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WORKS INVOLVING MARC KARSNER AND FRENCH MATHEMATICIANS

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Abstract. Works due to Marc Kasner are well known and famous in several domains. Here are short abstracts on only some domains Krasner's works: on hypergroups, ultrametric analysis and Hederic fields, mainly made with French mathematicians.

1. Hypergroups

Hypergroups were introduced by Frédéric Marty in 1934. Although, Marc Krasner, independently of Marty, in first draft of his Thesis (1935 handed to Chevalley for criticisms) and almost at the same time, introduced the concept of hypergroup, he gave Marty a priority and accepted his terminology. His Thesis [20] and two papers (under the same title) was published in [21], [22]. Hypergroups have been subsequently studied by different authors: Comer, Dresher, Eaton, Krasner, Ore, Utumi, among others. Marc Krasner used them in his theory of non-normal ramiffcation ([20]–[24]), Prenowitz and Jantosciak in geometry; Corsini for linearly ordered abelian groups. (Unfortunately, the name hypergroup was abusively used for an entirely different concept, in harmonic analysis that has no connection). Here a hypergroup is a non-empty set H endowed with a multivalued bi-nary operation which assigns to each pair (x, y) of elements of H a non-empty subset $xy \subset H$, obeying two rules ensuring associativity and reproduction. In order to extend the operation to subsets of H, given $A \subset H, B \subset H$, just write $AB = \bigcup(x, y)_{(x,y)\in A\times B}$, identifying each element x of H to the singleton x. Here are the two above mentioned rules (axioms). For each $x,\ y,\ z \in H$:

1) $(xy)z = x(yz)$;
2) $xH = Hx = H$.

If each product xy is a singleton, (that is, if the operation is univalued) then the hypergroup is just a group. Next, many other kinds of hypergroups appear. For instance, let G be a group and S a subgroup of G. Then let $H = \{xS : x \in G\}$ be the set of right classes of G modulo S, i.e. G/S.

Setting $(xS)(yS) = \{zS : zS \subset xSyS\}$ obtains a multivalued operation on H which is seen to be a hypergroup. Those are the so-called D-hypergroups. In 1941, Marc Krasner gave a characterization of D-hypergroups, using the second order language, in two Notes aux Comptes Rendus (the publication of the French Académie des Sciences) [23], [24]. Next, Labib Haddad and Yves Sureau (who was an alumno of Krasner) showed that no such characterization exists for D-hypergroups, in the first order language [19].

Hypergroups also found a peculiar application to kinship systems. The kinship system of the Murngins, a tribe among the Aborigines in Australia, so much intrigued Claude Lévy-Strauss that he asked André Weil for an advise. The tribe has 8 different clans. Marriages are governed by the following (somewhat complex) laws. The clan of each member of the tribe is determined by the clan of the mother. A woman can choose her husband in any of two clans determined by her own clan, similarly for men choosing a wife. There are two types of marriages: Normal marriages and Optional marriages. The clan of each member of the tribe must be the same as the clan of the optional mother in law and the normal mother in law. Moreover, the normal mother in law of the normal mother in law must be in the same class as the optional mother in law of the optional mother in law. André Weil used a subgroup G of the group of permutations on 8 elements, isomorphic to the product group $(\frac{\mathbb{Z}}{4\,\mathbb{Z}}) \times (\frac{\mathbb{Z}}{2\,\mathbb{Z}})$ to explain how the system operates. Endowing the group with a multivoque operation, Haddad and Sureau [18] turned it into a hypergroup which further explains the kinship laws, by amalgamation [23].

Marc Krasner, in conception with his work on valuation first introduced hyperrings in 1956 [26] and hyperfields in 1982 [30]. At the beginning the hyperrings and hyperfields were not well understood nor accepted, even criticized because of their generality, today they both find significant applications. Alain Connes and Caterina Consani show that the theory of hyperrings, due to M. Krasner, supplies a perfect framework to understand the algebraic structure of adéle class space $H_{\mathbf{K}} = A_{\mathbf{K}}/\mathbf{K}^{\times}$ of a global field $\mathbf{K}$.

2. Theory of analytic functions on a p-adic field

Fields complete with respect to an ultrametric absolute value are well known and p-adic fields are not the only examples but are probably the most famous. However, let us just notice that new examples are now appearing and present an increasing interest: the Levi-Civita fields, where Khodr Shamseddine and Martin Berz are among the main specialists [35]. In such constructions, we can get an algebraically closed field complete for an ultrametric absolute value. Let us first recall a basic interesting result, relating the topology, on algebraic extensions known as Krasner's Lemma [31].

Let K be a field of chracterisric 0*, complete for an ultrametric absolute value, let L be an algebraic extension of K provided with the continuation of the absolute value of K and let $a_1, \dots, a_n \in L$ be conjugate of degree n over K. Let $b \in L$ be such that $|b - a_1| < |b - a_j|$ $\forall j \geq 2$. Then $K[a_1] \subset K[b]$.*

Now, it is possible to examine power series and Laurent series on an algebraically closed field complete for an ultrametric absolute value. Such studies were made by the beginning of the 20th century but came up against a serious problem: inside a disk D of $\mathbb{K}$, of diameter r, every point b is a center of D and the disk of center b and diameter r is identical to D. Consequently, it is impossible to expand a power series converging in D, outside of D by analytic extension, as it is done in complex analysis. This gap was frequently recalled by Marc Krasner: "You can't get out!" he said...laughing a lot!...and everybody laughed with him!...

The solution imagined by Marc Krasner took something from *Runge's theorem* for complex holomorphic functions showing that on an open bounded subset D of $\mathbb{C}$, the holomorphic functions are the uniform limits of rational functions having no pole in the closure of D.

Then Krasner first defined in $\mathbb{K}$ a family of sets called *quasi-connected sets*, stable through a chained union and through intersection, and also satisfying a strong property on its holes.

The main property of the quasi-connected sets is the following: given $a,\ b \in D$, the set of circles of center a and diameter $\leq |a - b|$, that are not included in D is finite.

Krasner then defined the so-called *analytic elements on D* as the limits of sequences of rational functions with respect to the uniform convergence on D. Krasner first showed that if an element f on a quasi-connected set D is equal to zero on a disk included in D, then it is identically equal to zero. Next, Krasner proved that the analytic elements enjoy a property of analytic extension, from a quasi-connected set D to another one D' such that $D \cap D' \neq \emptyset$ thanks to the existence of an extension $\widehat{\mathbb{K}}$ of $\mathbb{K}$ having a non-countable residue class field. This analytic extension from D to $D \cup D'$ defined in such a way is unique! (see Krasner [27]–[29]).

Krasner's theory has proven to be very fruitful and essential tools for most works in functional analysis as well as the p-adic Nevanlinna theory and its applications, and the Corona problrm in the ultrametric disc. It was generalized by Philippe Robba, working with Elhanan Motzkin [31], [32]. Next, infraconnected subsets were defined and generalized the family of quasi-connected subsets [5], [6], [7]. It was then proven that an infraconnected subset owns the property of analyticity if and only if it has no T-filter with a non-empty beach and that property is essential for studying algebras of analytic elements denoted by $H(D)$ as well as for generalizing the theory of analytic functions. This is also essential to study the algebraic properties of algebras of analytic elements, particularly regarding their ideals [6], [7].

Another very important property shown by Krasner concerning analytic elements on quasi-connected sets is wonderful *Mittag-Leffler theorem* showing that the Banach space of analytic elements on a closed bounded quasi-connected set D is a topological direct sum of Banach subspaces respectively attached to each hole of D (Krasner 1964; [28]). This result was later generalized to *infraconnected sets* by Ph. Robba (1973; [32]). Many properties of the analytic elements derive from this: the absolute value of an analytic element admits a limit along certain filters called *circular filters* [7], [8] , [9], [16] and that limit constitutes a multiplicative semi-norm. That property is not explicitly formulated in that way in Krasner's work, but clearly appears in his papers and particularly in the proceedings of the Conference held at Clermont-Ferrand in 1964 [28].

Let me recall that Bernard Guennebaud [17] and I on one hand [8], [9] and later Vladimir Berkovich on the other hand [1] showed the importance of multiplicative semi-norms. All this is essential in the construction of the ultrametric holomorphic functional calculus that has made possible for example, the study of the ultrametric spectral theory by Nicolas Maïnetti and me [8], [9], [10] and the so-called ultrametric *Corona problem* [12] where T-sequences, studied by Marie-Claude Sarmant, are also involved [34].

Simultaneously to the work of Mark Krasner, John Tate defined affinoid subsets in order to obtain another theory of analytic functions through a sheaf of algebras and that theory also applies to functions of several variables [36]. But, in one variable, the inconvenient of Tate's theory is that the set of affinoid domains is very short comparatively to this of quasi-connected sets and does not let us obtain a suitable family of functions. I compared the two theories by defining *Krasner-Tate algebras*: Banach algebras of analytic elements $H(D)$ where D is an affinoid subset of the field $\mathbb{K}$ [5]. The sets D are then quite simple but that sample of Banach algebras is useful in many circumstances.

More generally, the use of analytic elements defined by Marc Krasner is very frequent and often implicit, for instance, when we study meromorphic functions, as for example through the p-adic *Nevanlinna theory* developed by Abdelbaki Boutabaa [2] with its various applications to problems of uniqueness and was generalized in different contexts [3].

By the sixties, Bernard Dwork began studying the p-adic differential equations and in the seventies [4], that became an important topic of studies (by Philippe Robba, Gilles Christol, Bruno Chiarelotto, Zoltan Mebkhout, Yves

André and several others) [32]. All colleagues have recognized the importance of foundations made by Marc Krasner in holomorphic functions theory.

3. Theory of analytic functions on a Hederic field

In the seventies, Marc Krasner considered valuated fields mentioned by Krull in 1928, whose valuation takes values in an ordered group of any rank superior to 1, i.e. a field $\mathbb{K}$ provided with a mapping denoted by $|\ .\ |$ whose set of values is a totally ordered multiplicative group G and an element 0 strictly inferior to all elements of G, satisfying further: $|a| = 0$ if and only if $a = 0$, $|ab| = |a|.|b|$, $|a + b| \leq \max(|a|, |b|)$.

One then set $d(a, b) = |a - b|$ and the mapping d defines a uniform structure on $\mathbb{K}$. So, one can consider the completion of $\mathbb{K}$ and hence consider such fields $\mathbb{K}$ complete with respect to that topology.

A preorder is defined on G as $a \preceq b$ if there exists $m,\ n \in \mathbb{Z}$ such that $b^m \leq a \leq b^n$ and the quotient of G by the equivalence relation defined by the preorder is called *framework* of G, denoted by $\mathcal{C}(G)$. Two main situations then appear, letting power series converge: either $\mathcal{C}(G)$ has a maximum, or there exists a cofinal sequence in $\mathcal{C}(G)$, i.e. a sequence (C_n) such that, for every $C \in \mathcal{C}(G)$, there exists $q \in \mathbb{N}$ such that $C \leq C_q$. The first case leads us to a trivial generalization of the classical ultrametric analysis but the second case leads us to a completely new analysis. That was the topic for doctoral thesis, advised by Marc Krasner, and defended under his supervision, to two young colleagues on such a field $\mathbb{K}$ satisfying the hypotheses of the second case: Rachel Hobeika and Yvette Perrin who called Hederic fields these fields [13], [14]. Otherwise Krasner's theories has been (and still are) the germ of many doctoral thesis.

Rachel Hobeika studied power series and Laurent series on $\mathbb{K}$, showing that the domain of convergence is either the whole field $\mathbb{K}$, or a unique point, or the empty set. Yvette Perrin then examined the analytic elements defined again as the uniform limits of sequences of rational functions. The theory, actually was made in several variables but for simplicity, here we will summarize it in one variable.

Methods are very different from those used in classical ultrametric analysis. Yvette Perrin showed that every analytic element that is null on a disk is null everywhere, hence the analytic continuation is uniform. Similarly to the process used for classical ultrametric fields, analytic functions on an open subset D are defined by a chained union of open subsets $(D_i)_{i \in J}$ and analytic elements f_i on D_i such that $f_i(x) = f_j(x)\ \forall x \in D_i \cap D_j$. This way, all open subsets D of $\mathbb{K}$ are analytic sets.

The set $F(D)$ of analytic functions on an open set D is a $\mathbb{K}$-algebra, the derivative of an analytic function also is an analytic function and the uniform limit of a sequence of analytic functions is an analytic function.

A subset Z of an open set D is the set of zeros of an analytic function $f \in F(D)$ if and only if it is discrete and countable. Every function $f \in F(D)$ satisfies the Weierstrass theorem: it is equal to a convergent product of polynomials having for zeros some zeros of f and an invertible function $h \in F(D)$.

If the residue characteristic of $\mathbb{K}$ is null, the functions $f \in F(D)$ satisfy a finite increasing theorem and admit a Taylor development.

Among many other properties mainly shown by Yvette Perrin, let us notice that every meromorphic function on D satisfies a Mittag-Leffler theorem. If D and D' are two open sets in $\mathbb{K}$, the algebras $F(D)$ and $F(D')$ are isomorphic if and only if there exists a bianalytic bijection from D onto D' and actually, the only bianalytic bijections from an open subset onto another one are the Möbius functions [15].

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