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# ON STABILITY, BOUNDEDNESS AND INTEGRABILITY OF SOLUTIONS OF CERTAIN SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. In this paper, the problems of stability, boundedness and integrability of solutions of a certain class of second order integro-differential equations are considered. By using a suitable Lyapunov-Krasovskii function(al), conditions to guarantee the stability of the null solution, boundedness and integrability of solutions were established. The results of this paper compliment in one way and generalize some of the known results existing in the literature.

## 1. INTRODUCTION

A differential equation is said to be an integro-differential equation (IDE) if it contains the integrals of the unknown function. When the current state of such an integro-differential equation now depends on the previous states, it is known to be a time-delay integro-differential equation.

Indeed, it is a well-known fact that stability and boundedness properties of solutions of second order (also higher order) ordinary differential equations and integro-differential equations with or without delay have many applications in many fields of science and technology such as biology, medicine, engineering, information system, control theory and financial mathematics. Therefore, the study of their qualitative properties has attracted the attention of many researchers, see ( [1] - [41]) and references contained in them. Readers are referred to [3] for an expository treatment of Volterra integral and differential equations.

In particular, Napoles [12] studied the problem of continuability and integrability of the first derivative of solutions of the following second order integro-differential equation

$$x'' + a(t)f(t,x,x')x' + g(t,x') + h(x) = \int_0^t C(t,s)x'(s)ds, \tag{1.1}$$

where $a(t)$ is a positive function defined on interval $I = [0,\infty)$. The direct method

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of Lyapunov was employed to show that, under certain conditions the solutions of (1.1) exist and are bounded.

Graef and Tunc [9] in 2014 considered a more general integro-differential equation with multiple delays given by

$$x'' + a(t)f(t,x,x')x' + g(t,x,x') + \sum_{i=1}^{n} h_i(x(t-\tau_i)) = \int_0^t C(t,\xi)x'(\xi)d\xi, \qquad (1.2)$$

where $\tau_i$ are positive constants. The duo used a Lyapunov-Krasovskii functional to prove some results on the problem of global continuability and boundedness of solutions of the time-delay IDE (1.2) under some predetermined assumptions. In [40], Zhao and Meng gave sufficient criteria for the stability of zero solutions of the following equations

$$x'' + a(t)f(t,x,x')x' + g(t,,x,x') + h(x(t-\tau)) = p(t,x(t))\int_0^t q(s,x'(s))ds \qquad (1.3)$$

and

$$x'' + a(t)f(t,x,x')x' + g(t,,x,x') + h(x) = p(x(t-\tau))\int_0^t q(s,x'(s))ds, \qquad (1.4)$$

where $\tau$ is a fixed positive constant.
Very recently, Mohammed [15] employed a suitable Lyapunov-Krasovskii functional to establish some new results on global existence, stability, asymptotic stability, boundedness of solutions and square integrability of the first derivatives of solutions of the following second order nonlinear delayed IDE

$$x'' + a(t)f(t,x,x')x' + b(t)p(x) + \sum_{i=1}^{n} h_i(x(t-\tau_i),x') = \int_0^t k(t,\xi)h(\xi,\frac{dx}{d\xi})d\xi, \qquad (1.5)$$

where $a(t)$ and $b(t)$ are continuous positive functions and $\tau_i,(i=1,2,\dots n)$ are fixed positive delay constants.

This work is motivated by the works of Graef and Tunc [9], Napoles [12], Mohammed [15] and Zhao and Meng [40]. Our goal in this paper is to give sufficient conditions which will ensure and guarantee stability of null solution, boundedness and integrability of solutions of the following integro-differential equations with multiple delays

$$x'' + a(t)f(t,x,x')x' + b(t)g(t,x,x') + c(t)h(x) = \sum_{i=1}^{n} P_i(t,x(t-\tau_i))\int_0^t K(s,x'(s))ds, \qquad (1.6)$$

where $\tau_i > 0$ are fixed delay constants and we allow $\tau^* = \max\{\tau_1,\tau_2,...,\tau_n\}$; the prime represents differentiation with respect to $t$, $t \in \mathbb{R}^+ = [0,\infty)$; $a(t),b(t),c(t) \in C(\mathbb{R}^+,\mathbb{R}^+)$ and the derivative $c'(t) = \frac{d}{dt}c(t)$ exists; $f,g \in C(\mathbb{R}^+ \times \mathbb{R}^2,\mathbb{R})$, $\mathbb{R} = (-\infty,\infty)$; $h \in C(\mathbb{R},\mathbb{R})$; $P_i,K \in C(\mathbb{R}^+ \times \mathbb{R},\mathbb{R})$ such that $0 \le s \le t < \infty$; $h(0) = 0, f(t,x,0) = 0, g(t,x,0) = 0, P_i(t,0) = 0$ and $K(s,0) = 0$. Also, it is assumed that the functions $f,g,h$ and $P_i$ are Lipschitz continuous in $x,\dot{x}$, $x(t-\tau_i)$ and the partial derivatives $\frac{\partial}{\partial x}P_i(\xi,x(\xi))$ exist and are continuous.

The remaining parts of the paper are presented as follows. The next Section features basic definitions of stability and boundedness of solutions of integro-differential equations, Section 3 gives the basic assumptions for this work. The main results of this work are presented in Section 4 while in Section 5 an example to validate our results is given.

## 2. Preliminary Definitions

In this section, we will give some basic definitions on the stability of solution of integro-differential equations. Consider the system of first order non-linear and non homogeneous Volterra integro-differential equations

$$X'(t) = -A(t)X(t) + \int_{t-\tau}^{t} B(t,s,h(X(s)))g(X(s))ds + E(t,X(t)), \tag{2.1}$$

where $t \in [0,\infty)$, $X \in \mathbb{R}^n, A(t), B(t,s,h(X(s))$ and $E(t,X(t))$ are continuous functions for the respective arguments explicitly displayed against them such that $0 \leq s \leq t < \infty, h(0) = 0, h(X) \neq 0,\ X \neq 0, B(t,s,0) = 0; h,g : \mathbb{R}^n \longrightarrow \mathbb{R}^n, g(0) = 0$ are continuous functions and $\tau > 0$ is a constant delay.
Let $X(t,t_0,\Phi), t \geq t_0$ be a solution of (2.1) on $[t_0 - \tau, \beta), \beta > 0$ such that $X(t) = \Phi(t)$ on $\Phi \in [t_0 - \tau, t_0]$ and $||\Phi(t)|| = \sup_{t\in[t_0-\tau,t_0]} ||\Phi(t)||$, where $\Phi : [t_0 - \tau, t_0] \longrightarrow \mathbb{R}^n$ is a continuous initial function.

The following basic definitions will be given for completeness sake.

**Definition 2.1.** [3] *The zero solution of the* (2.1) *is said to be* stable *if for each* $\varepsilon > 0$ *and* $t_0 \geq 0$. *there exists a* $\delta = \delta(t_0, \varepsilon) > 0$ *such that if* $||\Phi(t)|| < \delta$ *on* $[t_0 - \tau, t_0]$, *we have* $||X(t,\Phi)|| < \varepsilon, \forall t \geq t_0$.

**Definition 2.2.** [3] *The zero solution of the* (2.1) *are said to be* uniformly stable *if* $\delta$ *is independent of* $t_0$.

**Definition 2.3.** [3] *The zero solution of the* (2.1) *is said to be* asymptotically stable *if it is stable and for each* $t_0 \geq 0$, *there is a* $\delta > 0$ *such that* $t \geq t_0, ||\Phi(t)|| < \delta$ *on* $[0,t_0]$ *implies* $||X(t,\Phi)|| \to 0$ *as* $t \to \infty$.

**Definition 2.4.** [3] *The solutions* $X(t_0,X_0)$ *of the* (2.1) *is said to be* bounded *if for* $T > 0$, *there exists* $\Gamma$ *such that for* $t_0 > 0, ||\Phi(t)||_{t_0} < T$ *and for* $t \geq t_0, \Rightarrow ||X(t)|| < \Gamma$.

## 3. Basic Assumptions

In this section, we present the basic assumptions for our results.
To begin with, the second order IDE (1.6) is transformed to its equivalent system of first order equationse bellow:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -a(t)f(t,x,y)y - b(t)g(t,x,y) - c(t)h(x) + \sum_{i=1}^{n} P_i(t,x(t))\int_0^t K(s,y(s))ds\\ &\quad - \sum_{i=1}^{n} P_i(t,x(t))\int_{t-\tau_i}^{t}\int_0^t K(s,y(s))\frac{\partial P_i(\xi,x(\xi))}{\partial x}y(\xi)dsd\xi. \qquad (3.1)\end{aligned}$$

**Assumptions:**

Suppose that $a_0, a_1, b_0, b_1, c_0, c_1, \alpha_0, \alpha_1, f_0, f_1, g_0, g_1, K_0, K_1, D, M, N, M_i, N_i$ (i = 1,2, . . . n) are some positive constants, $d_0$ is a negative constant and functions $a(t), b(t), P_i(t), q(t) : \mathbb{R}^+ \to \mathbb{R}^+$ and in addition the following conditions hold:

(i) $a_0 \leq a(t) \leq a_1$, $b_0 \leq b(t) \leq b_1$, $0 < c_0 \leq c(t) \leq c_1$, $c'(t) \leq d_0$;
$h(0) = 0,\ \alpha_0 \leq \dfrac{h(x)}{x} \leq \alpha_1$ for $x \neq 0$;

(ii) $f(t,x,0) = 0,\ f_0 \leq \dfrac{f(t,x,y)}{y^2} \leq f_1$ for $y \neq 0$;
$g(t,x,0) = 0,\ g_0 \leq \dfrac{g(t,x,y)}{y} \leq g_1$, for $y \neq 0$;

(iii) $K(t,y(t)) \leq q(t)|y(t)|$, $K_0 \leq q(t) \leq K_1$ for all $t$;

(iv) $|P_i(t,x(t))| \leq |P_i(t)| \leq M_i$, $|\dfrac{\partial P_i(t,x(t))}{\partial x}| \leq N_i$;

(v) $3N\tau^* \int_0^\infty q(s)ds \leq 4a(t)f_0$; and

(vi) $M\int_0^\infty q(s)ds + \frac{1}{2}K_1\left(2 + D\tau^*\right)\int_0^\infty |P_i(\eta,x(\eta))|d\eta \leq 2b(t)g_0$,

where

$$M = \sum_{i=1}^{n} M_i,\ N = \sum_{i=1}^{n} N_i \text{ and } D = \sum_{i=1}^{n}\frac{N_i}{M_i}.$$

## 4. Main Results

**Theorem 4.1.** *If the conditions stated under the basic assumptions above are satisfied, then all the solutions of the system* (3.1) *are continuable and bounded.*

*Proof.* The proof of the this theorem rest on the following differentiable scalar function $V \equiv V(t,x(t),y(t))$ defined as

$$\begin{aligned}V = \frac{1}{2}y^2 + c(t)\int_0^x h(u)du + \sum_{i=1}^{n}\mu_i\int_0^t\int_t^\infty |P_i(\eta,x(\eta))|q(s)y^2(s)dsd\eta\\ + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} y^4(\theta)d\theta ds, \qquad (4.1)\end{aligned}$$

where $\mu_i$ and $\lambda_i$ are positive constants whose values are to be determined later. The function $V$ defined by equation (4.1) clearly vanishes for $x = y = 0$ and can easily be shown to be positive definite for $x \neq 0, y \neq 0$ following the stated assumptions of Theorem 4.1 in the following way:

$$
\begin{aligned}
V =&\frac{1}{2}y^2 + c(t)\int_0^x \frac{h(u)}{u}udu + \sum_{i=1}^{n}\mu_i \int_0^t \int_t^\infty |P_i(\eta, x(\eta))| q(s) y^2(s) ds d\eta \\
&+ \sum_{i=1}^{n}\lambda_i \int_{-\tau_i}^{0} ds \int_{t+s}^{t} y^4(\theta) d\theta, \\
\geq&\frac{1}{2}y^2 + \frac{1}{2}\alpha_0 c_0 x^2 + \sum_{i=1}^{n}\mu_i \int_0^t \int_t^\infty |P_i(\eta, x(\eta))| q(s) y^2(s) ds d\eta \\
&+ \sum_{i=1}^{n}\lambda_i \int_{-\tau_i}^{0} ds \int_{t+s}^{t} y^4(\theta) d\theta, \\
=&\frac{1}{2}y^2 + \frac{1}{2}\alpha_0 c_0 x^2,
\end{aligned}
$$

there exists a positive constant $d_1$ such that

$$
V(t) \geq d_1(x^2 + y^2), \tag{4.2}
$$

for all $x, y$ where $d_1 = \frac{1}{2}\min\{1, \alpha_0 c_0\} > 0$. Thus, the function $V$ is positive definite at all points $(x, y)$ and zero only at point $x = y = 0$. In addition $V(t) = 0$ if and only if $x^2(t) + y^2(t) = 0$ and $V(t) > 0$ if and only if $x^2(t) + y^2(t) \neq 0$, it follows that

$$
V(t) \to +\infty \text{ as } x^2(t) + y^2(t) \to \infty. \tag{4.3}
$$

Furthermore, there exist positive constants $d_2, d_3$ and $d_4$ such that

$$
V(t) \leq d_2(x^2 + y^2) + d_3 \int_0^t \int_0^\infty y^2(s) ds d\eta + d_4 \int_{-\tau^*}^{0} \int_{t+s}^{t} y^4(\theta) d\theta ds, \tag{4.4}
$$

for all $x, y$ where $d_2 := \frac{1}{2}\max\{1, c_1\alpha_1\}$, $d_3 := \sum_{i=1}^{n} M_i K_1 \mu_i$ and $d_4 := \sum_{i=1}^{n} \lambda_i$.

Next, we proceed to show that the derivative of the function $V$ is negative semi-definite.

$$
\begin{aligned}
\frac{d}{dt}V(t) = V'(t) =& yy' + c'(t)\int_0^x h(u)udu + c(t)h(x)y \\
&+ y^2(t)q(t)\sum_{i=1}^{n}\mu_i \int_t^\infty |P_i(\eta, x(\eta))| d\eta + \sum_{i=1}^{n}(\lambda_i \tau_i) y^4(t) \\
&- \sum_{i=1}^{n}\mu_i |P_i(t, x(t))| \int_0^t q(s) y^2(s) ds - \sum_{i=1}^{n}\lambda_i \int_{t-\tau_i}^{t} y^4(u) du, \\
=& -a(t) f(t, x, y) y^2(t) - b(t) g(t, x, y) y(t) + \sum_{i=1}^{n}(\lambda_i \tau_i) y^4(t) \\
&+ y\sum_{i=1}^{n} P_i(t, x(t)) \int_0^t K(s, y(s)) ds + c'(t)\int_0^x h(u) du - \sum_{i=1}^{n}\lambda_i \int_{t-\tau_i}^{t} y^4(u) du \\
&- y\sum_{i=1}^{n} P_i(t, x(t)) \int_{t-\tau_i}^{t} \int_0^t K(s, y(s)) \frac{\partial P_i(\xi, x(\xi))}{\partial x} y(\xi) ds d\xi
\end{aligned}
$$

$$+q(t)y^2(t)\sum_{i=1}^{n}\mu_i\int_t^{\infty}|P_i(\eta,x(\eta))|d\eta-\sum_{i=1}^{n}\mu_i|P_i(t,x(t))|\int_0^t q(s)y^2(s)ds.$$

By the assumption (iii) of the theorem, we have the following,

$$\begin{aligned} y\sum_{i=0}^{n}P_i(t,x(t))\int_0^t K(s,y(s))ds &\leq |y|\sum_{i=1}^{n}|P_i(t,x(t))|\int_0^t |K(s,y(s))|ds \\ &\leq \sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t |y(t)|(|q(s)||y(s)|)|ds. \end{aligned}$$

On using the inequality $2|ab|\leq(a^2+b^2)$, we obtain,

$$\begin{aligned} \sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t|y(t)|(|q(s)||y(s)|)|ds &\leq \frac{1}{2}\sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t q(s)(y^2(t)+y^2(s))ds \\ &\leq \frac{1}{2}\sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t q(s)y^2(s)ds+\frac{1}{2}y^2(t))\sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t q(s)ds. \end{aligned}$$

Also, by the assumption (iv) and inequality $2|ab|\leq(a^2+b^2)$, we equally obtain that

$$\begin{aligned} &|y|\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\int_0^t|K(s,y(s))||\frac{\partial P_i(\xi,x(\xi))}{\partial x}y(\xi)|dsd\xi \\ &\leq \frac{1}{2}\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\int_0^t N_i q(s)|y(s)|(y^2(t)+y^2(\xi))dsd\xi, \\ &\leq \frac{1}{4}\sum_{i=1}^{n}(N_i\tau_i)\int_0^t q(s)(y^2(s)+y^4(t))ds+\frac{1}{4}\sum_{i=1}^{n}N_i\int_{t-\tau_i}^{t}\int_0^t q(s)(y^2(s)+y^4(\xi))dsd\xi \\ &\leq \frac{1}{4}\sum_{i=1}^{n}(N_i\tau_i)\int_0^t q(s)y^2(s)ds+\frac{1}{2}y^4(t)\sum_{i=1}^{n}(N_i\tau_i)\int_0^t q(s)ds \\ &+\frac{1}{4}\sum_{i=1}^{n}N_i\int_{t-\tau_i}^{t}\int_0^t q(s)y^4(\xi)dsd\xi. \end{aligned}$$

Therefore,

$$\begin{aligned} V'(t) \leq &-a(t)f(t,x,y)y^{-2}(t)y^4(t)-b(t)g(t,x,y)y^{-1}(t)y^2(t)+\sum_{i=1}^{n}(\lambda_i\tau_i)y^4(t) \\ &+\frac{1}{2}\sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t q(s)y^2(s)ds+\frac{1}{2}y^2(t))\sum_{i=0}^{n}|P_i(t,x(t))|\int_0^t q(s)ds \\ &-\sum_{i=1}^{n}\lambda_i\int_{t-\tau_i}^{t}y^4(u)du+\frac{1}{4}\sum_{i=1}^{n}(N_i\tau_i)\int_0^t q(s)y^2(s)ds \\ &+\frac{1}{4}\sum_{i=1}^{n}N_i\int_{t-\tau_i}^{t}\int_0^t q(s)y^4(\xi)dsd\xi+q(t)y^2(t)\sum_{i=1}^{n}\mu_i\int_t^{\infty}|P_i(\eta,x(\eta))|d\eta \\ &-\sum_{i=1}^{n}\mu_i|P_i(t,x(t))|\int_0^t q(s)y^2(s)ds+\frac{1}{2}y^4(t)\sum_{i=1}^{n}(N_i\tau_i)\int_0^t q(s)ds. \end{aligned}$$

Applying the assumption (ii) of the theorem on the above inequality yields,

$$\begin{aligned} V'(t) \le &-\frac{1}{2}\Big[2a(t)f_0 - 2\sum_{i=1}^{n}(\lambda_i\tau_i) - \sum_{i=1}^{n}(N_i\tau_i)\int_0^{\infty} q(s)ds\Big]y^4(t)\\ &-\frac{1}{2}\Big[2b(t)g_0 - \sum_{i=1}^{n} M_i\int_0^{\infty} q(s)ds - 2K_1\sum_{i=1}^{n}\mu_i\int_0^{\infty}|P_i(\eta,x(\eta))|d\eta\Big]y^2(t)\\ &+\frac{1}{4}\sum_{i=1}^{n}\Big[2M_i(1-2\mu_i)+N_i\tau_i\Big]\int_0^t q(s)y^2(s)ds\\ &+\frac{1}{4}\sum_{i=1}^{n}\Big[N_i\int_0^{\infty} q(s)ds - 4\lambda_i\Big]\int_{t-\tau_i}^{t} y^4(u)du. \end{aligned}$$

By taking $\mu_i = \frac{2M_i+N_i\tau_i}{4M_i}$, $\lambda_i = \frac{N_i\int_0^{\infty} q(s)ds}{4}$ and using the assumptions (v) and (vi) of the Theorem 4.1, we obtain

$$\begin{aligned} V'(t) \le &-\frac{1}{2}\Big[2b(t)g_0 - \sum_{i=i}^{n} M_i\int_0^{\infty} q(s)ds\\ &- 2K_1\sum_{i=1}^{n}\Big(\frac{2M_i+N_i\tau_i}{4M_i}\Big)\int_0^{\infty}|P_i(\eta,x(\eta))|d\eta\Big]y^2(t)\\ &-\frac{1}{2}\Big[2a(t)f_0 - \frac{3}{2}\sum_{i=1}^{n}(N_i\tau_i)\int_0^{\infty} q(s)ds\Big]y^4(t),\\ \le &-\frac{1}{2}\Big[2b(t)g_0 - M\int_0^{\infty} q(s)ds - \frac{1}{2}K_1\Big(2+D\tau^*\Big)\int_0^{\infty}|P_i(\eta,x(\eta))|d\eta\Big]y^2(t)\\ &-\frac{1}{2}\Big[2a(t)f_0 - \frac{3}{2}N\tau^*\int_0^{\infty} q(s)ds\Big]y^4(t),\\ \le &-\frac{1}{2}\Big[2b(t)g_0 - M\int_0^{\infty} q(s)ds - \frac{1}{2}K_1\Big(2+D\tau^*\Big)\int_0^{\infty}|P_i(\eta,x(\eta))|d\eta\Big]y^2(t), \end{aligned}$$

there exists a positive $\delta$ such that

$$V'(t) \le -\delta y^2(t) \le 0, \tag{4.5}$$

for all $y$.

Now, the solutions $(x(t),y(t))$ of the system (3.1) will only fail to be defined after some time $T$ if the condition

$$\lim_{t\to T^-}(x^2(t)+y^2(t)) = +\infty, \tag{4.6}$$

is met. By taking $(x(t),y(t))$ to be a solution of the system (3.1) with initial condition $(x_0,y_0)$, it is clear that the Lyapunov function defined in (4.1) is positive semi definite, meaning that, $V = V(t,x(t),y(t)) \ge 0$ and its derivative $V'(t,x(t),y(t)) \le 0$, for all point $(x(t),y(t))$. Thus, the function $V(t)$ is bounded on the interval $[0,T]$. But, it has already been shown earlier that

$$d_1(x^2+y^2) \le V(t), \tag{4.7}$$

and

$$\dot{V}(t) \leq 0. \tag{4.8}$$

Therefore, on integrating inequality(4.8) from $t_0$ to $T$ we obtain

$$V(T) \leq V(t_0). \tag{4.9}$$

From inequalities (4.7) to (4.9), we can easily deduce that

$$d_1(x^2+y^2) \leq V(t) \leq V(T) \leq V(t_0),\ \forall t \geq T. \tag{4.10}$$

It is now clear from the inequality (4.10) that,

$$(x^2+y^2) \leq d_1^{-1}V(t_0) = N, \tag{4.11}$$

where $V(t_0) = \beta > 0$ and $N = d_1^{-1}\beta$. It then follows from the inequality (4.11) that,

$$|x(t)| \leq N^*,\ |y(t)| \leq N^*,\ N^* = \sqrt{N},\ \forall t \geq T > t_0 \geq 0.$$

We can easily conclude from the above that the condition stated in inequality (4.6) is not possible. Therefore, all the solutions of the system (3.1) and consequently equation (1.6) are bounded. Hence, the proof of the theorem is complete. □

**Theorem 4.2.** *Under the assumptions of Theorem 4.1, the trivial solution of the system* (3.1) *is asymptotically stable.*

*Proof.* It has been shown from the proof of Theorem 4.1 that

$$V(t,x(t),y(t)) \geq d_1(y^2+x^2) \geq 0,$$

and

$$V'(t) \leq -\delta y^2(t).$$

From these two inequalities, we established the stability of the trivial solution of equation (1.6). To prove the asymptotic stability of the trivial solution, we employed LaSalle's invariant principle.

Let us define

$$W \equiv= W(t,x,y) = \{(t,x(t),y(t)) : V'(t,x(t),y(t)) = 0\}.$$

Already, we have that

$$V'(t) \leq -\delta y^2(t),\ \delta > 0.$$

Going by the definition of $W$, it must then mean that, $y = 0$ and $y = 0$ also implies that $\frac{dx}{dt} = y = 0$. Integrating $\frac{dx}{dt} = y = 0$, we get $x = \eta, \eta \in \mathbb{R}, \eta \neq 0$. By putting $y = 0$ into the system (3.1) and following the assumptions of Theorem 2.1, we obtain

$$c(t)h(x) = 0.$$

But since $c(t) > 0$, then we must have $h(x) = 0$. However, $h(x) = 0$ only when $x = 0$. Therefore, $\eta = x = 0$. Hence, $x = y = 0$. Therefore, the largest invariant set contained in $W(t,x,y)$ is $(t,0,0)$. Thus, the zero solution of the system (3.1) is asymptotically stable and the proof of the theorem is established. □

**Theorem 4.3.** *Under the assumptions of Theorem 4.1, if* $(x(t),y(t))$ *is any solution of the system* (3.1) *with given initial condition* $(x_0,y_0)$*, then* $y(t) \in L^2[0,\infty)$*, meaning that the first derivative of the system* (3.1) *is square integrable.*

*Proof.* If $(x(t),y(t))$ is any solution of the system (3.1) with initial conditions $(x_0,y_0)$ then, from the proof of Theorem 4.1, we have

$$V'(t,x(t),y(t)) \leq -\delta y^2(t). \tag{4.12}$$

Integrating (4.12) from 0 to $t$ we have

$$0 \leq V(t,x(t),y(t)) \leq V(0,x(0),y(0)) - \delta \int_0^t y^2(u)du.$$

This implies that

$$\int_0^\infty y^2(u)du < \infty.$$

Thus, we conclude that $y^2(t) \in L^2[0,\infty)$ and the proof of Theorem (4.3) is now complete. □

**Theorem 4.4.** *If the assumptions of Section 3 hold, then the trivial solution of the system* (3.1) *is uniformly asymptotically stable.*

*Proof.* Let $(x_t,y_t)$ be any solution of the system (3.1), from inequalities (4.2), (4.4) and (4.5) the trivial solution of the system (3.1) is uniformly asymptotically stable. □

**Corollary 4.1.** *In addition to assumptions (i) and (ii) of Section 3, if* $\left|\sum_{i=1}^{n} P_i(t,x(t-\tau_i))\right| \leq \kappa,\ 0<\kappa<\infty$*, then the solutions of the system* (3.1) *are bounded, uniformly bounded and uniformly ultimately bounded.*

*Proof.* Let $(x_t,y_t)$ be any solution of the system (3.1), from inequality (4.2), estimate (4.3), inequalities (4.4) and (4.5) the solutions of the system (3.1) are bounded, uniformly bounded and uniformly ultimately bounded. □

## 5. Example

In this section, we shall consider a special case of equation (1.6) to establish the correctness of the results discussed in Section 4.

**Example 5.1.** *Consider the following second order integro-differential equation with delay*

$$\begin{aligned} x''(t)+\left(\frac{7+2t^2}{3+t^2}\right)\left(\frac{x'^2}{2}+\frac{x'^2}{1+|t|+|x|+|x'|}\right)x'(t)+(2+\sin 3t)\left(\frac{x'}{3}+\frac{x'}{1+t^2+x^2+x'^2}\right) \\ +e^{-2t}\left(\frac{3x+2x^3}{1+x^2}\right)=\sum_{i=1}^{n}\left(\frac{1}{5+t^2}-\frac{1}{3+x^2(t-\tau_i)}\right)\int_0^t\left(1+\frac{1}{2+s^2}\right)|x'(s)|ds. \end{aligned} \tag{5.1}$$

*Equation* (5.1) *as a system of first order differential equations is*

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\left(\frac{7+2t^2}{3+t^2}\right)\left(\frac{y^2}{2}+\frac{y^2}{1+|t|+|x|+|y|}\right)y(t)-(2+\sin 3t)\left(\frac{y}{3}+\frac{y}{1+t^2+x^2+y^2}\right) \\ &\quad -e^{-2t}\left(\frac{3x+2x^3}{1+x^2}\right)+\left(\frac{1}{5+t^2}-\frac{1}{3+x^2}\right)\int_0^t\left(1+\frac{1}{2+s^2}\right)|y(s)|ds \\ &\quad -\sum_{i=1}^{n}\left(\frac{1}{5+t^2}-\frac{1}{3+x^2}\right)\int_{t-\tau_i}^{t}\int_0^t\left(1+\frac{1}{2+s^2}\right)|y(s)|\left(\frac{2x(\xi)}{(3+x^2(\xi))^2}\right)y(\xi)dsd\xi. \end{aligned} \tag{5.2}$$

*Comparing equations* (3.1) *and* (5.2) *we have the following relations:*

(i) *the function*

$$a(t) := \frac{7+2t^2}{3+t^2} = 2+\frac{1}{3+t^2}.$$

*Since* $0 < \frac{1}{3+t} \leq \frac{1}{3}$ *for all* $t \geq 0$, *it follows that*

$$2 = a_0 \leq a(t) \leq a_1 = 2.33,$$

*(see Figure* 1.*) Also the function*

$$b(t) := 2+\sin 3t.$$

*Noting that* $-1 \leq \sin 3t \leq 1$ *for all* $t \in [-2\pi, 2\pi]$ *and on* $\mathbb{R} \supset [-2\pi, 2\pi]$, *we conclude that*

$$1 = b_0 \leq b(t) \leq b_1 = 3,$$

*for all* $t \in \mathbb{R}$. *The function* $b(t)$ *and its bounds are shown in Figure* 2.
*Next the function*

$$c(t) := e^{-2t} > 0,$$

*for all* $t$ *and*

$$c'(t) = -2e^{-2t} < 0,$$

*for all* $t$. *Moreover, the function*

$$h(x) := \frac{3x+2x^3}{1+x^2} = 2x+\frac{x}{1+x^2}.$$

*Clearly,* $h(0) = 0$, *since* $1+x^2 \geq 1$, *for all* $x$ *it follows that*

$$0 < \frac{1}{1+x^2} \leq 1$$

*for all* $x$ *and*

$$\lim_{x\to\infty}\frac{1}{1+x^2} = 0,$$

*we conclude that*

$$2 = \alpha_0 \leq \frac{h(x)}{x} \leq \alpha_1 = 3,$$

*for all* $x \neq 0$.

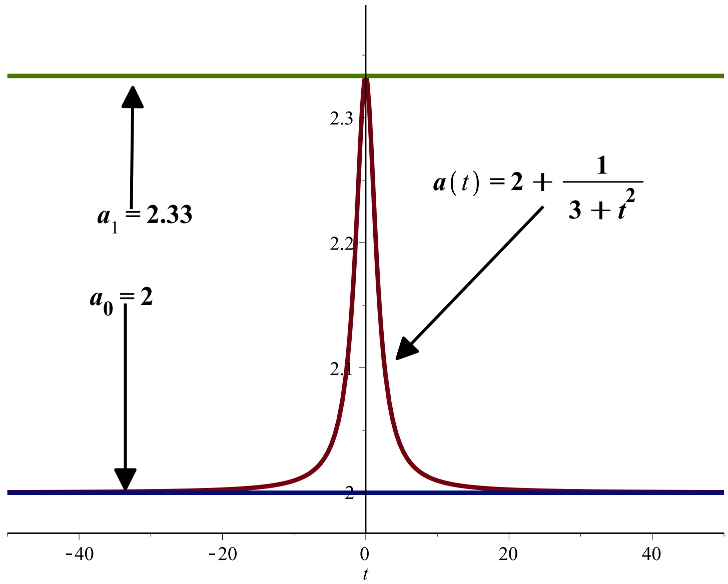


FIGURE 1. *The function* $a(t)$ *for* $t \in [-50, 50]$.

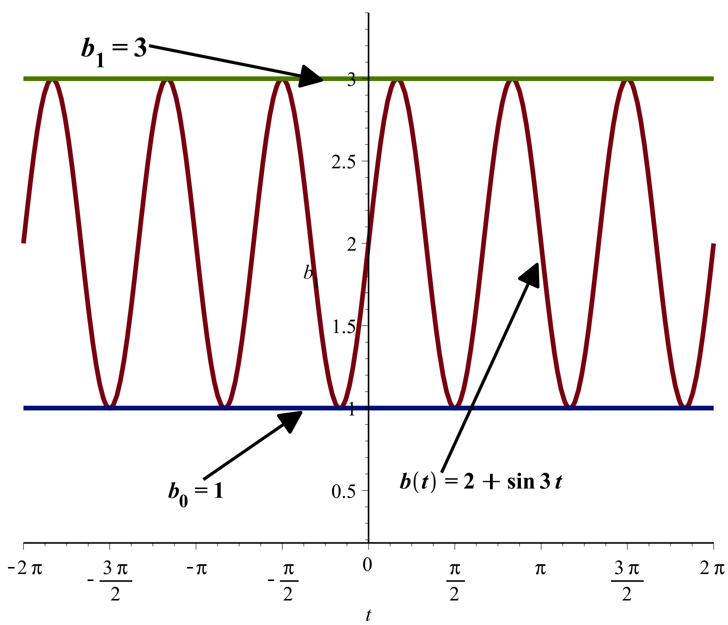


FIGURE 2. *The function* $b(t)$ *for* $t \in [-2\pi, 2\pi]$.

(ii) *The function*

$$f(t,x,y) := \frac{1}{2}y^2 + \frac{y^2}{1+|t|+|x|+|y|},$$

*it can be seen that* $f(t,x,0) = 0$ *and that*

$$0 < \frac{1}{1+|t|+|x|+|y|} < 1,$$

*for all* $t,x$ *and* $y$. *Therefore,*

$$\frac{1}{2} = f_0 \leq \frac{f(t,x,y)}{y^2} \leq f_1 = \frac{3}{2}$$

*for all* $t,x,y \neq 0$. *In a similar manner the function*

$$g(t,x,y) := \frac{1}{3}y + \frac{y}{1+t^2+x^2+y^2},$$

*satisfies* $g(t,x,0) = 0$ *and*

$$\frac{1}{3} = g_0 \leq \frac{g(t,x,y)}{y} \leq g_1 = \frac{4}{3},$$

*for all* $t,x,y \neq 0$.

(iii) *The function*

$$K(t,y(t)) := \left(1 + \frac{1}{2+t^2}\right)|y|.$$

*Now, from inequalities in assumption (iii), we have*

$$q(t) := 1 + \frac{1}{2+t^2}.$$

*It is not difficult to show that*

$$1 = K_0 \leq q(t) \leq K_1 = \frac{3}{2}$$

*for all* $t$. *Therefore, the function*

$$K(t,y(t)) = \left(1 + \frac{1}{2+t^2}\right)|y| \leq K_1|y(t)|,$$

*satisfies assumption (iii) with* $K_1 = \frac{3}{2}$. *The shape and path of* $K(t,y)$ *and* $q(t)$ *are shown in Figures* 3. *and* 4. *respectively.*

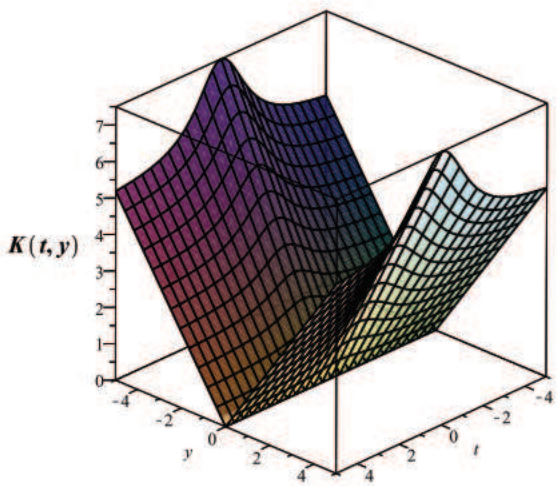


FIGURE 3. *The function* $K(t,y)$ *for* $t,y \in [-4,4]$.

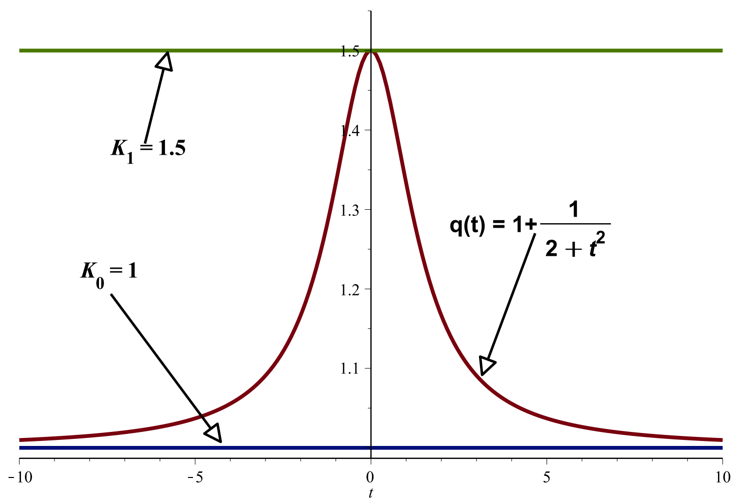


FIGURE 4. *The function* $q(t)$ *for* $t \in [-10,10]$.

(iv) *The function*

$$\sum_{i=1}^{n} P_i(t,x(t)) := \frac{1}{5+t^2} - \frac{1}{3+x^2}.$$

*Since* $\dfrac{1}{3+x^2} > 0$ *for all* $x$ *and* $\lim\limits_{x\to\infty} \dfrac{1}{3+x^2} = 0$ *it follows that*

$$\sum_{i=1}^{n} P_i(t,x(t)) = \frac{1}{5+t^2} - \frac{1}{3+x^2} \leq \frac{1}{5+t^2}$$

*and*

$$\sum_{i=1}^{n} |P_i(t,x(t))| \leq \sum_{i=1}^{n} |P_i(t)| = \left|\frac{1}{5+t^2}\right|.$$

*Considering the fact that* $5+t^2 > 5$ *for all* $t$, *it follows that*

$$\sum_{i=1}^{n} |P_i(t,x(t))| \leq \sum_{i=1}^{n} |P_i(t)| \leq \sum_{i=1}^{n} M_i = \frac{1}{5}$$

*for all* $t$. *In addition*

$$\frac{\partial}{\partial x}P(t,x(t)) = \frac{2x}{(3+x^2)^2},$$

*and*

$$\left|\frac{\partial}{\partial x}P(t,x(t))\right| = \left|\frac{2x}{(3+x^2)^2}\right| \leq N = 0.125$$

*for all* $x \in [-10,10]$ *see Figure* 5., *the inequality holds for all* $x \in \mathbb{R}$.

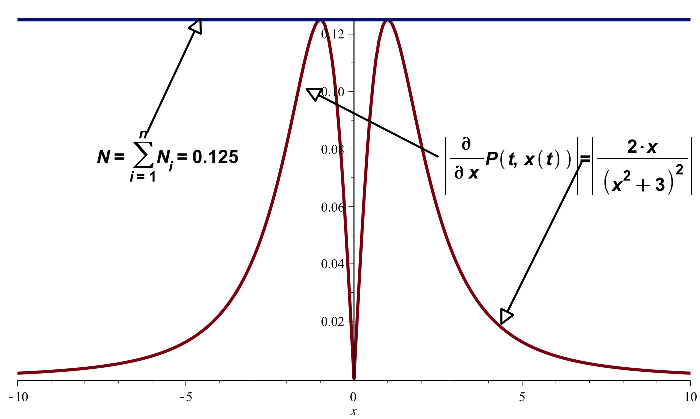


FIGURE 5. *The* $\left|\frac{\partial}{\partial x}P(t,x(t))\right|$ *for* $x \in [-10,10]$.

(v) *From assumption (v) we have*

$$3N\tau^* K_0 \leq 3N\tau^* \int_0^{\infty} q(s)ds \leq 4a(t)f_0 \leq 4a_1 f_0,$$

*so that* $\tau^* \leq 12.44$.

(vi) *Finally, the inequality in assumption (vi) is*

$$M\int_0^{\infty} q(s)ds + \frac{1}{2}K_1\left(2+D\tau^*\right)\int_0^{\infty} |P_i(\eta,x(\eta))|d\eta \leq 2b(t)g_0.$$

*After substituting the estimates we find that* $1.607 < 2$.

*The estimates in items (i) to (vi) fulfill all the assumptions of Theorems 4.1, 4.2, 4.3, 4.4 and Corollary 4.1, hence the conclusions follow immediately.*

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