_0 in $L_0^X(\Omega)$ such that

$$E[\alpha\left(|\lambda|\|\Phi u_0 - \Phi u_1\|\right)]^p < \infty \tag{43}$$

where $u_1 = \frac{\Phi u_0 - \eta}{\lambda}$.

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